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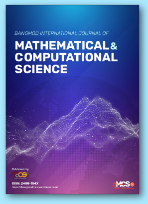

A TRIGONOMETRICALLY ADAPTED EMBEDDED PAIR OF EXPLICIT RUNGE-KUTTA-NYSTRÖM METHODS TO SOLVE PERIODIC SYSTEMS

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Abstract In this paper a 5(3) pair of explicit trigonometrically adapted Runge-Kutta-Nyström methods with four stages is derived based on an explicit pair appeared in the literature. The new adapted method is able to integrate exactly the usual test equation: $y'' = -w^2 y$. The local truncation error of the new method is obtained, proving that the algebraic order of convergence is maintained. The stability interval of the new method is obtained, showing that the proposed method is absolutely stable. The numerical experiments performed demonstrate the robustness of the new embedded pair in comparison with some standard codes available in the literature.

MSC: 65L05, 65L06
Keywords: Trigonometrically-fitted method; Runge-Kutta-Nyström; Periodic Problems; Initial Value Problems

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1. Introduction

Deriving efficient methods for solving numerically the special second-order initial-value problem of the form

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0, \tag{1.1}$$

whose solutions are periodic, where $y \in \Re^d$ and $f : \Re \times \Re^d \to \Re^d$ is sufficiently differentiable, has attracted the interest of many researchers. Problem (1.1) is encountered in the areas of quantum chemistry, fluid mechanics, physical chemistry, astronomy and many others. To solve (1.1) directly, the class of Runge-Kutta-Nyström (RKN) methods has been largely used. Related to an efficient use of these methods, the embedded technique was firstly proposed by Fehlberg in [1] in order to provide an estimate of the error committed on each step when applying Runge-Kutta type methods. Van de Vyver in [2] developed a 5(3) pair with four stages of explicit RKN methods for solving (1.1). Similarly, Franco have developed a 5(3) pair of explicit ARKN methods with four stages in [3]. A lot of RKN adapted methods have been derived by different researchers, among them we mention those by Simos [4], Kalogiratou and Simos [5], Van de Vyver [6], Liu [7] and Demba et al. [8, 9]. Senu et al. [10] constructed an explicit pair of embedded RKN methods designed for solving periodic problems, Franco et al. [11] developed two pairs of embedded explicit RKN methods for solving oscillatory problems, Anastassi and Kosti [12] constructed a 6(4) embedded RKN optimized pair for numerically solving periodic problems. Recently, Demba et al. in [13] derived a new explicit exponentially-fitted embedded RKN method for solving the problem in (1.1). Most Recently, Demba et al. in [14] derived a new explicit phase-fitted and amplification-fitted embedded RKN method for solving the problem in (1.1). In this work, we formulate a new trigonometrically adapted 5(3) embedded pair of explicit RKN methods based on the 5(3) pair of explicit type given in [2] for solving the problem in (1.1). The derived method solves exactly the usual test equation: $y'' = -w^2 y$. The numerical experiments performed bring out the efficiency of the developed method compared with standard embedded RKN codes of orders 5(3) with four stages.

The rest of this paper is as follows: The definition of an explicit RKN pair, the concept of a trigonometrically adapted RKN method, and the development of a trigonometrically-fitted explicit RKN method are addressed in Section 2. Section 3 is devoted to the derivation of the new code. The algebraic order of the new embedded pair derived in this paper is analyzed, and we give details about the linear stability of the derived pair in Section 4. Some numerical results are presented in Section 5. Comments on the obtained results are given in Section 6, and finally, in Section 7 we give a conclusion.

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2. Elementary concepts

An explicit RKN method with r-stages for solving the problem in (1.1) is generally expressed by the formulas:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{l=1}^{r} b_l f(x_n + c_l h, Y_l), \tag{2.1}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l f(x_n + c_l h, Y_l), \tag{2.2}$$

$$Y_l = y_n + c_l hy'_n + h^2 \sum_{j=1}^{l-1} a_{lj} f(x_n + c_j h, Y_j), \tag{2.3}$$

where y_{n+1} and y'_{n+1} denote approximations for $y(x_{n+1})$ and $y'(x_{n+1})$, respectively, and $x_{n+1} = x_n + h,\ n = 0, 1, ...,$ being h a fixed stepsize.
The above explicit method may be formulated compactly using the Butcher tableau, in the form

$$\begin{array}{c|c} c & A \\ \hline & b \\ & d \end{array}$$

where $A = (a_{ij})_{r\times r}$ is a lower triangular matrix of coefficients, $c = (c_1, c_2, ..., c_r)^T$ is the vector of stages, and $b = (b_1, b_2, ..., b_r)$, $d = (d_1, d_2, ..., d_r)$ are two vectors related to the coefficients of the method. For short, this can be denoted as (c, A, b, d).
A $m(n)$ embedded-type pair of RKN methods comprises two of such methods, one given by (c, A, b, d) with order m and another one $(c, A, \hat{b}, \hat{d})$ with same coefficients c and A with order n $(n < m)$. The higher order method provides the approximate values y_{n+1}, y'_{n+1}, while a second pair of approximate values $\hat{y}_{n+1}$, $\hat{y}'_{n+1}$ is provided by the lower order method. This second approximation is obtained for the sole purpose of providing an estimate of the local truncation error.

A pair of RKN embedded methods may be expressed by the following Butcher tableau:

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & d^T \\ \hline & \hat{b}^T \\ & \hat{d}^T \end{array}$$

In this paper, a variable stepsize approach is considered on the basis of a cheap local error estimation provided by the embedding procedure. The local error estimate at the point $x_{n+1} = x_n + h$ is obtained through the differences $\eta_{n+1} = \hat{y}_{n+1} - y_{n+1}$ and $\eta'_{n+1} = \hat{y}'_{n+1} - y'_{n+1}$.
Let $\text{Est}_{n+1} = \max(\|\eta_{n+1}\|_\infty, \|\eta'_{n+1}\|_\infty)$ denote the local error estimate used to manage the step-length h on each iteration. In order to advance the solution of the problem in hand we use the step-length control strategy presented in [7]:

- if $Est_{n+1} < Tol/100$, then $h_{n+1} = 2h_n$,
- if $Tol/100 \leq Est_{n+1} < Tol$, then $h_{n+1} = h_n$,
- if $Est_{n+1} \geq Tol$, then $h_{n+1} = h_n/2$ and repeat the step.

being Tol the user tolerance.

Definition 2.1. An explicit Runge-Kutta-Nyström method as given in the equations (2)–(4) is said to be trigonometrically adapted if it can integrate exactly a second order problem with a set of fundamental solutions as $\{\sin(wx), \cos(wx)\}$, where $w > 0$ is a parameter known as the principal frequency of the problem.

Applying the RKN method in (2)–(4) to the test equation $y^{''} = -w^2 y$ we obtain:

$$y_{n+1} = y_n + hy'_n + h^2 \sum_{l=1}^{r} b_l(-w^2 Y_l), \tag{2.4}$$

$$y'_{n+1} = y'_n + h \sum_{l=1}^{r} d_l(-w^2 Y_l), \tag{2.5}$$

where

$$Y_1 = y_n, \tag{2.6}$$

$$Y_l = y_n + c_l h y'_n + h^2 \sum_{j=1}^{l-1} a_{lj}(-w^2 Y_j), \; l = 2, 3, ..., r. \tag{2.7}$$

Let us assume that the exact solution of the test equation is given by $y(x) = e^{Iwx}$ with $I = \sqrt{-1}$ the imaginary unit. We compute the values of y_n, y_{n+1}, y'_n and y'_{n+1} for this exact solution and introduce them in the equations of the method (5)–(8). If we use the Euler's identity $e^{Iv} = \cos(v) + I\sin(v)$ and compare the real and imaginary parts, we get the following system:

$$\begin{cases} \cos(v) = 1 - v^2 \sum\limits_{l=1}^{r} b_l(1 - v^2 \sum\limits_{j=1}^{l-1} a_{lj} Y_j e^{-Iwx}), \\ \sin(v) = v - v^2 \sum\limits_{l=1}^{r} b_l c_l v, \\ \sin(v) = v \sum\limits_{l=1}^{r} d_l(1 - v^2 \sum\limits_{j=1}^{l-1} a_{lj} Y_j e^{-Iwx}), \\ \cos(v) = 1 - v^2 \sum\limits_{l=1}^{r} d_l c_l. \end{cases} \tag{2.8}$$

where $v = wh$.

3. Derivation of the new embedded pair

This section is devoted to obtain a new 5(3) pair of explicit trigonometrically-fitted embedded RKN methods based on the RKN5(3) embedded pair derived by Van de Vyver in [2]. The coefficients of the method in [2] are shown in Table 1.

TABLE 1. Coefficients of the RKN5(3) pair in [2]

0				
$\frac{1}{5}$	$\frac{1}{50}$			
$\frac{2}{3}$	$-\frac{1}{27}$	$\frac{7}{27}$		
1	$\frac{3}{10}$	$-\frac{2}{35}$	$\frac{9}{35}$	
	$\frac{1}{24}$	$\frac{25}{84}$	$\frac{9}{56}$	0
	$\frac{1}{24}$	$\frac{125}{336}$	$\frac{27}{56}$	$\frac{5}{48}$
	$-\frac{5}{24}$	$\frac{125}{168}$	$-\frac{9}{56}$	$\frac{1}{8}$
	$-\frac{1}{12}$	$\frac{25}{42}$	$\frac{9}{28}$	$\frac{1}{6}$

To obtain the adapted method in the embedding procedure, we consider firstly the coefficients of the method of third order in the RKN5(3) pair. We solve the system of equations in (2.8) considering these coefficients but taking four of them as unknowns, specifically we take $\hat{b}_2, \hat{b}_3, \hat{d}_2, \hat{d}_3$ as unknowns. In this way we obtain the following values:

$$\begin{aligned}
\hat{b}_2 &= -\frac{1}{840v^3\left(2\,v^2-45\right)}\Big(-54000\cos(v)\,v-27000\,v+28425\,v^3+81000\sin(v)-1265\,v^5\\
&\quad -93\,v^7-18000\sin(v)\,v^2+4200\,v^3\cos(v)+420\,v^4\sin(v)\Big),\\
\hat{b}_3 &= -\frac{9}{280v^3\left(2\,v^2-45\right)}\Big(600\cos(v)\,v+2400\,v-485\,v^3-3000\sin(v)+30\,v^5-2\,v^7+60\sin(v)\,v^2\Big),\\
\hat{d}_2 &= -\frac{1}{420v^2\left(2\,v^2-45\right)}\Big(27000\sin(v)\,v-62\,v^6-365\,v^4+13500\,v^2-9000\,v^2\cos(v)\\
&\quad +40500\cos(v)-40500-2100\,v^3\sin(v)+210\,v^4\cos(v)\Big),\\
\hat{d}_3 &= -\frac{3}{140v^2\left(2\,v^2-45\right)}\Big(-900\sin(v)\,v-4\,v^6+60\,v^4-765\,v^2+90\,v^2\cos(v)-4500\cos(v)\\
&\quad +4500\Big).
\end{aligned} \tag{3.1}$$

The corresponding Taylor series expansions in powers of v are given by

$$\begin{aligned}
\hat{b}_2 &= \frac{125}{168} - \frac{1}{140}v^2 - \frac{17}{7056}v^4 - \frac{1319}{12700800}v^6 - \frac{63331}{12573792000}v^8 - \frac{6313049}{29422673280000}v^{10} \\
&\quad - \frac{178769447}{18536284166400000}v^{12} - \frac{97066415543}{226884118196736000000}v^{14} + \cdots \quad , \\
\hat{b}_3 &= -\frac{9}{56} + \frac{1}{140}v^2 - \frac{163}{176400}v^4 - \frac{89}{1984500}v^6 - \frac{121937}{62868960000}v^8 - \frac{6379199}{73556683200000}v^{10} \\
&\quad - \frac{356901019}{92681420832000000}v^{12} - \frac{48543732709}{283605147745920000000}v^{14} + \cdots \quad , \\
\hat{d}_2 &= \frac{25}{42} - \frac{1}{280}v^2 - \frac{1}{720}v^4 - \frac{1189}{12700800}v^6 - \frac{6871}{1143072000}v^8 - \frac{447823}{2263282560000}v^{10} \\
&\quad - \frac{182797897}{18536284166400000}v^{12} - \frac{5718403829}{13346124599808000000}v^{14} + \cdots \quad , \\
\hat{d}_3 &= \frac{9}{28} + \frac{1}{280}v^2 - \frac{1}{1800}v^4 - \frac{3323}{63504000}v^6 - \frac{331}{178605000}v^8 - \frac{994871}{11316412800000}v^{10} \\
&\quad - \frac{178970647}{46340710416000000}v^{12} - \frac{11471890783}{66730622999040000000}v^{14} + \cdots \quad . \qquad (3.2)
\end{aligned}$$

As $v \to 0$, the coefficients $\hat{b}_2$, $\hat{b}_3$, $\hat{d}_2$ and $\hat{d}_3$ of the third order adapted method give the coefficients of the counterpart original method in the RKN5(3) approach. Similarly, if we use the coefficients of the fifth order method, except for b_1, b_2, d_1, d_2 which are taken as unknowns in the equations (2.8), the solution of this system results in

$$\begin{aligned}
b_1 &= -\frac{1}{120v^3}\left(120\cos(v)\,v + 480\,v + 2\,v^5 - 57\,v^3 + 12\,v^2\sin(v) - 600\sin(v)\right), \\
b_2 &= -\frac{1}{168v^3}\left(840\sin(v) - 840\,v - 7\,v^5 + 90\,v^3\right), \\
d_1 &= \frac{1}{360v^2}\left(360\sin(v)\,v + v^6 - 36\,v^2\cos(v) + 591\,v^2 - 33\,v^4 - 1800 + 1800\cos(v)\right), \\
d_2 &= -\frac{1}{1008v^2}\left(5040\cos(v) - 5040 - 210\,v^4 + 2145\,v^2 + 7\,v^6\right). \qquad (3.3)
\end{aligned}$$

The corresponding Taylor series expansions in power of v of the above coefficients are given by

$$
\begin{aligned}
b_1 &= \frac{1}{24}-\frac{11}{25200}v^4+\frac{1}{113400}v^6-\frac{1}{7983360}v^8+\frac{19}{15567552000}v^{10}-\frac{1}{118879488000}v^{12}\\
&\quad+\frac{19}{444609285120000}v^{14}+\cdots \qquad ,\\
b_2 &= \frac{25}{84}+\frac{1}{1008}v^4-\frac{1}{72576}v^6+\frac{1}{7983360}v^8-\frac{1}{1245404160}v^{10}+\frac{1}{261534873600}v^{12}\\
&\quad-\frac{1}{71137485619200}v^{14}+\cdots \qquad ,\\
d_1 &= \frac{1}{24}+\frac{13}{201600}v^6-\frac{1}{907200}v^8+\frac{31}{2395008000}v^{10}-\frac{23}{217945728000}v^{12}+\frac{1}{1609445376000}v^{14}+\cdots \qquad ,\\
d_2 &= \frac{125}{336}-\frac{1}{8064}v^6+\frac{1}{725760}v^8-\frac{1}{95800320}v^{10}+\frac{1}{17435658240}v^{12}-\frac{1}{4184557977600}v^{14}+\cdots \qquad . \qquad (3.4)
\end{aligned}
$$

As $v \to 0$, the coefficients b_1, b_2, d_1 and d_2 of the fifth order adapted method give the coefficients of the counterpart original method in the RKN5(3) approach.

The newly obtained coefficients depending on v together with the rest of coefficients of the original RKN5(3) method form the new adapted embedded method, which will be named as TFEERKN5(3).

4. Algebraic Order and Stability Analysis

This section is devoted to present the local truncation errors of the proposed pair and to get the algebraic orders of convergence. This is accomplished by using the usual tool of Taylor expansions. The local truncation errors (LTE) of the solution y and its first derivative $y^{'}$ are given by:

$$
\begin{aligned}
LTE &= y(x_0+h)-y_1,\\
LTE_{der} &= y^{'}(x_0+h)-y_1^{'}.
\end{aligned} \qquad (4.1)
$$

Proposition 4.1. *For the lower order method these formulas result in:*

$$
\begin{aligned}
LTE &= -\tfrac{h^4}{24}(f_{xx}+2y'f_{xy}+(y')^2f_{yy}+f_yy'')+O(h^5),\\
LTE_{der} &= \tfrac{h^4}{24}(f_{xxx}+3y'f_{yxx}+3y''f_{xy}+3(y')^2f_{xyy}+3y'f_{yy}y''\\
&\quad+(y')^3f_{yyy}+f_yf_x+(f_y)^2y')+O(h^5),
\end{aligned}
$$

where the functions in the right hand sides are evaluated at x_0, from which we can infer that the lower order method has algebraic order three.

Proposition 4.2. *Similarly, for the higher order method, we have:*

$$\begin{aligned} LTE &= \tfrac{h^6}{21600}(4(y')^3 f_{xyyy} + 3(y'')^2 f_{yy} + 6y'' f_{yxx} + 6(y')^2 f_{xxyy} + (y')^4 f_{yyyy} \\ &\quad +4y' f_{xxxy} + 12 f_y f_{xx} + 12(f_y)^2 y'' + 6(y')^2 f_{yyy} y'' + 12(y')^2 f_{yy} f_y \\ &\quad +12y' f_{xyy} y'' + 24 f_y y' f_{xy} + f_{xxxx} - 12w^4 y'') + O(h^7), \\ LTE_{der} &= \tfrac{h^6}{720}(f_{xxxxx} + 18y' f_{yy} f_y y'' + 15(y'')^2 f_{xyy} + 10y'' f_{xxxy} + 10y'' f_{xxxy} \\ &\quad +10y'(f_{xy})^2 + (f_y)^2 f_x + 5 f_{xx} f_{xy} + f_y f_{xxx} + 5(y')^4 f_{xyyyy} + 5y' f_{xxxxy} \\ &\quad +10 f_{yxx} f_x + 10(y')^2 f_{xxxyy} + 10(y')^3 f_{xxyyy} + 5(y')^3 (f_{yy})^2 + (y')^5 f_{yyyyy} \\ &\quad +15y' f_{yyy}(y'')^2 + 11(y')^3 f_{yyy} f_y + 30y' f_{xxyy} y'' + 30(y')^2 f_{xyyy} y'' + 8 f_y y'' f_{xy} \\ &\quad +10y'' f_{yy} f_x + 10(y')^3 f_{yyyy} y'' + 10(y')^2 f_{yyy} f_x + 23(y')^2 f_y f_{xyy} + 15(y')^2 f_{yy} f_{xy} \\ &\quad +20y' f_{xyy} f_x + 13 f_y y' f_{yxx} + (f_y)^3 y' + 5y' f_{yy} f_{xx}) + O(h^7), \end{aligned}$$

where the functions in the right hand sides are evaluated at x_0, from which we can infer that the higher order method has algebraic order five.

4.1. Stability Analysis

Applying the RKN method in (2)–(4) to the test equation $y^{''} = -w^2 y$, the linear stability is derived. In particular, for the method given in Table 1, letting $\tilde{h} = -w^2 h^2$, the approximate solution verifies the recurrence equation:

$$L_{n+1} = E(\tilde{h}) L_n,$$

where

$$L_{n+1} = \begin{bmatrix} y_{n+1} \\ hy^{'}_{n+1} \end{bmatrix}, \; L_n = \begin{bmatrix} y_n \\ hy^{'}_n \end{bmatrix}, \; E(\tilde{h}) = \begin{bmatrix} 1 + \tilde{h} b^T N^{-1} e & wh(1 + \tilde{h} b^T N^{-1} c) \\ -whd^T N^{-1} e & 1 + \tilde{h} d^T N^{-1} c \end{bmatrix},$$

$N = I - \tilde{h}A$, $A = (a_{ij})_{4\times 4}$ is the corresponding matrix of coefficients, I is the identity matrix of order four, and

$$b = [b_1, b_2, b_3, b_4]^T, \; d = [d_1, d_2, d_3, d_4]^T,$$
$$e = [1, 1, 1, 1]^T, \; c = [c_1, c_2, c_3, c_4]^T.$$

It is assumed that for sufficiently small values of v, $E(\tilde{h})$ has complex conjugate eigenvalues [15]. Under this assumption, a periodic numerical solution is obtained. The periodic behavior depends on the eigenvalues of the stability matrix $E(\tilde{h})$, whose characteristic equation can be expressed as:

$$\lambda^2 - tr(E(\tilde{h}))\lambda + det(E(\tilde{h})) = 0. \tag{4.1}$$

Definition 4.3. An interval $(-\tilde{h}_b, 0)$ corresponding to the method in (2)–(4) is said to be an interval of absolute stability if for all $\tilde{h} \in (-\tilde{h}_b, 0)$, it holds that $|\lambda_{1,2}| < 1$, where $\lambda_{1,2}$ are the solutions of the equation in (4.1).

Proposition 4.4. *Using the Maple package, from the above definition we find out that the lower order method of the new pair TFEERKN5(3) has* $(-38.71, 0)$ *as the absolute stability interval, while the higher order method of the new pair TFEERKN5(3) has* $(-78.46, 0)$ *as the interval of absolute stability. Hence, TFEERKN5(3) is absolutely stable.*

5. Numerical Examples

To evaluate the performance of the derived method, we consider some standard pairs of RKN methods for numerical comparisons:

- TFEERKN5(3): The new derived adapted RKN embedded pair,
- RKN5(3): The 5(3) explicit pair of RKN methods given by Van de Vyver in [2],
- ARKN5(3): The 5(3) explicit pair of ARKN method derived by Franco in [16],
- EFRKN5(3): The 5(3) explicit pair of RKN methods derived by Van de Vyver in [6],
- EEERKN5(3): An embedded exponentially-fitted explicit RKN5(3) pair derived by Demba et al. in [13],
- PFAFRKN5(3): A Phase-Fitted and Amplification-Fitted explicit RKN5(3) pair presented by Demba et al. in [14].

We will used them to integrate the following periodic initial value problems that have been used to test the efficiency of different methods appeared in the literature.

Problem 1. (Inhomogeneous Problem) in [17]

$$y'' = -v^2 y + (v^2 - 1)\sin x,\ y(0) = 1,\ y'(0) = v + 1,\ x \in [0, 10].$$

We consider the case of $v = 10$, for which the exact solution is given by

$$y(x) = \sin(10x) + \cos(10x) + \sin(x).$$

For the application of the adapted method derive in this paper and the adapted methods derived by Franco in [16], Van de Vyver in [6], and Demba et al. in [13, 14], we consider $w = 10$.

Problem 2. (Almost Periodic Problem) in [18]

$$\begin{aligned} y_1'' &= -y_1 + \epsilon\cos(\Psi x), y_1(0) = 1, y_1'(0) = 0,\\ y_2'' &= -y_2 + \epsilon\sin(\Psi x), y_2(0) = 0, y_2'(0) = 1,\ x \in [0, 5]. \end{aligned}$$

The exact solution is

$$y_1(x) = \tfrac{(1-\epsilon-\Psi^2)}{(1-\Psi^2)}\cos(x) + \tfrac{\epsilon}{(1-\Psi^2)}\cos(\Psi x),$$

$$y_2(x) = \tfrac{(1-\epsilon\Psi-\Psi^2)}{(1-\Psi^2)}\sin(x) + \tfrac{\epsilon}{(1-\Psi^2)}\sin(\Psi x),$$

where $\epsilon = 0.001$ and $\Psi = 0.1$.

Now, in the adapted methods we consider the value $w = 1$.

Problem 3. (Almost Periodic Orbital Problem) in [19]

$$\begin{aligned} y_1'' &= -y_1 + 0.001\cos(x),\ y_1(0) = 1,\ y_1'(0) = 0,\\ y_2'' &= -y_2 + 0.001\sin(x),\ y_2(0) = 0,\ y_2'(0) = 0.9995,\ x \in [0, 10]. \end{aligned}$$

The exact solution is

$$y_1(x) = \cos(x) + 0.0005\, x\cos(x),$$

$$y_2(x) = \sin(x) - 0.0005\, x\sin(x).$$

Now we take $w = 1.0$ to apply our method and the ones in [16], [6] and [13, 14].

Problem 4. (Linear Problem) in [20]

$$y'' + y = 2\Omega\cos(x),\ \ y(0) = 1,\ \ y'(0) = 0,\ \ x \in [0, 10].$$

with $\Omega = 10^{-6}$.

The exact solution is

$$y(x) \quad = \quad \cos(x) + \Omega x \sin(x).$$

Now we take $w = 1$ to apply our method and the ones in [6, 13, 14, 16].

Problem 5. (Non-linear System) in [18]

$$\begin{aligned} y_1'' + w^2 y_1 &= \frac{2y_1y_2 - \sin(2wx)}{({y_1}^2 + {y_2}^2)^{\frac{3}{2}}}, y_1(0) = 1, y_1'(0) = 0, \\ y_2'' + w^2 y_2 &= \frac{{y_1}^2 - {y_2}^2 - \sin(2wx)}{({y_1}^2 + {y_2}^2)^{\frac{3}{2}}}, y_2(0) = 0, y_2'(0) = w, \quad x \in [0, 10], \end{aligned}$$

with a known solution given by

$$\begin{aligned} y_1(x) &= \cos(wx), \\ y_2(x) &= \sin(wx). \end{aligned}$$

To use the adapted methods we have taken the parameter value $w = 5$.

The numerical results are given in Tables 2 to 6, considering different tolerances. The tables contain the number of steps, NSTEP; the number of function evaluations, NFE; the number of rejected steps, RSTEP; the maximum absolute errors, MAXER, and the computational time in seconds. We can see that the proposed method presents very good results concerning the number of steps, number of functions evaluation, maximum global error and computational time.

Table 2. Numerical data corresponding to Problem 1

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	TFEERKN5(3)	146	689	35	4.186947(-5)	0.141
10^{-3}	RKN5(3)	285	1161	7	2.095181(-4)	0.189
	ARKN5(3)	1004	4151	45	1.237791(-1)	0.187
	EFRKN5(3)	539	2258	34	7.588179(-4)	0.203
	EEERKN5(3)	302	1274	22	8.262920(-5)	0.109
	PFAFRKN5(3)	288	1170	6	4.575831(-5)	0.143
	TFEERKN5(3)	499	2191	65	4.427588(-8)	0.142
10^{-6}	RKN5(3)	1732	7036	36	1.130375(-7)	0.203
	ARKN5(3)	7120	28624	48	1.894206(-3)	1.284
	EFRKN5(3)	4126	16675	57	9.883194(-6)	1.060
	EEERKN5(3)	1792	7303	45	5.930698(-9)	0.478
	PFAFRKN5(3)	1797	7203	5	2.654823(-7)	0.347
	TFEERKN5(3)	1645	6808	76	1.069855(-11)	0.447
10^{-9}	RKN5(3)	10432	41842	38	2.346656(-11)	0.835
	ARKN5(3)	85821	343536	84	1.248406(-5)	14.406
	EFRKN5(3)	41842	167623	85	7.976877(-8)	12.849
	EEERKN5(3)	10804	43363	49	4.685141(-12)	5.517
	PFAFRKN5(3)	11361	45465	7	2.044276(-11)	1.747
	TFEERKN5(3)	9913	39757	35	1.864464(-11)	2.548
10^{-12}	RKN5(3)	61066	244471	69	8.936074(-11)	7.474
	ARKN5(3)	969212	3877190	114	1.236898(-7)	5.873
	EFRKN5(3)	296195	1185047	89	1.810908(-9)	18.557
	EEERKN5(3)	64844	259916	180	1.116336(-10)	13.323
	PFAFRKN5(3)	61566	246387	41	8.915368(-11)	12.095
	TFEERKN5(3)	58888	236140	196	5.049960(-12)	13.865
10^{-15}	RKN5(3)	224161	896854	70	5.122236(-12)	15.081
	ARKN5(3)	7146145	28584937	119	1.002275(-8)	39.575
	EFRKN5(3)	3534626	14138900	132	6.336190(-9)	36.334
	EEERKN5(3)	375894	1504896	440	5.179079(-12)	15.080
	PFAFRKN5(3)	228540	914190	10	5.105028(-12)	15.005

Table 3. Numerical data corresponding to Problem 2

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
10^{-2}	TFEERKN5(3)	4	16	0	3.640016(-7)	0.036
	RKN5(3)	8	32	0	3.757843(-4)	0.053
	ARKN5(3)	14	56	0	5.933544(-2)	0.038
	EFRKN5(3)	13	52	0	1.335143(-4)	0.047
	EEERKN5(3)	8	32	0	9.621090(-5)	0.077
	PFAFRKN5(3)	8	32	0	1.599658(-5)	0.062
10^{-4}	TFEERKN5(3)	5	20	0	4.099943(-7)	0.038
	RKN5(3)	27	108	0	3.508553(-7)	0.045
	ARKN5(3)	53	215	1	3.571642(-3)	0.067
	EFRKN5(3)	27	108	0	2.140877(-5)	0.053
	EEERKN5(3)	27	108	0	1.960642(-8)	0.078
	PFAFRKN5(3)	27	108	0	2.987395(-9)	0.068
10^{-6}	TFEERKN5(3)	7	28	0	4.517335(-8)	0.047
	RKN5(3)	57	228	0	7.598835(-9)	0.083
	ARKN5(3)	225	906	2	1.944919(-4)	0.053
	EFRKN5(3)	113	455	1	1.076621(-6)	0.059
	EEERKN5(3)	57	228	0	1.910067(-10)	0.082
	PFAFRKN5(3)	57	228	0	2.978306(-11)	0.078
10^{-8}	TFEERKN5(3)	12	48	0	1.000491(-9)	0.043
	RKN5(3)	121	484	0	1.627435(-10)	0.055
	ARKN5(3)	968	3881	3	1.041741(-5)	0.048
	EFRKN5(3)	3888	969	4	1.430596(-8)	0.100
	EEERKN5(3)	121	484	0	1.882161(-12)	0.109
	PFAFRKN5(3)	121	484	0	2.979839(-13)	0.090
10^{-10}	TFEERKN5(3)	35	140	0	5.345724(-13)	0.048
	RKN5(3)	522	2091	1	1.079137(-13)	0.053
	ARKN5(3)	4169	16688	4	5.561668(-7)	0.092
	EFRKN5(3)	4172	16709	7	7.324931(-10)	0.154
	EEERKN5(3)	524	2102	2	4.551914(-15)	0.144
	PFAFRKN5(3)	524	2091	1	7.105427(-15)	0.093

Table 4. Numerical data corresponding to Problem 3

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
	TFEERKN5(3)	8	32	0	2.174479(-5)	0.049
10^{-2}	RKN5(3)	13	52	0	8.984682(-4)	0.061
	ARKN5(3)	25	100	0	1.113811(-1)	0.059
	EFRKN5(3)	25	100	0	2.019958(-4)	0.059
	EEERKN5(3)	13	52	0	2.214262(-4)	0.071
	PFAFRKN5(3)	13	52	0	1.752577(-5)	0.074
	TFEERKN5(3)	15	60	0	5.741456(-7)	0.047
10^{-4}	RKN5(3)	53	212	0	7.096185(-7)	0.048
	ARKN5(3)	105	423	1	6.065366(-3)	0.051
	EFRKN5(3)	53	212	0	3.711559(-5)	0.063
	EEERKN5(3)	53	212	0	4.400680(-8)	0.094
	PFAFRKN5(3)	53	212	0	3.498385(-9)	0.074
	TFEERKN5(3)	30	120	0	1.298783(-8)	0.045
10^{-6}	RKN5(3)	113	452	0	1.512986(-8)	0.073
	ARKN5(3)	449	1802	2	3.267460(-4)	0.056
	EFRKN5(3)	225	903	1	1.949030(-6)	0.060
	EEERKN5(3)	113	452	0	4.433294(-10)	0.094
	PFAFRKN5(3)	113	452	0	4.083911(-11)	0.094
	TFEERKN5(3)	66	264	0	2.620753(-10)	0.043
10^{-8}	RKN5(3)	243	972	0	3.243754(-10)	0.044
	ARKN5(3)	1936	7753	3	1.759887(-5)	0.051
	EFRKN5(3)	1936	7756	4	2.621184(-8)	0.085
	EEERKN5(3)	243	972	0	4.296230(-12)	0.047
	PFAFRKN5(3)	243	972	0	5.360157(-13)	0.062
	TFEERKN5(3)	262	1048	0	2.052802(-13)	0.044
10^{-10}	RKN5(3)	1043	4175	1	2.949863(-13)	0.047
	ARKN5(3)	8337	33360	4	9.478899(-7)	0.089
	EFRKN5(3)	8340	33381	7	1.410515(-9)	0.187
	EEERKN5(3)	1044	4182	2	1.514899(-13)	0.297
	PFAFRKN5(3)	1044	4175	1	7.105427(-15)	0.174

TABLE 5. Numerical data corresponding to Problem 4

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
10^{-3}	TFEERKN5(3)	7	28	0	1.623637(-7)	0.085
	RKN5(3)	19	76	0	1.372783(-4)	0.128
	ARKN5(3)	36	147	1	4.484465(-2)	0.136
	EFRKN5(3)	23	92	0	3.740529(-4)	0.115
	EEERKN5(3)	19	76	0	2.376574(-5)	0.091
	PFAFRKN5(3)	20	80	0	1.437980(-6)	0.085
10^{-6}	TFEERKN5(3)	11	44	0	1.647419(-8)	0.089
	RKN5(3)	103	418	2	1.504347(-7)	0.146
	ARKN5(3)	396	1599	5	3.748960(-4)	0.222
	EFRKN5(3)	186	756	4	3.476864(-6)	0.221
	EEERKN5(3)	107	437	3	5.004367(-9)	0.123
	PFAFRKN5(3)	113	452	0	1.100668(-12)	0.098
10^{-9}	TFEERKN5(3)	27	108	0	7.918688(-11)	0.096
	RKN5(3)	542	2186	6	3.538836(-11)	0.195
	ARKN5(3)	2756	11051	9	7.325320(-6)	0.467
	EFRKN5(3)	1601	6440	12	3.457818(-8)	0.519
	EEERKN5(3)	621	2496	4	2.204903(-13)	0.175
	PFAFRKN5(3)	710	2843	1	1.009193(-13)	0.102
10^{-12}	TFEERKN5(3)	165	762	34	8.826273(-14)	0.145
	RKN5(3)	2202	8823	5	1.456613(-13)	0.266
	ARKN5(3)	33644	134618	14	4.733259(-8)	12.620
	EFRKN5(3)	18805	75271	17	2.380514(-10)	12.421
	EEERKN5(3)	2374	9556	20	1.451617(-13)	0.899
	PFAFRKN5(3)	4490	17966	2	3.119727(-13)	1.062
10^{-15}	TFEERKN5(3)	838	3646	98	1.039724(-13)	0.288
	RKN5(3)	13705	54844	8	2.833983(-13)	1.577
	ARKN5(3)	391068	1564329	19	3.979006(-10)	28.795
	EFRKN5(3)	204249	817080	28	3.424261(-12)	24.212
	EEERKN5(3)	14676	58815	37	2.891576(-13)	10.523
	PFAFRKN5(3)	28398	113604	4	2.738920(-13)	11.968

Table 6. Numerical data corresponding to Problem 5

TOL	METHOD	NSTEP	NFE	RSTEP	MAXER	TIME(s)
10^{-2}	TFEERKN5(3)	54	234	6	1.998157(-4)	0.104
	RKN5(3)	98	398	2	4.928421(-4)	0.142
	ARKN5(3)	391	1588	8	6.218929(-2)	0.257
	EFRKN5(3)	194	785	3	6.684847(-4)	0.219
	EEERKN5(3)	98	398	2	2.296670(-3)	0.144
	PFAFRKN5(3)	99	405	3	5.517730(-4)	0.128
10^{-4}	TFEERKN5(3)	214	877	7	3.468820(-7)	0.034
	RKN5(3)	210	846	2	1.049238(-5)	0.129
	ARKN5(3)	1672	6715	9	3.397381(-3)	0.483
	EFRKN5(3)	837	3369	7	2.007584(-5)	0.373
	EEERKN5(3)	420	1698	6	4.018595(-7)	0.128
	PFAFRKN5(3)	417	1677	3	6.507917(-6)	0.098
10^{-6}	TFEERKN5(3)	455	1844	8	8.873090(-10)	0.159
	RKN5(3)	898	3601	3	7.221669(-9)	0.179
	ARKN5(3)	7188	28782	10	1.825373(-4)	2.133
	EFRKN5(3)	3598	14422	10	1.023593(-6)	0.1.652
	EEERKN5(3)	904	3643	9	3.203667(-10)	0.372
	PFAFRKN5(3)	898	3601	3	6.726153(-8)	0.265
10^{-8}	TFEERKN5(3)	1941	7791	9	3.453481(-12)	0.899
	RKN5(3)	1936	7753	3	1.556580(-10)	0.434
	ARKN5(3)	30958	123865	11	9.784898(-6)	14.391
	EFRKN5(3)	15485	61979	13	5.438598(-8)	11.996
	EEERKN5(3)	3877	15541	11	2.198630(-12)	1.844
	PFAFRKN5(3)	3871	15499	5	4.235956(-11)	1.269
10^{-10}	TFEERKN5(3)	4176	16734	10	2.598831(-12)	1.981
	RKN5(3)	8337	33360	4	6.799068(-12)	1.781
	ARKN5(3)	133381	533560	12	5.245856(-7)	26.553
	EFRKN5(3)	66699	266847	17	2.959362(-9)	24.758
	EEERKN5(3)	8346	33424	13	6.796678(-12)	4.097
	PFAFRKN5(3)	8339	33374	6	6.800039(-12)	2.755

To further ascertain the efficiency of the proposed TFEERKN5(3) method, we use the graphical approach to display the performance of TFEERKN5(3) in comparison with other existing embedded RKN methods of orders 5(3) with four stages. Figures 1 to 5 show the efficiency curves for the considered problems, where one can observe the good behavior of the proposed method. We utilized the following tolerances: Tol=$1/10^{3i}$, $i = 1, 2, 3, 4$, for problems 1 and 4, and Tol=$1/10^{2i}$, $i = 1, 2, 3, 4$, for problems 2, 3 and 5.

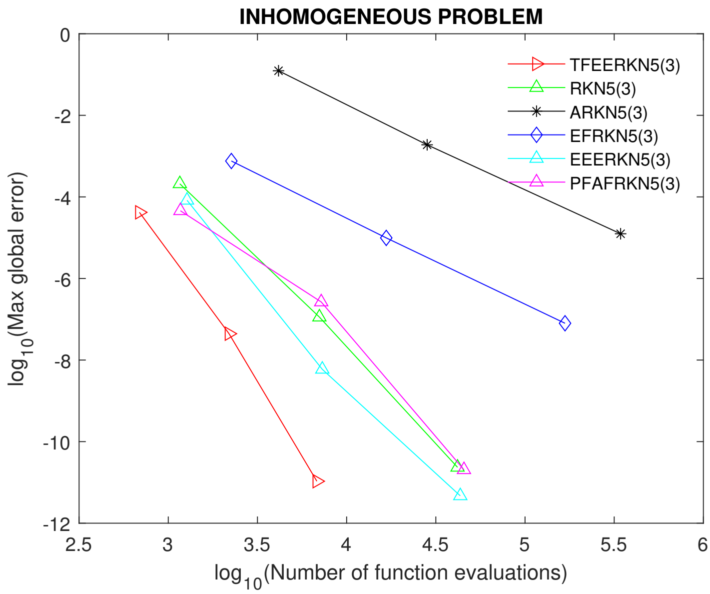

FIGURE 1. Efficiency curves corresponding to Problem 1

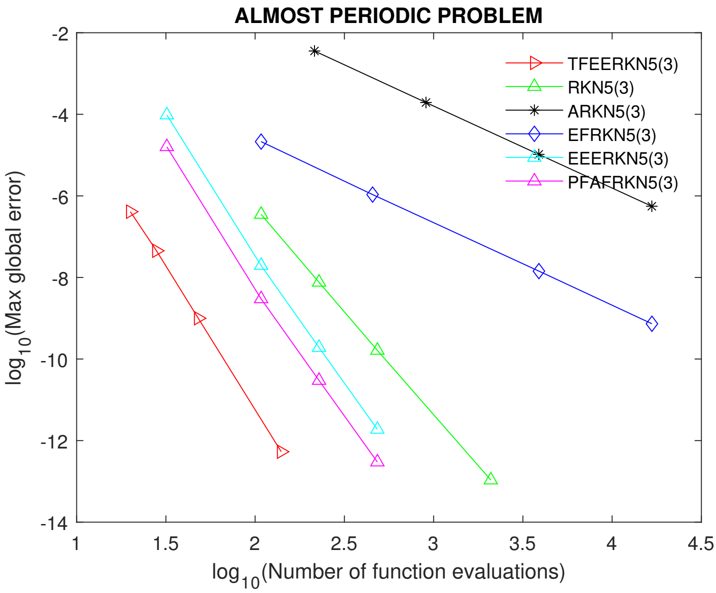

FIGURE 2. Efficiency curves corresponding to Problem 2

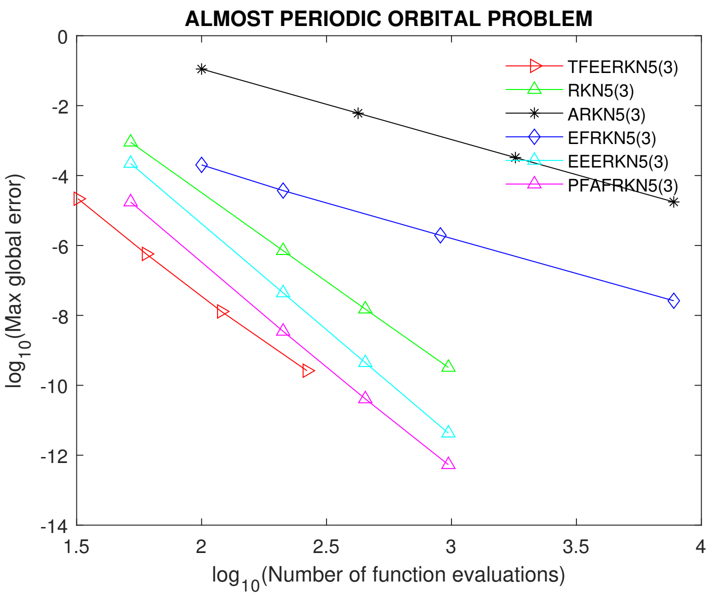

FIGURE 3. Efficiency curves corresponding to Problem 3

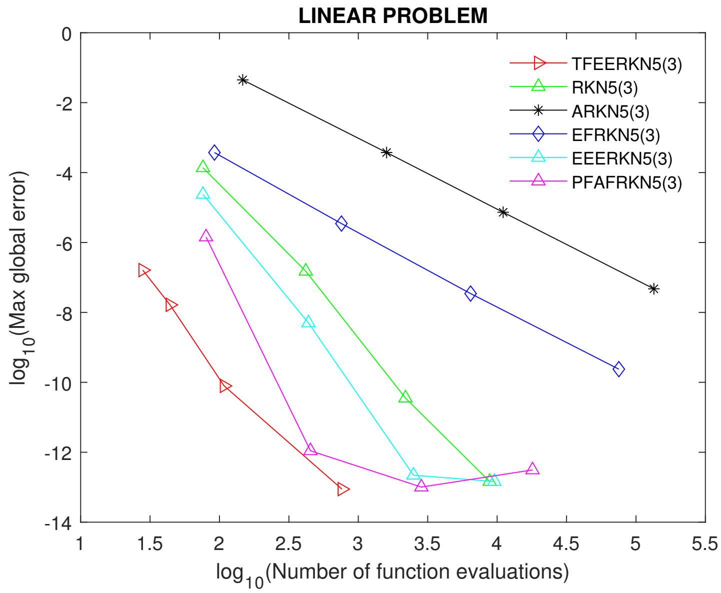

FIGURE 4. Efficiency curves corresponding to Problem 4

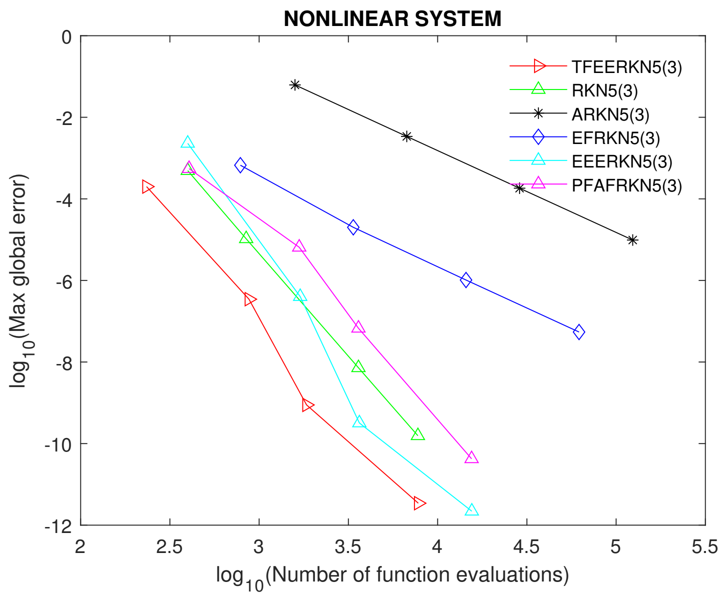

FIGURE 5. Efficiency curves corresponding to Problem 5

6. Discussion

Our proposed method (TFEERKN5(3)) has least error norm, least number of functions evaluation per steps, and least computational time; signifying that it is highly efficient and accurate for solving the kind of problems considered, as shown in Tables 2 − 6 and Figures 1 − 5. Therefore, we can state that the TFEERKN5(3) is more suitable for solving (1.1) than the some existing embedded RKN methods of orders 5(3) with four stages in the literature.

7. Conclusion

In this work, we have used the methodology for constructing a trigonometrically-fitted method based on the 5(3) embedded pair of Van de Vvyver in [2] and obtained a new embedded pair of explicit trigonometrically-fitted RKN methods. The developed method contains eight coefficients that depend on a parameter which is given by the product of the parameter of the method w, and the step-length h [21, 22]. We computed the local truncation error for both the higher and lower order methods in the new pair TFEERKN5(3), confirming that the fifth algebraic order of convergence is achieved. In addition, the stability intervals for both the higher and lower order methods have been obtained, signifying that the proposed method TFEERKN5(3) is absolutely stable. The numerical results obtained clearly show that TFEERKN5(3) is more accurate and efficient for solving problem (1.1) than other existing methods in the literature.

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