BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE

Volume 10 (2024)
Pages 30–47

BANGMOD INTERNATIONAL JOURNAL OF
MATHEMATICAL & COMPUTATIONAL SCIENCE ISSN: 2408-154X
https://bangmodjmcs.com/

EXISTENCE OF SOLUTION OF MULTI-TERM FRACTIONAL ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

Dahiru Umar[1,2,*], **Sirajo Lawan Bichi**[2]

[1] *Department of Mathematics, Modibbo Adama University Yola, Nigeria*
E-mails: dah4bab@mau.edu.ng
[2] *Department of Mathematical Sciences, Bayero University Kano, Nigeria*
E-mails: slawanbh@gmail.com

**Corresponding author.*

Received: 4 April 2024 / Accepted: 6 July 2024

Abstract This paper considered a multi-term fractional order Fredholm integro-differential equation. The multi-term fractional order Fredholm integro-differential equation was transformed into its corresponding integral equation form with the help of Riemann-Liouville fractional integral by which, Schauder's fixed point theorem is utilised in the study and establishing the existence of solution for the multi-term fractional order Fredholm integro-differential equation. Moreover, some examples were considered to prove the claim of the established existence of solution theorem.

MSC: 26A33; 34A08; 45B05; 47H10

Keywords: Fredholm integro-differential equation; fractional integro-differential equation; Existence of solution; Schauder's fixed point theorem

Published online: 12 July 2024

Please cite this article as: D. Umar et al., Eexistence of solution of multi-term fractional order fredholm integro-differential equation, Bangmod J-MCS., Vol. 10 (2024) 30–47.

1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators over real and complex domain and their applications. It does not mean the calculus of fractions nor the fractions of calculus [1]. It is therefore, the extension of classical calculus, involving derivatives and integrals of real or complex order [2, 3]. The concept of this differentiation for non-integer numbers dated as far back as in the days classical calculus in 1695. In a famous correspondence between Leibniz and L'Hopital, L'Hopital asked Leibniz about the possibility that the order n in the notation $\frac{d^n y}{dx^n}$, for the n^{th} derivative of the function y, be a non-integer $n = \frac{1}{2}$ [4]. Ever since, a number of brilliant scientists motivated to focus their attention on fractional calculus [5], such as Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1873), Riemann (1847), Grunward (1867-1872), Letnikov (1868-1872), Hadamard (1892), Heaviside (1892-1912), Riesz (1949), Caputo (1965) and they have all contributed to the growth of this field. One of the first applications of fractional calculus appear in 1823 by Niels Abel, through the solution of an integral equation used in the formulation of the Tautochrone problem [4].

Mostly, physical problems are better translated by fractional derivatives because fractional operators considers the evolution of the problems into account [6]. However, it is quite challenging to find analytical solutions for these fractional differential equations (FDEs). Therefore, numerical approximation plays a vital role in finding the approximate solution to these equations and as such, numerous scholars have introduced and developed numerical approaches to find approximations to the solutions to this class of equations [7]. Furthermore, in the past few decades, fractional calculus found its application in various field of science such as Physics [8], Chemical Reactions [9], Electrochemistry [10], Biology [11] Optics [12], in engineering such as Bio-engineering [13], Chemical kinetics [14, 15], Fluid Mechanics [16], Chaotic systems [17, 18], Viscoelasticity [19], and in social sciences such as Finance [20], Optimal Control problems [21, 22], Social Sciences [23], Economics [24, 25]. Furthermore, several scholars have created and studied the existence and uniqueness of solutions of these equations such as [26] proved the existence of solution for fractional integro-differential equations with nonlocal condition via resolvent operators. Also [2, 27–45] studied the existence of solutions of various types of fractional differential equations and fractional integro-differential equations of boundary value problems and initial value problems.

In this study, we considered the multi-term fractional order Fredholm integro-differential equation of the form

$$D^{\beta}\xi(x) = \sum_{j=0}^{k} c_j(x) D^{\delta_j}\xi(x) + h(x) + \int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \tag{1.1}$$

subject to the initial condition

$$\begin{aligned} \xi^{(n)}(0) &= d_n,\ n = 0,1,2,...,m-1, \\ m-1 &< \delta_0 < \delta_1 < \cdots < \delta_j < \beta \leq m,\ m \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where D is the differential operator of order β defined in Caputo sense, $\xi : Q \longrightarrow \mathbb{R}$, $Q = [0,1]$ is a continuous function which needs to be determined, c_j, $h : Q \longrightarrow \mathbb{R}$ are given continuous functions, $K : Q \times Q \longrightarrow \mathbb{R}$ is the kernel of integration which is also continuous, $H : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function.

https://doi.org/10.58715/bangmodjmcs.2024.10.3

2. Preliminaries

Definition 2.1 (Compact Map [46]). Let X and Y be Banach spaces and let $\Omega \subseteq X$. A map $F : \Omega \longrightarrow Y$ is said to be compact if it is continuous and $F(\Omega)$ is relatively compact (i.e, for every $(x_n)_n \subseteq \Omega$, there exists a subsequence $\left(x_{n_j}\right)_j$ of $(x_n)_n$ such that $F\left(x_{n_j}\right)_j$ is convergent).

Definition 2.2 (Uniformly Bounded [3]). A set M is called uniformly bounded if there exists a constant $K > 0$ such that $\|m\|_\infty \leq K$ for every $m \in M$.

Definition 2.3 (Equicontinuous [3]). A set M is called equicontinuous if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, for all $m \in M$ and all $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$, we have $|m(x_1) - m(x_2)| < \epsilon$.

Definition 2.4 (Fixed Point [47]). Given a map $T : A \longrightarrow B$,every solution ξ of the equation

$$T\xi = \xi$$

is called a fixed point of T.

Definition 2.5 (Banach Space [48]). A Banach space is a normed vector space which is complete (as a metric space).

Definition 2.6 (Riemann-Liouville Fractional Integral [3]). Riemann-Liouville fractional integral of order α of a function ξ is defined as

$$I^\beta \xi(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \xi(\sigma)\, d\sigma, \qquad x > 0,\ \beta \in \mathbb{R}^+, \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.7 (Caputo Fractional Derivative [3]). The fractional derivative of $\xi(x)$ in the Caputo sense is defined by

$$\begin{aligned} D^\beta \xi(x) &= I^{m-\beta} D^m \xi(x) \\ &= \frac{1}{\Gamma(m-\beta)} \int_0^x (x-\sigma)^{m-\beta-1} \frac{d^m \xi(\sigma)}{ds^m} d\sigma,\ m-1 < \beta \leq m. \end{aligned} \tag{2.2}$$

With the Properties

1. $I^\alpha D^\alpha \xi(x) = \xi(x) - \sum_{n=0}^{m-1} \frac{\xi^{(n)}(0)}{n!} x^n,\ m-1 < \alpha < m,$
2. $I^\alpha D^\beta \xi(x) = I^{\alpha-\beta} \xi(x),\ \beta < \alpha,$
3. $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}.$

Theorem 2.8 (Arzelà-Ascoli [3]). *Let M be a subset of $C[a, b]$ equipped with the norm $(\|\cdot\|_\infty)$. Then M is relatively compact in $C[a, b]$ if, and only if, M is equicontinuous and uniformly bounded.*

Theorem 2.9 (Schauder's Fixed Point Theorem [46]). *Let C be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : C \longrightarrow C$ be compact. Then T has a fixed point.*

3. Preliminary results

Lemma 3.1. *Let $\xi : Q \longrightarrow \mathbb{R}$ and $h : Q \longrightarrow \mathbb{R}$ be continuous functions. Then, a function ξ is a solution to the fractional integro-differential equation (1.1) – (1.2) if, and only if,*

$$\begin{aligned}\xi(x) &= \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n + \sum_{j=0}^{k} \frac{c_j(x)}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} \xi(\sigma)\, d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} h(\sigma)\, d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) d\sigma.\end{aligned}$$

Proof. Using equation (2.1) on equation (1.1) and by property (i), (ii) and (iii) we have,

$$\begin{aligned} I^\beta\left(D^\beta \xi(x)\right) &= I^\beta\left(\sum_{j=0}^{k} c_j(x) D^{\delta_j} \xi(x)\right) + I^\beta(h(x)) + I^\beta\left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau\right) \\ &= \sum_{j=0}^{k} I^\beta c_j(x) D^{\delta_j} \xi(x) + I^\beta(h(x)) + I^\beta\left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau\right) \\ \xi(x) &= \sum_{n=0}^{m-1} \frac{\xi^{(n)}(0)}{n!} x^n + \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(x) \xi(\sigma)\, d\sigma + \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} h(\sigma)\, d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) d\sigma.\end{aligned}$$

Substituting the initial condition, we obtain

$$\begin{aligned}\xi(x) &= \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n + \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(x) \xi(\sigma)\, d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} h(\sigma)\, d\sigma \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) d\sigma. \end{aligned} \tag{3.1}$$

Thus, ξ solves (1.1) – (1.2) if, and only if, ξ solves (3.1). ∎

Lemma 3.2. *Let X and Y be normed linear spaces and let $f : A \longrightarrow Y$ be a Lipschitz map, $A \subseteq X$. Then f sends bounded sets to bounded sets.*

Proof. For any set $E \subseteq A$, if there exists $M > 0 : \|x\|_X \leq M$ for all $x \in E$, then we show that there exists $\tilde{M} > 0 : \|f(x)\|_Y \leq \tilde{M}$ for all $x \in E$. Since f is Lipschitz, that is, there exists $L > 0$ such that

$$\|f(x) - f(y)\|_Y \leq L \|x - y\|_X \text{ for all } x, y \in A.$$

Let x_0 be a fixed element of A.Then, if there exists $M > 0 : \|x\|_X \leq M$ for all $x \in E$, then,

$$\begin{array}{rcl} \|f(x)\|_Y & = & \|f(x) - f(x_0) + f(x_0)\|_Y \\ & \leq & \|f(x) - f(x_0)\| + \|f(x_0)\| \\ & \leq & L\|x - x_0\| + \|f(x_0)\| \\ & \leq & L(\|x\| + \|x_0\|) + \|f(x_0)\| \\ & \leq & L(M + \|x_0\|) + \|f(x_0)\|. \end{array}$$

Thus, $\|f(x)\|_Y \leq \tilde{M}$ for all $x \in E$, where $\tilde{M} = L(M + \|x_0\|) + \|f(x_0)\|$. Hence, $f(E)$ is bounded. Thus, f sends bounded sets to bounded sets. ∎

Lemma 3.3. *For any* $0 \leq a < b$,

$$b^\alpha - a^\alpha \leq (b-a)^\alpha, \alpha \in (0,1).$$

Proof. Define $f : [0, b) \longrightarrow \mathbb{R}$ by

$$f(t) = (b-t)^\alpha - b^\alpha + t^\alpha, t \in [0, b). \tag{3.2}$$

Then from equation (3.2)

$$\begin{array}{rcl} f(t) & = & (b-t)^\alpha - b^\alpha + t^\alpha \\ & \geq & 0. \end{array}$$

Case I. Suppose $\alpha = 0$, then for any $0 \leq a < b$ we have from equation (3.2)

$$\begin{array}{rcl} f(a) & = & 1 \\ & \geq & 0. \end{array}$$

Case II (a). Suppose $\alpha > 0$, and $f(t)$ is increasing and we have from equation (3.2)

$$\begin{array}{rcl} f'(t) & = & -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \\ & \geq & 0, \end{array}$$

this implies

$$\begin{array}{rcl} (b-t)^{\alpha-1} - t^{\alpha-1} & \leq & 0 \\ (b-t)^{\alpha-1} & \leq & t^{\alpha-1} \\ b-t & \leq & t \\ b & \leq & 2t, \end{array}$$

therefore

$$t \geq \frac{b}{2}.$$

(b) Suppose $\alpha > 0$, and $f(t)$ is decreasing and we have from equation (3.2)

$$\begin{array}{rcl} f'(t) & = & -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \\ & \leq & 0, \end{array}$$

this implies

$$\begin{aligned}(b-t)^{\alpha-1}-t^{\alpha-1} &\geq 0\\ (b-t)^{\alpha-1} &\geq t^{\alpha-1}\\ b-t &\geq t\\ b &\geq 2t,\end{aligned}$$

therefore

$$t \leq \frac{b}{2}.$$

So, f is decreasing on $\left[0, \frac{b}{2}\right]$ and increasing on $[\frac{b}{2}, b)$. Hence, $\inf_{t\in[0,b)} f(t) = f(0) = 0$ and $\lim_{t\longrightarrow b^-} f(t) = f(b) = 0$, i.e, $\inf_{t\in[0,b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t\in[0,b)} f(t) = 0$. Therefore, for any $0 \leq a < b$, $f(a) \geq \inf_{t\in[0,b)} f(t) = 0$, i.e,

$$(b-a)^{\alpha} - b^{\alpha} + a^{\alpha} \geq 0,$$

that is,

$$b^{\alpha} - a^{\alpha} \leq (b-a)^{\alpha}.$$

Case III (a). Suppose $\alpha < 0$, and $f(t)$ is increasing and we have from equation (3.2)

$$\begin{aligned}f'(t) &= -\alpha\left((b-t)^{\alpha-1}-t^{\alpha-1}\right)\\ &\geq 0\end{aligned}$$

this implies

$$\begin{aligned}(b-t)^{\alpha-1}-t^{\alpha-1} &\geq 0\\ (b-t)^{\alpha-1} &\geq t^{\alpha-1}\\ b-t &\geq t\\ b &\geq 2t,\end{aligned}$$

therefore

$$t \leq \frac{b}{2}.$$

(b) Suppose $\alpha < 0$, and $f(t)$ is decreasing and we have from equation (3.2)

$$\begin{aligned}f'(t) &= -\alpha\left((b-t)^{\alpha-1}-t^{\alpha-1}\right)\\ &\leq 0\end{aligned}$$

this implies

$$\begin{aligned}(b-t)^{\alpha-1}-t^{\alpha-1} &\leq 0\\ (b-t)^{\alpha-1} &\leq t^{\alpha-1}\\ b-t &\leq t\\ b &\leq 2t,\end{aligned}$$

therefore

$$t \geq \frac{b}{2}.$$

https://doi.org/10.58715/bangmodjmcs.2024.10.3

So, f is increasing on $\left[0, \frac{b}{2}\right]$ and decreasing on $[\frac{b}{2}, b)$. Hence, $\inf_{t\in[0,b)} f(t) = f(0) = 0$ and $\lim_{t\longrightarrow b^-} f(t) = f(b) = 0$, i.e, $\inf_{t\in[0,b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t\in[0,b)} f(t) = 0$. Therefore, for any $0 \le a < b$, $f(a) \ge \inf_{t\in[0,b)} f(t) = 0$, i.e,

$$(b-a)^{\alpha} - b^{\alpha} + a^{\alpha} \ge 0,$$

that is,

$$b^{\alpha} - a^{\alpha} \le (b-a)^{\alpha}.$$

∎

4. Main Results

Throughout this work, we denote by

i: $\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $c \in C(Q,\mathbb{R})$, $\|c\|_{\infty} = \sup_{x\in Q} |c(x)|$.

ii: $\Pi := \sum_{j=0}^{k} \frac{\|c_j\|_{\infty}}{\Gamma(\beta-\delta_j+1)}$, where $c_j : Q \longrightarrow \mathbb{R}$, are given continuous functions.

We also make the following hypotheses:

(k_1) there exists a constant $M > 0$ such that for any $y_1, y_2 \in C(Q,\mathbb{R})$ we have

$$|H(\xi_1(x)) - H(\xi_2(x))| \le M \|\xi_1(x) - \xi_2(x)\|_{\infty} \ x \in Q,$$

(k_2) there exists a constant $\tilde{K}$ such that

$$\tilde{K} = \sup_{\sigma\in[0,1]} \int_0^1 |K(\sigma,\tau)| \, d\tau < \infty, \ 0 \le \tau \le 1.$$

Theorem 4.1. (*Existence of Solution*). *Assume that* (k_1) *and* (k_2) *holds, if*

$$\left(\Pi + \frac{\tilde{K}M}{\Gamma(\beta+1)}\right) < 1, \tag{4.1}$$

then there exists a solution $\xi \in C(Q,\mathbb{R})$ *to problem* (1.1) − (1.2).

Proof. Let T be an operator such that $T : C(Q,\mathbb{R}) \longrightarrow C(Q,\mathbb{R})$ defined by

$$\begin{aligned}(T\xi)(x) \quad = \quad & \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n + \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma) d\sigma \\ & + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} h(\sigma) \, d\sigma \\ & + \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau)) \, d\tau \right) d\sigma. \end{aligned} \tag{4.2}$$

Our objective is to utilize Schauder's fixed point theorem in proving the existence of solution of the multi-term fractional order Fredholm integro-differential equation. To do that, we will show that T satisfies the following;

i: T is continuous,

ii: T sends bounded sets to bounded sets,

iii: T sends bounded sets to equicontinuous sets.

First, we note that T is well defined. Indeed, since $x \longmapsto \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n$, $x \longmapsto \sum_{j=0}^{k} c_j(x) \left(I^{\beta-\delta_j}\xi\right)(x)$, $x \longmapsto \left(I^{\beta}h\right)(x)$, $x \longmapsto I^{\beta}\left(\int_0^1 K(\sigma,\tau) H(\xi(\tau)) d\tau\right)$ are continuous in $[0,1]$, the right hand of equation (4.2) is well defined and $x \longmapsto (T\xi)(x)$ is continuous. Thus, for $\xi \in C(Q,\mathbb{R})$, $T\xi \in C(Q,\mathbb{R})$.

Let $u, v \in C(Q,\mathbb{R})$. Then for any $x \in [0,1]$, we have by letting $E = |(Tu)(x) - (Tv)(x)|$

$$\begin{aligned}
E &= \left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma)(u(\sigma)-v(\sigma))\, d\sigma + \right. \\
&\quad \left. \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau)(H(u(\tau))-H(v(\tau)))\, d\tau\right) d\sigma \right| \\
&\le \left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma)(u(\sigma)-v(\sigma))\, d\sigma \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau)(H(u(\tau))-H(v(\tau)))\, d\tau\right) d\sigma \right| \\
&\le \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)|\, |u(\sigma)-v(\sigma)|\, d\sigma + \\
&\quad \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \left(\int_0^1 |K(\sigma,\tau)|\, |H(u(\tau))-H(v(\tau))|\, d\tau\right) d\sigma \\
&\le \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|u-v\|_\infty}{\Gamma(\beta-\delta_j)} \int_0^x (x-\sigma)^{\beta-\delta_j-1}\, d\sigma \\
&\quad + \frac{\tilde{K}M \|u-v\|_\infty}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1}\, d\sigma.
\end{aligned}$$

By property (iii) we have

$$|(Tu)(x) - (Tv)(x)| \le \left(\Pi + \frac{\tilde{K}M}{\Gamma(\beta+1)}\right) \|u-v\|_\infty, \text{ for all } x \in [0,1].$$

Thus,

$$\|(Tu) - (Tv)\|_\infty \le \Psi \|u-v\|_\infty,$$

where

$$\Psi = \Pi + \frac{\tilde{K}M}{\Gamma(\beta+1)} \in \mathbb{R}.$$

It follows that T is not just continuous, but Lipschitz continuous.

Next, we conclude by *Lemma 2* that operator T (i.e T being Lipschitz) maps bounded sets to bounded sets in $C(Q,\mathbb{R})$.

Furthermore, we show that T maps bounded sets to equicontinuous sets of $C(Q,\mathbb{R})$. Let $\xi \in B_\epsilon = \{\xi \in C(Q,\mathbb{R}) : \|\xi\|_\infty \le \epsilon\}$, $\sup_{x\in Q} |H(\xi(x))| \le M\|\xi\|_\infty + |H(L)|$, (with

$\xi(c) = L)$ and let $x_1, x_2 \in [0,1]$ with $x_1 < x_2$. Setting $E = (T\xi)(x_2) - (T\xi)(x_1)$, then we have from (4.2)

$$\begin{aligned}
|E| &= \Bigg| \textstyle\sum_{n=0}^{m-1} \frac{d_n}{n!}\left(x_2^n - x_1^n\right) \\
&\quad + \textstyle\sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma)d\sigma \\
&\quad - \int_0^{x_1} (x_1-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma)d\sigma \Bigg) \\
&\quad + \tfrac{1}{\Gamma(\beta)} \left(\int_0^{x_2} (x_2-\sigma)^{\beta-1} h(\sigma)\, d\sigma - \int_0^{x_1} (x_1-\sigma)^{\beta-1} h(\sigma)\, d\sigma \right) \\
&\quad + \tfrac{1}{\Gamma(\beta)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) \\
&\quad - \int_0^{x_1} (x_1-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) \Bigg) d\sigma \Bigg| \\
&\le \left| \sum_{n=0}^{m-1} \frac{d_n}{n!}\left(x_2^n - x_1^n\right) \right| \\
&\quad + \Bigg| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma)d\sigma \\
&\quad - \int_0^{x_1} (x_1-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma)d\sigma \Bigg) \Bigg| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \left(\int_0^{x_2} (x_2-\sigma)^{\beta-1} h(\sigma)\, d\sigma - \int_0^{x_1} (x_1-\sigma)^{\beta-1} h(\sigma)\, d\sigma \right) \right| \Bigg) \\
&\quad + \Bigg| \frac{1}{\Gamma(\beta)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) \\
&\quad - \int_0^{x_1} (x_1-\sigma)^{\beta-1} \left(\int_0^1 K(\sigma,\tau) H(\xi(\tau))\, d\tau \right) \Bigg) d\sigma \Bigg|.
\end{aligned}$$

Introducing a special zero to the right hand side, we have

$$\begin{aligned}
|E| &\le \left| \sum_{n=0}^{m-1} \frac{d_n}{n!}\left(x_2^n - x_1^n\right) \right| + \\
&\quad \Bigg| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma) \\
&\quad - \int_0^{x_1} (x_2-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma) + \int_0^{x_1} (x_2-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma) \\
&\quad - \int_0^{x_1} (x_1-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi(\sigma) \Bigg) d\sigma \Bigg| \\
&\quad + \Bigg| \frac{1}{\Gamma(\beta)} \Bigg(\int_0^{x_2} (x_2-\sigma)^{\beta-1} h(\sigma) - \int_0^{x_1} (x_2-\sigma)^{\beta-1} h(\sigma)
\end{aligned}$$

https://doi.org/10.58715/bangmodjmcs.2024.10.3
Bangmod J-MCS 2024

$$
\begin{aligned}
&+\int_0^{x_1}(x_2-\sigma)^{\beta-1}h(\sigma)-\int_0^{x_1}(x_1-\sigma)^{\beta-1}h(\sigma)\Big)d\sigma\Big|\\
&+\Big|\frac{1}{\Gamma(\beta)}\left(\int_0^{x_2}(x_2-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)H(\xi(\tau))d\tau\right)\right.\\
&-\int_0^{x_1}(x_2-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)H(\xi(\tau))d\tau\right)\\
&+\int_0^{x_1}(x_2-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)H(\xi(\tau))d\tau\right)\\
&\left.-\int_0^{x_1}(x_1-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)H(\xi(\tau))d\tau\right)\right)d\sigma\Big|.
\end{aligned}
$$

By simplification, we have

$$
\begin{aligned}
|E| \quad \leq \quad & \sum_{n=0}^{m-1}\frac{|d_n|}{n!}(x_2^n-x_1^n)\\
&+\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\left(\int_{x_1}^{x_2}(x_2-\sigma)^{\beta-\delta_j-1}|c_j(\sigma)||\xi(\sigma)|d\sigma\right.\\
&+\int_0^{x_1}(x_2-\sigma)^{\beta-\delta_j-1}|c_j(\sigma)||\xi(\sigma)|d\sigma\\
&\left.-\int_0^{x_1}(x_1-\sigma)^{\beta-\delta_j-1}|c_j(\sigma)||\xi(\sigma)|d\sigma\right)\\
&+\frac{1}{\Gamma(\beta)}\left(\int_{x_1}^{x_2}(x_2-\sigma)^{\beta-1}|h(\sigma)|d\sigma\right.\\
&\left.+\int_0^{x_1}(x_2-\sigma)^{\beta-1}|h(\sigma)|d\sigma-\int_0^{x_1}(x_1-\sigma)^{\beta-1}|h(\sigma)|d\sigma\right)\\
&+\frac{1}{\Gamma(\beta)}\left(\int_{x_1}^{x_2}(x_2-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))d\tau|\right)d\sigma\right.\\
&+\int_0^{x_1}(x_2-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))|d\tau\right)d\sigma\\
&\left.-\int_0^{x_1}(x_1-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))|d\tau\right)d\sigma\right).
\end{aligned}
$$

The above inequality can be written as

$$
|(T\xi)(x_2)-(T\xi)(x_2)|\leq A_1+A_2+A_3+A_4, \tag{4.3}
$$

where

$$
A_1=\sum_{n=0}^{m-1}\frac{|d_n|}{n!}(x_2^n-x_1^n)
$$

$$\begin{aligned} A_2 &= \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \left(\int_{x_1}^{x_2} (x_2-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right. \\ &\quad + \int_0^{x_1} (x_2-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \\ &\quad \left. - \int_0^{x_1} (x_1-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right), \end{aligned}$$

$$\begin{aligned} A_3 &= \frac{1}{\Gamma(\beta)} \left(\int_{x_1}^{x_2} (x_2-\sigma)^{\beta-1} |h(\sigma)| d\sigma \right. \\ &\quad + \int_0^{x_1} (x_2-\sigma)^{\beta-1} |h(\sigma)| d\sigma \\ &\quad \left. - \int_0^{x_1} (x_1-\sigma)^{\beta-1} |h(\sigma)| d\sigma \right), \end{aligned}$$

$$\begin{aligned} A_4 &= \frac{1}{\Gamma(\beta)} \left(\int_{x_1}^{x_2} (x_2-\sigma)^{\beta-1} \left(\int_0^1 |K(\sigma,\tau)| |H(\xi(\tau))| d\tau \right) d\sigma \right. \\ &\quad + \int_0^{x_1} (x_2-\sigma)^{\beta-1} \left(\int_0^1 |K(\sigma,\tau)| |H(\xi(\tau))| d\tau \right) d\sigma \\ &\quad \left. - \int_0^{x_1} (x_1-\sigma)^{\beta-1} \left(\int_0^1 |K(\sigma,\tau)| |H(\xi(\tau))| d\tau \right) d\sigma \right). \end{aligned}$$

On Simplifying the term A_1 by cosidering that $(x_2^n - x_1^n) \leq (x_2 - x_1)$ for $0 \leq n \leq m-1$, since $x_2, x_1 \in [0,1]$ and $x_1 < x_2$ and taking $d_{n^*} = \max_{0 \leq n \leq m-1} \{d_n\}$, we have

$$\begin{aligned} A_1 &= \sum_{n=0}^{m-1} \frac{|d_n|}{n!} (x_2^n - x_1^n) \\ &\leq \left(\frac{|d_1|}{1!} + \frac{|d_2|}{2!} + \ldots + \frac{|d_{m-1}|}{(m-1)!} \right) (x_2 - x_1) \\ &= \frac{|d_{n^*}|}{n^*!} (m-1)(x_2 - x_1). \end{aligned}$$

On simplifying the term A_2 and by propeerty (iii) and *Lemma 3* we have

$$\begin{aligned} A_2 &= \sum_{j=0}^{k} \frac{1}{\Gamma(\beta-\delta_j)} \left(\int_{x_1}^{x_2} (x_2-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right. \\ &\quad \left. + \int_0^{x_1} (x_2-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma - \int_0^{x_1} (x_1-\sigma)^{\beta-\delta_j-1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right) \\ &\leq \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|\xi\|_\infty}{\Gamma(\beta-\delta_j+1)} \left((x_2-x_1)^{\beta-\delta_j} + \left(x_2^{\beta-\delta_j} - (x_2-x_1)^{\beta-\delta_j} \right) - x_1^{\beta-\delta_j} \right) \\ &= \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|\xi\|_\infty}{\Gamma(\beta-\delta_j+1)} \left(x_2^{\beta-\delta_j} - x_1^{\beta-\delta_j} \right) \end{aligned}$$

$$\leq \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|\xi\|_\infty}{\Gamma(\beta-\delta_j+1)} (x_2-x_1)^{\beta-\delta_j}.$$

Simplifying the term A_3, we use property (iii) and *Lemma 3* as follows

$$\begin{aligned}
A_3 &= \frac{1}{\Gamma(\beta)}\bigg(\int_{x_1}^{x_2}(x_2-\sigma)^{\beta-1}|h(\sigma)|\,d\sigma+\int_0^{x_1}(x_2-\sigma)^{\beta-1}|h(\sigma)|\,d\sigma \\
&\quad -\int_0^{x_1}(x_1-\sigma)^{\beta-1}|h(\sigma)|\,d\sigma\bigg) \\
&\leq \frac{\|h\|_\infty}{\Gamma(\beta+1)}\left((x_2-x_1)^\beta+\left(x_2^\beta-(x_2-x_1)^\beta\right)-x_1^\beta\right) \\
&= \frac{\|h\|_\infty}{\Gamma(\beta+1)}\left(x_2^\beta-x_1^\beta\right) \\
&\leq \frac{\|h\|_\infty}{\Gamma(\beta+1)}(x_2-x_1)^\beta.
\end{aligned}$$

To simplify term A_4, we use property (iii) and *Lemma 3* as follows

$$\begin{aligned}
A_4 &= \frac{1}{\Gamma(\beta)}\bigg(\int_{x_1}^{x_2}(x_2-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))|\,d\tau\right)d\sigma \\
&\quad +\int_0^{x_1}(x_2-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))|\,d\tau\right)d\sigma \\
&\quad -\int_0^{x_1}(x_1-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||H(\xi(\tau))|\,d\tau\right)d\sigma\bigg) \\
&\leq \frac{\tilde{K}(M\|\xi\|_\infty+|H(L)|)}{\Gamma(\beta+1)}\left((x_2-x_1)^\beta+\left(x_2^\beta-(x_2-x_1)^\beta\right)-x_1^\beta\right) \\
&= \frac{\tilde{K}(M\|\xi\|_\infty+|H(L)|)}{\Gamma(\beta+1)}\left(x_2^\beta-x_1^\beta\right) \\
&\leq \frac{\tilde{K}(M\|\xi\|_\infty+|H(L)|)}{\Gamma(\beta+1)}(x_2-x_1)^\beta.
\end{aligned}$$

Thus, equation (4.3) is

$$\begin{aligned}
|(T\xi)(x_2)-(T\xi)(x_1)| &\leq \frac{|d_{n^*}|}{n^*!}(m-1)(x_2-x_1)+\sum_{j=0}^{k}\frac{\|c_j\|_\infty\|\xi\|_\infty}{\Gamma(\beta-\delta_j+1)}(x_2-x_1)^{\beta-\delta_j} \\
&\quad +\frac{\|h\|_\infty}{\Gamma(\beta+1)}(x_2-x_1)^\beta+\frac{\tilde{K}(M\|\xi\|_\infty+|H(L)|)}{\Gamma(\beta+1)}(x_2-x_1)^\beta.
\end{aligned}$$

We see that the right hand side of the above equation is independent of ξ and tends to zero as $x_2-x_1\longrightarrow 0$. This leads to $|(T\xi)(x_2)-(T\xi)(x_1)|\longrightarrow 0$ as $x_2\longrightarrow x_1$ uniformly in ξ. Therefore, the set $\{T\xi:\xi\in B_\epsilon\}$ is equicontinuous and finally, we need to show that there exists a closed convex bounded subset C of X such that $Tc\subseteq C$.

Consider $B_\epsilon=\{\xi\in C(Q,\mathbb{R}):\|\xi\|_\infty\leq\epsilon\}$, we will show that for some $\epsilon>0$, $TB_\epsilon\subseteq B_\epsilon$. For contradiction, suppose that $TB_\epsilon\nsubseteq B_\epsilon$ for all $\epsilon>0$.

Let μ be a positive integer, then there exists $\xi_\mu\in B_\mu$ such that $\|T\xi_\mu\|_\infty>\mu$.

Consider

$$\begin{aligned}
\left|\left(T\xi_{\mu}\right)(x)\right| &\leq \left|\sum_{n=0}^{m-1}\frac{d_n}{n!}x^n\right| + \left|\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^x (x-\sigma)^{\beta-\delta_j-1} c_j(\sigma)\xi_{\mu}(\sigma)d\sigma\right| \\
&+\left|\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1} h(\sigma)\, d\sigma\right| \\
&+\left|\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau) H\left(\xi_{\mu}(\tau)\right)d\tau\right)d\sigma\right| \\
&\leq \sum_{n=0}^{m-1}\sup_{x\in[0,1]}\frac{|d_n|}{n!}x^n \\
&\sum_{j=0}^{k}\frac{1}{\Gamma\left(\beta-\delta_j\right)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}\sup_{\sigma\in Q}|c_j(\sigma)|\sup_{\sigma\in Q}|\xi_{\mu}(\sigma)|\, d\sigma \\
&+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\sup_{\sigma\in Q}|h(\sigma)|\, d\sigma \\
&+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 \sup_{\sigma\in Q}|K(\sigma,\tau)|\sup_{\tau\in Q}|H\left(\xi_{\mu}(\tau)\right)|\, dt\right)d\sigma \\
&\leq \sum_{n=0}^{m-1}\frac{|d_n|}{n!} + \sum_{j=0}^{k}\frac{\|c_j\|_{\infty}\|\xi_{\mu}\|_{\infty}}{\Gamma\left(\beta-\delta_j+1\right)} \\
&+\frac{\|h\|_{\infty}}{\Gamma(\beta+1)} + \frac{\tilde{K}\left(M\mu+|H(L)|\right)}{\Gamma(\beta+1)}, \text{ for all } x\in[0,1].
\end{aligned}$$

Thus,

$$\|(T\xi_n)\|_{\infty} \leq \sum_{n=0}^{m-1}\frac{|d_n|}{n!} + \Pi\mu + \frac{\|h\|_{\infty}}{\Gamma(\beta+1)} + \frac{\tilde{K}\left(M\mu+|H(L)|\right)}{\Gamma(\beta+1)}.$$

Observe that if

$$\begin{aligned}
\mu &< \|(T\xi_{\mu})\|_{\infty} \\
&< \sum_{n=0}^{m-1}\frac{|d_n|}{n!} + \Pi\mu + \frac{\|h\|_{\infty}}{\Gamma(\beta+1)} + \frac{\tilde{K}\left(M\mu+|H(L)|\right)}{\Gamma(\beta+1)}.
\end{aligned}$$

Dividing through by μ we have,

$$1 < \sum_{n=0}^{m-1}\frac{|d_n|}{\mu n!} + \Pi + \frac{\|h\|_{\infty}}{\mu\Gamma(\beta+1)} + \frac{\tilde{K}\left(M\mu+|H(L)|\right)}{\mu\Gamma(\beta+1)}.$$

Letting $\mu \longrightarrow \infty$ we obtain

$$1 < \Pi + \frac{\tilde{K}M}{\Gamma(\beta+1)}.$$

Which is a contradiction to equation (4.1). Hence, for some μ_0, $TB_{\mu_0} \subseteq B_{\mu_0}$.

Let $C := B_{\mu_0}$ and let $\hat{T} := T|c$, i.e, $T : C \longrightarrow C$ with $\hat{T}\xi = T\xi$. By Arzelà-Ascoli thus, for any $(\xi_n)_n \subseteq C$, since C is bounded $(\xi_n)_n$ is bounded and by $\left(\hat{T}\xi_n\right)_n \equiv (T\xi_n)_n$ is equicontinuous. Then there exists a subsequence $\left(\hat{T}\xi_{n_j}\right)_j$ of $\left(\hat{T}\xi_n\right)_n$ which is convergent. Hence, $\hat{T}$ is compact. By of Schauder's fixed point theorem, there exists a fixed point ξ of T in $C(Q,\mathbb{R})$. Then ξ is a solution of equation (1.1) – (1.2). ∎

5. Example

Example 1 [49]. Consider the fractional integro-differential equation of Fredholm type

$$D^{\frac{3}{4}}\xi(x)=\frac{x^{0.25}}{\Gamma(1.25)}+(x\cos x-\sin x)\,\xi(x)+\int_0^1 x\sin\tau\xi(\tau)\,d\tau. \tag{5.1}$$

Subject to $\xi(0)=0$ with exact solution $\xi(x)=x$.

Solution: Equation (5.1) can be written as

$$\begin{aligned}
|(T\xi_2)(x)-(T\xi_1)(x)| &= \left|\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{-\frac{1}{4}}\begin{pmatrix}\sigma\cos\sigma-\sin\sigma+\\ \sigma-\sigma\cos\sigma\end{pmatrix}(\xi_2(\sigma)-\xi_1(\sigma))\,d\sigma\right|\\
&= \left|\frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{-\frac{1}{4}}(\sigma-\sin\sigma)(\xi_2(\sigma)-\xi_1(\sigma))\,d\sigma\right|\\
&\le \frac{1}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{-\frac{1}{4}}|\sigma-\sin\sigma|\,|\xi_2(\sigma)-\xi_1(\sigma)|\,d\sigma\\
&\le \frac{\|\xi_2-\xi_1\|_\infty}{\Gamma\left(\frac{3}{4}\right)}\int_0^x (x-\sigma)^{-\frac{1}{4}}\left(\sigma-\frac{(-1)^n\sigma^{2n+1}}{(2n+1)!}\right)d\sigma.
\end{aligned}$$

By *Property (iii)* we have,

$$\|T\xi_2-T\xi_1\|_\infty\le\left(\frac{\Gamma(2)}{\Gamma(2.75)}-\frac{(-1)^n\Gamma(2n+2)}{(2n+1)!\Gamma\left(2n+\frac{11}{4}\right)}\right)\|\xi_2-\xi_1\|_\infty, x\in[0,1],\ n\in\mathbb{N}.$$

Thus,

$$\|T\xi_2-T\xi_1\|_\infty\le(0.621\,75)\,\|\xi_2-\xi_1\|_\infty. \tag{5.2}$$

Since $0.621\,75<1$, we say that the problem satisfies the condition of *Theorem 4.1*.

Example 2 [21]. Consider the multi-term fractional order integro-differential equation

$$aD^2\xi(x)+b(x)D^{\delta_1}\xi(x)+c(x)D\xi(x)+e(x)D^{\delta_2}\xi(x)+k(x)\xi(x)+\lambda\int_0^1 K(\sigma,\tau)\xi(\tau)\,d\tau=h(x). \tag{5.3}$$

Subject to $\xi(0)=2,\ \xi'(0)=0$ with exact solution $\xi(x)=2-\frac{x^2}{2}$,

where $h(x)=-a-\frac{b(x)}{\Gamma(3-\delta_1)}x^{2-\delta_1}-c(x)x-\frac{e(x)}{\Gamma(3-\delta_2)}x^{2-\delta_2}+k(x)\left(2-\frac{x^2}{2}\right)$, $\lambda=0$ $a=1,\ b(x)=x^{\frac{1}{2}},\ c(x)=x^{\frac{1}{3}},\ e(x)=x^{\frac{1}{4}},\ k(x)=x^{\frac{1}{5}},\ \delta_2=0.333$ and $\delta_1=1.234$.

Solution: Equation (5.3) can be written as

$$\begin{aligned}
|(T\xi_2)(x)-(T\xi_1)(x)| &\leq \frac{1}{\Gamma(2)\Gamma(1.766)}\int_0^x (x-\sigma)^{2-1}\left|\sigma^{1.266}\right||\xi_2(\sigma)-\xi_1(\sigma)|\,d\sigma \\
&+\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)^{2-1}\left|\sigma^{1.333}\right||\xi_2(\sigma)-\xi_1(\sigma)|\,d\sigma \\
&+\frac{1}{\Gamma(2)\Gamma(1.667)}\int_0^x (x-\sigma)^{2-1}\left|\sigma^{0.917}\right||\xi_2(\sigma)-\xi_1(\sigma)|\,d\sigma \\
&+\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)^{2-1}\left|\sigma^{0.2}\right||\xi_2(\sigma)-\xi_1(\sigma)|\,d\sigma \\
&\leq \frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)\Gamma(1.766)}\int_0^x (x-\sigma)\,\sigma^{1.266}d\sigma \\
&+\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{1.333}d\sigma \\
&+\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)\Gamma(1.667)}\int_0^x (x-\sigma)\,\sigma^{0.917}d\sigma \\
&+\frac{\|\xi_2-\xi_1\|_\infty}{\Gamma(2)}\int_0^x (x-\sigma)\,\sigma^{0.2}d\sigma.
\end{aligned}$$

By *Property (iii), we have*

$$\begin{aligned}
|(T\xi_2)(x)-(T\xi_1)(x)| \leq & \left(\frac{\Gamma(2.266)\,x^{3.266}}{\Gamma(1.766)\Gamma(4.266)}+\frac{\Gamma\left(\frac{7}{3}\right)x^{\frac{10}{3}}}{\Gamma\left(\frac{13}{3}\right)}\right. \\
&\left.+\frac{\Gamma(1.917)\,x^{2.917}}{\Gamma(1.667)\Gamma(3.917)}+\frac{\Gamma\left(\frac{6}{5}\right)x^{\frac{11}{5}}}{\Gamma\left(\frac{16}{5}\right)}\right)\|\xi_2-\xi_1\|_\infty .
\end{aligned}$$

Thus,

$$\|T\xi_2-T\xi_1\|_\infty \leq (0.851\,87)\,\|\xi_2-\xi_1\|_\infty .$$

Since $0.851\,87<1$, we say that the problem satisfies the condition of the *Theorem 4.1*.

6. Conclussion

This research focused on multi-term fractional order Fredholm integro-differential equation, whereby the multi-term fractional order Fredholm integro-differential equation was converted to its corresponding integral equation using Riemann-Liouville fractional integral. Schauder's fixed point theorem is further used in the study of the existence of the solution for the multi-term fractional order Fredholm integro-differential equation. Furthermore, examples were considered to demonstrate the validity of the proposed existence of solution theorem for the solution of multi-term fractional order Fredholm integro-differential equations.

References

[1] I. Podlubny, Fractional differential equations, mathematics in science and engineering. 1999.

[2] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Vol. 1, Gordon and Breach Science Publishers, Yverdon-les-Bains, Switzerland, 1993.
[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Vol. 204, 2006.
[4] B. Ross, The development of fractional calculus 1695–1900, Historia Mathematica 4 (1) (1977) 75–89, https://doi.org/10.1016/0315-0860(77)90039-8.
[5] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional calculus: models and numerical methods (Vol. 3), World Scientific, 2012, https://doi.org/10.1142/9789814355216.
[6] R. Almeida, D. Tavares, D.F. Torres, The variable-order fractional calculus of variations, Cham, Switzerland: Springer International Publishing, 2019, https://doi.org/10.1007/978-3-319-94006-9.
[7] S. Nemati, P.M. Lima, D.F. Torres, Numerical solution of variable-order fractional differential equations using Bernoulli polynomials, Fractal and Fractional 5 (4) (2021) 219, https://doi.org/10.3390/fractalfract5040219.
[8] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific, 2000, https://doi.org/10.1142/9789812817747.
[9] J. Zabadal, M. Vilhena, P. Livotto, Simulation of chemical reactions using fractional derivatives, Nuovo Cimento. B 116 (5) (2001) 529–545.
[10] K.B. Oldham, Fractional differential equations in electrochemistry. Advances in Engineering software 41 (1) (2010) 9–12, https://doi.org/10.1016/j.advengsoft.2008.12.012.
[11] V.S. Erturk, Z.M. Odibat, S. Momani, An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of CD4+T-cells, Computers & Mathematics with Applications 62 (3) (2011) 996–1002, https://doi.org/10.1016/j.camwa.2011.03.091.
[12] H. Bulut, T.A. Sulaiman, H.M. Baskonus, H. Rezazadeh, M. Eslami, M. Mirzazadeh, Optical solitons and other solutions to the conformable space–time fractional Fokas–Lenells equation, Optik 172 (2018) 20–27, https://doi.org/10.1016/j.ijleo.2018.06.108.
[13] R.L. Magin, Fractional Calculus in Bioengineering: A Tool to Model Complex Dynamics, In Proceedings of the 13th International Carpathian Control Conference (ICCC), IEEE (2012) 464–469, https://doi.org/10.1109/CarpathianCC.2012.6228688.
[14] J. Singh, D. Kumar, D. Baleanu, On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag-Leffler type kernel, Chaos: An Interdisciplinary Journal of Nonlinear Science 27 (10) (2017) 103–113, https://doi.org/10.1063/1.4995032.
[15] C.E. Stoenoiu, S.D. Bolboaca, L. Jäntschi, Model formulation & interpretation-from experiment to theory. Int. J. Pure Appl. Math. 47 (2008) 9–16.
[16] L. Zheng, X. Zhang, Modeling and Analysis of Modern Fluid Problems, Academic Press, London, UK, 2017.
[17] M. Hajipour, A. Jajarmi, D. Baleanu, An efficient nonstandard finite difference scheme for a class of fractional chaotic systems, Journal of Computational and Nonlinear Dynamics 13 (2) (2018), https://doi.org/10.1115/1.4038444.
[18] L. Huang, Y. Bae, Chaotic dynamics of the fractional-Love model with an external environment, Entropy 20 (1) (2018) 53, https://doi.org/10.3390/e20010053.

[19] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, World Scientific, 2022.
[20] H. Fallahgoul, S. Focardi, F. Fabozzi, Fractional Calculus and Fractional Processes with Applications to Financial Economics: Theory and Application, Academic Press, 2016, https://doi.org/10.1016/B978-0-12-804248-9.50002-4.
[21] N.I. Mahmudov, Finite-approximate controllability of evolution equations. Appl. Comput. Math. 16 (2) (2017) 159–167, https://doi.org/10.1016/j.chaos.2020.110277.
[22] D. Baleanu, A. Jajarmi, M. Hajipour, A new formulation of the fractional optimal control problems involving Mittag–Leffler nonsingular kernel, Journal of Optimization Theory and Applications 175 (3) (2017) 718–737, https://doi.org/10.1007/s10957-017-1186-0.
[23] D. Baleanu, A.M. Lopes, Handbook of Fractional Calculus with Applications. Applications in Engineering, Life and Social Sciences, Part A, Southampton: Comput Mech Publicat, 7 (2019).
[24] V.E. Tarasov, Mathematical Economics: Application of Fractional Calculus, Mathematics 8 (5) (2020) 660, https://doi.org/10.3390/math8050660.
[25] H. Ming, J. Wang, M. Fečkan, The application of fractional calculus in Chinese economic growth models, Mathematics 7 (8) (2019) 665, https://doi.org/10.3390/math7080665.
[26] K. Balachandran, S. Kiruthika, Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators, Computers & Mathematics with Applications 62 (3) (2011) 1350–1358, https://doi.org/10.1016/j.camwa.2011.05.001
[27] D. Umar, S.L. Bichi, Uniqueness of solution of multi-term fractional order Volterra Integro-differential equations with convergence analysis, International Journal of Mathematical Analysis and Modelling 7 (1) (2024) 12–22.
[28] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, Journal of Mathematical Analysis and Applications 265 (2) (2002) 229–248. https://doi.org/10.1006/jmaa.2000.7194.
[29] S.B. Yuste, L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, SIAM Journal on Numerical Analysis 42 (5) (2005) 1862–1874, https://doi.org/10.1137/030602666.
[30] A.A. Kilbas, S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, Differential Equations 41 (2005) 84–89, https://doi.org/10.1007/s10625-005-0137-y.
[31] S. Pilipovic, M. StojanoviÄ‡, Fractional differential equations through Laguerre expansions in abstract spaces: Error estimates, Integral Transforms and Special Functions 17 (12) (2006) 877–887, https://doi.org/10.1080/10652460601042571.
[32] R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Applicandae Mathematicae 109 (2010) 973–1033, https://doi.org/10.1007/s10440-008-9356-6
[33] D. Baleanu, O.G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Computers & Mathematics with Applications 59 (5) (2010) 1835–1841, https://doi.org/10.1016/j.camwa.2009.08.028.

[34] Y. Tian, Z. Bai, Existence results for the three-point impulsive boundary value problem involving fractional differential equations, Computers & Mathematics with Applications 59 (8) (2010) 2601–2609, https://doi.org/10.1016/j.camwa.2010.01.028.
[35] Z. Wei, Q. Li, J. Che, Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative, Journal of Mathematical Analysis and Applications 367 (1) (2010) 260–272, https://doi.org/10.1016/j.jmaa.2010.01.023.
[36] A. Anguraj, P. Karthikeyan, J.J. Trujillo, Existence of solutions to fractional mixed integrodifferential equations with nonlocal initial condition, Advances in Difference Equations 2011 (2011) 1–12, https://doi.org/10.1155/2011/690653.
[37] A. Aghajani, J. Banaś, Y. Jalilian, Existence of solutions for a class of nonlinear Volterra singular integral equations, Computers & Mathematics with Applications 62 (3) (2011) 1215–1227, https://doi.org/10.1016/j.camwa.2011.03.049.
[38] D. Idczak, R. Kamocki, On the existence, uniqueness, and formula for the solution of RL fractional Cauchy problem in $\mathbb{R}^n$, Fractional Calculus and Applied Analysis 14 (2011) 538–553, https://doi.org/10.2478/s13540-011-0033-5.
[39] M. Kostić, Abstract time-fractional equations: existence and growth of solutions, Fractional Calculus and Applied Analysis 14 (2) (2011) 301–316, https://doi.org/10.2478/s13540-011-0018-4.
[40] R.P. Agarwal, B. Ahmad, Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions, Computers & Mathematics with Applications 62 (3) (2011) 1200–1214, https://doi.org/10.1016/j.camwa.2011.03.001.
[41] A. Aghajani, Y. Jalilian, J.J. Trujillo, On the existence of solutions of fractional integro-differential equations, Fractional Calculus and Applied Analysis 15 (2012) 44–69, https://doi.org/10.2478/s13540-012-0005-4.
[42] Z. Hu, W. Liu, Solvability for fractional order boundary value problems at resonance, Boundary Value Problems 2011 (1) (2011) 1–10, https://doi.org/10.1186/1687-2770-2011-20.
[43] A. Hamoud, K. Ghadle, S. Atshan, The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method, Khayyam Journal of Mathematics 5 (1) (2019) 21–39, https://doi.org/10.22034/KJM.2018.73593.
[44] W. Rui, Existence of solutions of nonlinear fractional differential equations at resonance, Electronic Journal of Qualitative Theory of Differential Equations 2011 (66) (2011) 1–12, https://doi.org/10.14232/ejqtde.2011.1.66.
[45] J. Caballero, J. Harjani, K. Sadarangani, On existence and uniqueness of positive solutions to a class of fractional boundary value problems, Boundary Value Problems 2011 (1) (2011) 1–9, https://doi.org/10.1186/1687-2770-2011-25.
[46] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin Heidelberg, New York, 1985.
[47] E. Zeidler, Nonlinear Functional Analysis and its Applications, Part 1: Fixed-Point Theorems, Springer, 1986, https://doi.org/10.1007/978-1-4612-4838-5_1.
[48] V. Berinde, Iterative Approximation of Fixed Points, Springer, Berlin, Heidelberg, 2007, https://doi.org/10.1109/SYNASC.2007.49.
[49] D. Varol Bayram, A. Daşcıoğlu, A method for fractional Volterra integro-differential equations by Laguerre polynomials, Advances in Difference Equations 2018 (1) (2018) 1–11, https://doi.org/10.1186/s13662-018-1924-0.