EXISTENCE OF SOLUTION OF MULTI-TERM FRACTIONAL ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

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Abstract This paper considered a multi-term fractional order Fredholm integro-differential equation. The multi-term fractional order Fredholm integro-differential equation was transformed into its corresponding integral equation form with the help of Riemann-Liouville fractional integral by which, Schauder’s fixed point theorem is utilised in the study and establishing the existence of solution for the multi-term fractional order Fredholm integro-differential equation. Moreover, some examples were considered to prove the claim of the established existence of solution theorem.

MSC: 26A33; 34A08; 45B05; 47H10

Keywords: Fredholm integro-differential equation; fractional integro-differential equation; Existence of solution; Schauder’s fixed point theorem

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1. INTRODUCTION

Fractional calculus deals with the study of fractional order integral and derivative operators over real and complex domain and their applications. It does not mean the calculus of fractions nor the fractions of calculus [1]. It is therefore, the extension of classical calculus, involving derivatives and integrals of real or complex order [2, 3]. The concept of this differentiation for non-integer numbers dated as far back as in the days classical calculus in 1695. In a famous correspondence between Leibniz and L’Hopital, L’Hopital asked Leibniz about the possibility that the order $n$ in the notation $\frac{d^n y}{dx^n}$, for the $n^{th}$ derivative of the function $y$, be a non-integer $n = \frac{1}{2}$ [4]. Ever since, a number of brilliant scientists motivated to focus their attention on fractional calculus [5], such as Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1873), Riemann (1847), Grunward (1867-1872), Letnikov (1868-1872), Hadamard (1892), Heaviside (1892-1912), Riesz (1949), Caputo (1965) and they have all contributed to the growth of this field. One of the first applications of fractional calculus appear in 1823 by Niels Abel, through the solution of an integral equation used in the formulation of the Tautochrone problem [4].

Mostly, physical problems are better translated by fractional derivatives because fractional operators considers the evolution of the problems into account [6]. However, it is quite challenging to find analytical solutions for these fractional differential equations (FDEs). Therefore, numerical approximation plays a vital role in finding the approximate solution to these equations and as such, numerous scholars have introduced and developed numerical approaches to find approximations to the solutions to this class of equations [7]. Furthermore, in the past few decades, fractional calculus found its application in various field of science such as Physics [8], Chemical Reactions [9], Electrochemistry [10], Biology [11] Optics [12], in engineering such as Bio-engineering [13], Chemical kinetics [14, 15], Fluid Mechanics [16], Chaotic systems [17, 18], Viscoelasticity [19], and in social sciences such as Finance [20], Optimal Control problems [21, 22], Social Sciences [23], Economics [24, 25]. Furthermore, several scholars have created and studied the existence and uniqueness of solutions of these equations such as [26] proved the existence of solution for fractional integro-differential equations with nonlocal condition via resolvent operators. Also [2, 27–45] studied the existence of solutions of various types of fractional differential equations and fractional integro-differential equations of boundary value problems and initial value problems.

In this study, we considered the multi-term fractional order Fredholm integro-differential equation of the form

$$D^{\beta} \xi (x) = \sum_{j=0}^{k} c_j (x) D^{\delta_j} \xi (x) + h (x) + \int_{0}^{1} K (\sigma, \tau) H (\xi (\tau)) d\tau$$

subject to the initial condition

$$\xi^{(n)} (0) = d_n, \quad n = 0, 1, 2, ..., m - 1,$$

$$m - 1 < \delta_0 < \delta_1 < \cdots < \delta_j < \beta \leq m, \quad m \in \mathbb{N},$$

where $D$ is the differential operator of order $\beta$ defined in Caputo sense, $\xi : Q \rightarrow \mathbb{R}$, $Q = [0, 1]$ is a continuous function which needs to be determined, $c_j, h : Q \rightarrow \mathbb{R}$ are given continuous functions, $K : Q \times Q \rightarrow \mathbb{R}$ is the kernel of integration which is also continuous, $H : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function.
2. PRELIMINARIES

**Definition 2.1** (Compact Map [46]). Let $X$ and $Y$ be Banach spaces and let $\Omega \subseteq X$. A map $F : \Omega \rightarrow Y$ is said to be compact if it is continuous and $F(\Omega)$ is relatively compact (i.e., for every $(x_n)_n \subseteq \Omega$, there exists a subsequence $(x_{n_j})_j$ of $(x_n)_n$ such that $F(x_{n_j})_j$ is convergent).

**Definition 2.2** (Uniformly Bounded [3]). A set $M$ is called uniformly bounded if there exists a constant $K > 0$ such that $\|m\| \leq K$ for every $m \in M$.

**Definition 2.3** (Equicontinuous [3]). A set $M$ is called equicontinuous if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, for all $m \in M$ and all $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$, we have $|m(x_1) - m(x_2)| < \epsilon$.

**Definition 2.4** (Fixed Point [47]). Given a map $T : A \rightarrow B$, every solution $\xi$ of the equation $T\xi = \xi$ is called a fixed point of $T$.

**Definition 2.5** (Banach Space [48]). A Banach space is a normed vector space which is complete (as a metric space).

**Definition 2.6** (Riemann-Liouville Fractional Integral [3]). Riemann-Liouville fractional integral of order $\alpha$ of a function $\xi$ is defined as

$$I^\beta \xi(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\sigma)^{\beta-1} \xi(\sigma) \, d\sigma, \quad x > 0, \beta \in \mathbb{R}^+, \quad (2.1)$$

where $\mathbb{R}^+$ is the set of positive real numbers.

**Definition 2.7** (Caputo Fractional Derivative [3]). The fractional derivative of $\xi(x)$ in the Caputo sense is defined by

$$D^\beta \xi(x) = I^{m-\beta} D^m \xi(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-\sigma)^{m-\beta-1} \frac{d^m \xi(\sigma)}{d\sigma^m} \, d\sigma, \quad m-1 < \beta \leq m. \quad (2.2)$$

With the Properties

1. $I^\alpha D^\alpha \xi(x) = \xi(x) - \sum_{n=0}^{m-1} \frac{\xi^{(n)}(0)}{n!} x^n$, $m-1 < \alpha < m$,
2. $I^\alpha D^\beta \xi(x) = I^{\alpha-\beta} \xi(x)$, $\beta < \alpha$,
3. $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}$.

**Theorem 2.8** (Arzelà-Ascoli [3]). Let $M$ be a subset of $C[a, b]$ equipped with the norm ($\|\|_\infty$). Then $M$ is relatively compact in $C[a, b]$ if, and only if, $M$ is equicontinuous and uniformly bounded.

**Theorem 2.9** (Schauder’s Fixed Point Theorem [46]). Let $C$ be a nonempty, closed, bounded and convex subset of a Banach space $X$ and let $T : C \rightarrow C$ be compact. Then $T$ has a fixed point.
3. Preliminary results

Lemma 3.1. Let $\xi: Q \rightarrow \mathbb{R}$ and $h: Q \rightarrow \mathbb{R}$ be continuous functions. Then, a function $\xi$ is a solution to the fractional integro-differential equation (1.1) – (1.2) if, and only if,

$$
\xi(x) = \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n + \sum_{j=0}^{k} \frac{c_j(x)}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} \xi(\sigma) d\sigma + \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} h(\sigma) d\sigma + \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 K(\sigma, \tau) H(\xi(\tau)) d\tau \right) d\sigma.
$$

Proof. Using equation (2.1) on equation (1.1) and by property (i), (ii) and (iii) we have,

$$
I^\beta(D^\beta \xi(x)) = I^\beta \left( \sum_{j=0}^{k} c_j(x) D^\delta_j \xi(x) \right) + I^\beta(h(x)) + I^\beta \left( \int_0^1 K(\sigma, \tau) H(\xi(\tau)) d\tau \right)
$$

Substituting the initial condition, we obtain

$$
\xi(x) = \sum_{n=0}^{m-1} \frac{\xi^{(n)}(0)}{n!} x^n + \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} c_j(x) \xi(\sigma) d\sigma + \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} h(\sigma) d\sigma + \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 K(\sigma, \tau) H(\xi(\tau)) d\tau \right) d\sigma.
$$

Thus, $\xi$ solves (1.1) – (1.2) if, and only if, $\xi$ solves (3.1).

Lemma 3.2. Let $X$ and $Y$ be normed linear spaces and let $f: A \rightarrow Y$ be a Lipschitz map, $A \subseteq X$. Then $f$ sends bounded sets to bounded sets.

Proof. For any set $E \subseteq A$, if there exists $M > 0 : \|x\|_X \leq M$ for all $x \in E$, then we show that there exists $\tilde{M} > 0 : \|f(x)\|_Y \leq \tilde{M}$ for all $x \in E$. Since $f$ is Lipschitz, that is, there exists $L > 0$ such that

$$
\|f(x) - f(y)\|_Y \leq L \|x - y\|_X \text{ for all } x, y \in A.
$$
Let \( x_0 \) be a fixed element of \( A \). Then, if there exists \( M > 0 : \|x\|_X \leq M \) for all \( x \in E \), then,

\[
\|f(x)\|_Y = \|f(x) - f(x_0) + f(x_0)\|_Y \\
\leq \|f(x) - f(x_0)\| + \|f(x_0)\| \\
\leq L \|x - x_0\| + \|f(x_0)\| \\
\leq L (\|x\| + \|x_0\|) + \|f(x_0)\| \\
\leq L (M + \|x_0\|) + \|f(x_0)\|. 
\]

Thus, \( \|f(x)\|_Y \leq \tilde{M} \) for all \( x \in E \), where \( \tilde{M} = L (M + \|x_0\|) + \|f(x_0)\| \). Hence, \( f(E) \) is bounded. Thus, \( f \) sends bounded sets to bounded sets. \( \square \)

**Lemma 3.3.** For any \( 0 \leq a < b \),

\[ b^\alpha - a^\alpha \leq (b-a)^\alpha , \alpha \in (0,1) . \]

**Proof.** Define \( f : [0, b) \longrightarrow \mathbb{R} \) by

\[
f(t) = (b-t)^\alpha - b^\alpha + t^\alpha , t \in [0, b) . \tag{3.2}
\]

Then from equation (3.2)

\[
f(t) = (b-t)^\alpha - b^\alpha + t^\alpha \\
\geq 0.
\]

Case I. Suppose \( \alpha = 0 \), then for any \( 0 \leq a < b \) we have from equation (3.2)

\[
f(a) = 1 \\
\geq 0.
\]

Case II (a). Suppose \( \alpha > 0 \), and \( f(t) \) is increasing and we have from equation (3.2)

\[
f'(t) = -\alpha \left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \\
\geq 0,
\]

this implies

\[
(b-t)^{\alpha-1} - t^{\alpha-1} \leq 0 \\
(b-t)^{\alpha-1} \leq t^{\alpha-1} \\
b - t \leq t \\
b \leq 2t,
\]

therefore

\[
t \geq \frac{b}{2}.
\]

(b) Suppose \( \alpha > 0 \), and \( f(t) \) is decreasing and we have from equation (3.2)

\[
f'(t) = -\alpha \left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \\
\leq 0,
\]
this implies
\[
(b - t)^{\alpha - 1} - t^{\alpha - 1} \geq 0
\]
\[
(b - t)^{\alpha - 1} \geq t^{\alpha - 1}
\]
\[
b - t \geq t
\]
\[
b \geq 2t,
\]
therefore
\[
t \leq \frac{b}{2}.
\]
So, \( f \) is decreasing on \([0, \frac{b}{2}]\) and increasing on \([\frac{b}{2}, b]\). Hence, \( \inf_{t \in [0, b]} f(t) = f(0) = 0 \) and
\[
\lim_{t \to b^-} f(t) = f(b) = 0, \text{ i.e., } \inf_{t \in [0, b]} f(t) = f(0) \text{ or } f(b). \text{ Thus, } \inf_{t \in [0, b]} f(t) = 0. \text{ Therefore, for any } 0 \leq a < b, \ f(a) \geq \inf_{t \in [0, b]} f(t) = 0, \text{ i.e.,}
\]
\[
(b - a)^\alpha - b^\alpha + a^\alpha \geq 0,
\]
that is,
\[
b^\alpha - a^\alpha \leq (b - a)^\alpha.
\]
Case III (a). Suppose \( \alpha < 0 \), and \( f(t) \) is increasing and we have from equation (3.2)
\[
f'(t) = -\alpha \left( (b - t)^{\alpha - 1} - t^{\alpha - 1} \right)
\]
\[
\geq 0
\]
this implies
\[
(b - t)^{\alpha - 1} - t^{\alpha - 1} \geq 0
\]
\[
(b - t)^{\alpha - 1} \geq t^{\alpha - 1}
\]
\[
b - t \geq t
\]
\[
b \geq 2t,
\]
therefore
\[
t \leq \frac{b}{2}.
\]
(b) Suppose \( \alpha < 0 \), and \( f(t) \) is decreasing and we have from equation (3.2)
\[
f'(t) = -\alpha \left( (b - t)^{\alpha - 1} - t^{\alpha - 1} \right)
\]
\[
\leq 0
\]
this implies
\[
(b - t)^{\alpha - 1} - t^{\alpha - 1} \leq 0
\]
\[
(b - t)^{\alpha - 1} \leq t^{\alpha - 1}
\]
\[
b - t \leq t
\]
\[
b \leq 2t,
\]
therefore
\[
t \geq \frac{b}{2}.
\]
So, $f$ is increasing on $[0, \frac{b}{2}]$ and decreasing on $[\frac{b}{2}, b]$. Hence, $\inf_{t \in [0, b)} f(t) = f(0) = 0$ and
$$
limit_{t \to b^-} f(t) = f(b) = 0,$$
i.e., $\inf_{t \in [0, b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t \in [0, b)} f(t) = 0$. Therefore, for any $0 \leq a < b$, $f(a)$ is increasing on $[0, \frac{b}{2})$ and decreasing on $[\frac{b}{2}, b)$. Hence, $\inf_{t \in [0, b)} f(t) = f(0) = 0$ and
$$
limit_{t \to b^-} f(t) = f(b) = 0,$$
i.e., $\inf_{t \in [0, b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t \in [0, b)} f(t) = 0$.

Thus, $\inf_{t \in [0, b)} f(t) = 0$.

Therefore, for any $0 < a < b$, $f(a)$ is increasing on $[0, \frac{b}{2})$ and decreasing on $[\frac{b}{2}, b)$.


4. **Main Results**

Throughout this work, we denote by
\begin{align*}
i: & \quad \|\cdot\|_\infty \text{ the sup norm on } C(Q, \mathbb{R}), \text{i.e. for } c \in C(Q, \mathbb{R}), \|c\|_\infty = \sup_{x \in Q} |c(x)|. \\
iI: & \quad \Pi := \sum_{j=0}^{k} \frac{\|c_j\|_\infty}{\Gamma(\beta - \delta_j + 1)}, \text{ where } c_j : Q \to \mathbb{R}, \text{ are given continuous functions.}
\end{align*}

We also make the following hypotheses:

$$(k_1)$$ there exists a constant $M > 0$ such that for any $y_1, y_2 \in C(Q, \mathbb{R})$ we have
$$|H(\xi_1(x)) - H(\xi_2(x))| \leq M \|\xi_1(x) - \xi_2(x)\|_\infty \quad x \in Q,$$

$$(k_2)$$ there exists a constant $\tilde{K}$ such that
$$\tilde{K} = \sup_{\sigma \in [0, 1]} \int_{0}^{1} \left| K(\sigma, \tau)e^\tau \right| d\tau < \infty, \quad 0 \leq \tau \leq 1.$$

**Theorem 4.1.** (Existence of Solution). Assume that $(k_1)$ and $(k_2)$ holds, if
\begin{equation}
\left(\Pi + \frac{\tilde{K} M}{\Gamma(\beta + 1)}\right) < 1, \tag{4.1}
\end{equation}
then there exists a solution $\xi \in C(Q, \mathbb{R})$ to problem (1.1) – (1.2).

**Proof.** Let $T$ be an operator such that $T : C(Q, \mathbb{R}) \to C(Q, \mathbb{R})$ defined by
\begin{align*}
(T\xi)(x) &= \sum_{n=0}^{m-1} \frac{d^n}{n!} x^n + \sum_{j=0}^{K} \frac{1}{\Gamma(\beta - \delta_j)} \int_{0}^{x} (x - \sigma)^{\beta - \delta_j - 1} c_j(\sigma) \xi(\sigma) d\sigma \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{0}^{x} (x - \sigma)^{\beta - 1} h(\sigma) d\sigma \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{0}^{x} (x - \sigma)^{\beta - 1} \left( \int_{0}^{1} K(\sigma, \tau) H(\xi(\tau)) d\tau \right) d\sigma. \tag{4.2}
\end{align*}

Our objective is to utilize Schauder’s fixed point theorem in proving the existence of solution of the multi-term fractional order Fredholm integro-differential equation. To do that, we will show that $T$ satisfies the following:

i: $T$ is continuous,

ii: $T$ sends bounded sets to bounded sets,

iii: $T$ sends bounded sets to equicontinuous sets.
First, we note that $T$ is well defined. Indeed, since $x \mapsto \sum_{n=0}^{m-1} \frac{d^n}{dt^n} x^n$, $x \mapsto \sum_{j=0}^{k} c_j(x) (I^{\beta-\delta_j} \xi)(x)$, $x \mapsto (I^\beta h)(x)$, $x \mapsto I^\beta \left( \int_0^1 K(\sigma, \tau) H(\xi(\tau)) d\tau \right)$ are continuous in $[0, 1]$, the right hand of equation (4.2) is well defined and $x \mapsto (T \xi)(x)$ is continuous. Thus, for $\xi \in C(Q, \mathbb{R})$, $T \xi \in C(Q, \mathbb{R})$.

Let $u, v \in C(Q, \mathbb{R})$. Then for any $x \in [0, 1]$, we have by letting $E = |(Tu)(x) - (Tv)(x)|$

\[
E = \left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} c_j(\sigma) (u(\sigma) - v(\sigma)) d\sigma \right| + \left| \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 K(\sigma, \tau) (H(u(\tau)) - H(v(\tau))) d\tau \right) d\sigma \right|
\]

\[
\leq \left| \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} c_j(\sigma) (u(\sigma) - v(\sigma)) d\sigma \right| + \left| \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 K(\sigma, \tau) (H(u(\tau)) - H(v(\tau))) d\tau \right) d\sigma \right|
\]

\[
\leq \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} c_j(\sigma) |u(\sigma) - v(\sigma)| d\sigma + \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 |K(\sigma, \tau)| |H(u(\tau)) - H(v(\tau))| d\tau \right) d\sigma
\]

\[
\leq \sum_{j=0}^{k} \frac{\|c_j\|_\infty \|u - v\|_\infty}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} d\sigma + \frac{\tilde{K} M \|u - v\|_\infty}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} d\sigma.
\]

By property (iii) we have

\[
|(Tu)(x) - (Tv)(x)| \leq \left( \Pi + \frac{\tilde{K} M}{\Gamma(\beta + 1)} \right) \|u - v\|_\infty, \text{ for all } x \in [0, 1].
\]

Thus,

\[
\|(Tu) - (Tv)\|_\infty \leq \Psi \|u - v\|_\infty,
\]

where

\[
\Psi = \Pi + \frac{\tilde{K} M}{\Gamma(\beta + 1)} \in \mathbb{R}.
\]

It follows that $T$ is not just continuous, but Lipschitz continuous.

Next, we conclude by Lemma 2 that operator $T$ (i.e. $T$ being Lipschitz) maps bounded sets to bounded sets in $C(Q, \mathbb{R})$.

Furthermore, we show that $T$ maps bounded sets to equicontinuous sets of $C(Q, \mathbb{R})$. Let $\xi \in B_\epsilon = \{ \xi \in C(Q, \mathbb{R}) : \|\xi\|_\infty \leq \epsilon \}$, $\sup_{x \in Q} |H(\xi(x))| \leq M \|\xi\|_\infty + |H(L)|$, (with

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\( \xi (c) = L \) and let \( x_1, x_2 \in [0, 1] \) with \( x_1 < x_2 \). Setting \( E = (T \xi) (x_2) - (T \xi) (x_1) \), then we have from (4.2)

\[
|E| = \left| \sum_{n=0}^{m-1} \frac{d_n}{n!} (x_2^n - x_1^n) \right|
+ \left| \sum_{j=0}^{k} \frac{1}{\Gamma (\beta - \delta_j)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - \delta_j - 1} c_j (\sigma) \xi (\sigma) d\sigma - \int_0^{x_1} (x_1 - \sigma)^{\beta - \delta_j - 1} c_j (\sigma) \xi (\sigma) d\sigma \right) \right|
+ \left| \frac{1}{\Gamma (\beta)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - 1} h (\sigma) d\sigma - \int_0^{x_1} (x_1 - \sigma)^{\beta - 1} h (\sigma) d\sigma \right) \right|
+ \left| \frac{1}{\Gamma (\beta)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - 1} \left( \int_0^{1} K (\sigma, \tau) H (\xi (\tau)) d\tau \right) d\sigma \right) \right|

Introducing a special zero to the right hand side, we have

\[
|E| \leq \left| \sum_{n=0}^{m-1} \frac{d_n}{n!} (x_2^n - x_1^n) \right|
+ \left| \sum_{j=0}^{k} \frac{1}{\Gamma (\beta - \delta_j)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - \delta_j - 1} c_j (\sigma) \xi (\sigma) d\sigma \right.
- \int_0^{x_1} (x_1 - \sigma)^{\beta - \delta_j - 1} c_j (\sigma) \xi (\sigma) d\sigma \right)
+ \left| \frac{1}{\Gamma (\beta)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - 1} h (\sigma) d\sigma - \int_0^{x_1} (x_1 - \sigma)^{\beta - 1} h (\sigma) d\sigma \right) \right|
+ \left| \frac{1}{\Gamma (\beta)} \left( \int_0^{x_2} (x_2 - \sigma)^{\beta - 1} \left( \int_0^{1} K (\sigma, \tau) H (\xi (\tau)) d\tau \right) d\sigma \right) \right|
\]
\[
\begin{align*}
&\quad + \int_0^x (x_2 - \sigma)^{\beta-1} h(\sigma) - \int_0^x (x_1 - \sigma)^{\beta-1} h(\sigma)\ d\sigma \\
&\quad + \int_0^x (x_2 - \sigma)^{\beta-1} \left( \int_0^x K(\sigma, \tau) H(\xi(\tau))\ d\tau \right) \\
&\quad - \int_0^x (x_2 - \sigma)^{\beta-1} \left( \int_0^x K(\sigma, \tau) H(\xi(\tau))\ d\tau \right) \\
&\quad + \int_0^x (x_2 - \sigma)^{\beta-1} \left( \int_0^x K(\sigma, \tau) H(\xi(\tau))\ d\tau \right) \\
&\quad - \int_0^x (x_1 - \sigma)^{\beta-1} \left( \int_0^x K(\sigma, \tau) H(\xi(\tau))\ d\tau \right)\ d\sigma.
\end{align*}
\]

By simplification, we have

\[
|E| \leq \sum_{n=0}^{m-1} \frac{|d_n|}{n!} \left( x_2^n - x_1^n \right)
\]

\[
+ \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta-\delta_j} \left| c_j(\sigma)\right| |\xi(\sigma)|\ d\sigma \\
+ \int_0^x (x_2 - \sigma)^{\beta-\delta_j} \left| c_j(\sigma)\right| |\xi(\sigma)|\ d\sigma \\
- \int_0^x (x_1 - \sigma)^{\beta-\delta_j} \left| c_j(\sigma)\right| |\xi(\sigma)|\ d\sigma \\
+ \frac{1}{\Gamma(\beta)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta-1} |h(\sigma)|\ d\sigma \\
+ \int_0^x (x_2 - \sigma)^{\beta-1} |h(\sigma)|\ d\sigma - \int_0^x (x_1 - \sigma)^{\beta-1} |h(\sigma)|\ d\sigma \\
+ \frac{1}{\Gamma(\beta)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta-1} \left( \int_0^x |K(\sigma, \tau)| |H(\xi(\tau))|\ d\tau \right)\ d\sigma \\
+ \int_0^x (x_2 - \sigma)^{\beta-1} \left( \int_0^x |K(\sigma, \tau)| |H(\xi(\tau))|\ d\tau \right)\ d\sigma \\
- \int_0^x (x_1 - \sigma)^{\beta-1} \left( \int_0^x |K(\sigma, \tau)| |H(\xi(\tau))|\ d\tau \right)\ d\sigma \right).
\]

The above inequality can be written as

\[
|(T\xi)(x_2) - (T\xi)(x_2)| \leq A_1 + A_2 + A_3 + A_4, \tag{4.3}
\]

where

\[
A_1 = \sum_{n=0}^{m-1} \frac{|d_n|}{n!} \left( x_2^n - x_1^n \right)
\]
\[ A_2 = \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \
+ \int_{0}^{x_1} (x_2 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \
- \int_{0}^{x_1} (x_1 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right), \]

\[ A_3 = \frac{1}{\Gamma(\beta)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta - 1} |h(\sigma)| d\sigma \
+ \int_{0}^{x_1} (x_2 - \sigma)^{\beta - 1} |h(\sigma)| d\sigma \
- \int_{0}^{x_1} (x_1 - \sigma)^{\beta - 1} |h(\sigma)| d\sigma \right), \]

\[ A_4 = \frac{1}{\Gamma(\beta)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta - 1} \left( \int_{0}^{1} |K(\sigma, \tau)||H(\xi(\tau))| d\tau \right) d\sigma \
+ \int_{0}^{x_1} (x_2 - \sigma)^{\beta - 1} \left( \int_{0}^{1} |K(\sigma, \tau)||H(\xi(\tau))| d\tau \right) d\sigma \
- \int_{0}^{x_1} (x_1 - \sigma)^{\beta - 1} \left( \int_{0}^{1} |K(\sigma, \tau)||H(\xi(\tau))| d\tau \right) d\sigma \right). \]

On Simplifying the term \( A_1 \) by considering that \( (x_n^2 - x_1^2) \leq (x_2 - x_1) \) for \( 0 \leq n \leq m - 1 \), since \( x_2, x_1 \in [0, 1] \) and \( x_1 < x_2 \) and taking \( d_{n^*} = \max_{0 \leq n \leq m - 1} \{d_n\} \), we have

\[ A_1 = \sum_{n=0}^{m-1} \frac{|d_n|}{n!} (x_2^n - x_1^n) \leq \left( \frac{|d_1|}{1!} + \frac{|d_2|}{2!} + \ldots + \frac{|d_{m-1}|}{(m-1)!} \right) (x_2 - x_1) = \frac{|d_{n^*}|}{n^*!} (m - 1) (x_2 - x_1). \]

On simplifying the term \( A_2 \) and by property (iii) and Lemma 3 we have

\[ A_2 = \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \left( \int_{x_1}^{x_2} (x_2 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \
+ \int_{0}^{x_1} (x_2 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma - \int_{0}^{x_1} (x_1 - \sigma)^{\beta - \delta_j - 1} |c_j(\sigma)| |\xi(\sigma)| d\sigma \right) \leq \sum_{j=0}^{k} \frac{||c_j||_\infty ||\xi||_\infty}{\Gamma(\beta - \delta_j + 1)} \left( (x_2 - x_1)^{\beta - \delta_j} + (x_2^{\beta - \delta_j} - (x_2 - x_1)^{\beta - \delta_j} - x_1^{\beta - \delta_j}) \right) = \sum_{j=0}^{k} \frac{||c_j||_\infty ||\xi||_\infty}{\Gamma(\beta - \delta_j + 1)} \left( x_2^{\beta - \delta_j} - x_1^{\beta - \delta_j} \right). \]
Thus, equation (4.3) is

\[ \| (T\xi)(x_2) - (T\xi)(x_1) \| \leq \frac{|d_n^r|}{n^r}(m - 1)(x_2 - x_1)\sum_{j=0}^{k} \frac{\|c_j\|_\infty \|\xi\|_\infty}{\Gamma(\beta - \delta_j + 1)}(x_2 - x_1)^{\beta - \delta_j} + \frac{\|h\|_\infty}{\Gamma(\beta + 1)}(x_2 - x_1)^\beta + \frac{\tilde{K}(M \|\xi\|_\infty + |H(L)|)}{\Gamma(\beta + 1)}(x_2 - x_1)^\beta. \]

We see that the right hand side of the above equation is independent of \( \xi \) and tends to zero as \( x_2 - x_1 \to 0 \). This leads to \( |(T\xi)(x_2) - (T\xi)(x_1)| \to 0 \) as \( x_2 \to x_1 \) uniformly in \( \xi \). Therefore, the set \( \{ T\xi : \xi \in B_\epsilon \} \) is equicontinuous and finally, we need to show that there exists a closed convex bounded subset \( C \) of \( X \) such that \( TC \subseteq C \).

Consider \( B_\epsilon = \{ \xi \in C (Q, \mathbb{R}) : \|\xi\|_\infty \leq \epsilon \} \), we will show that for some \( \epsilon > 0 \), \( TB_\epsilon \subseteq B_\epsilon \). For contradiction, suppose that \( TB_\epsilon \not\subseteq B_\epsilon \) for all \( \epsilon > 0 \).

Let \( \mu \) be a positive integer, then there exists \( \xi_\mu \in B_\mu \) such that \( \|T\xi_\mu\|_\infty > \mu \).
Consider
\[
||(T\xi_{\mu}) (x)|| \leq \sum_{n=0}^{m-1} \frac{d_n}{n!} x^n + \sum_{j=0}^{k} \frac{1}{\Gamma(\beta - \delta_j)} \int_0^x (x - \sigma)^{\beta - \delta_j - 1} c_j(\sigma)\xi_{\mu}(\sigma) d\sigma \\
+ \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} h(\sigma) d\sigma \\
+ \frac{1}{\Gamma(\beta)} \int_0^x (x - \sigma)^{\beta - 1} \left( \int_0^1 K(\sigma, \tau) H(\xi_{\mu}(\tau)) d\tau \right) d\sigma \\
\leq \sum_{n=0}^{m-1} \frac{|d_n|}{n!} x^n \\
+ \sum_{j=0}^{k} \frac{\|c_j\|_{\infty} \|\xi_{\mu}\|_{\infty}}{\Gamma(\beta - \delta_j + 1)} \\
+ \frac{\|h\|_{\infty}}{\Gamma(\beta + 1)} + \frac{\tilde{K}(M\mu + |H(L)|)}{\Gamma(\beta + 1)}, \text{ for all } x \in [0, 1].
\]
Thus,
\[
||(T\xi_n)||_{\infty} \leq \sum_{n=0}^{m-1} \frac{|d_n|}{n!} + \Pi\mu + \frac{\|h\|_{\infty}}{\Gamma(\beta + 1)} + \frac{\tilde{K}(M\mu + |H(L)|)}{\Gamma(\beta + 1)}.
\]
Observe that if
\[
\mu < ||(T\xi_{\mu})||_{\infty} \\
< \sum_{n=0}^{m-1} \frac{|d_n|}{n!} + \Pi\mu + \frac{\|h\|_{\infty}}{\Gamma(\beta + 1)} + \frac{\tilde{K}(M\mu + |H(L)|)}{\Gamma(\beta + 1)}.
\]
Dividing through by \(\mu\) we have,
\[
1 < \sum_{n=0}^{m-1} \frac{|d_n|}{\mu n!} + \Pi + \frac{\|h\|_{\infty}}{\mu \Gamma(\beta + 1)} + \frac{\tilde{K}(M\mu + |H(L)|)}{\mu \Gamma(\beta + 1)}.
\]
Letting \(\mu \rightarrow \infty\) we obtain
\[
1 < \Pi + \frac{\tilde{K}M}{\Gamma(\beta + 1)}.
\]
Which is a contradiction to equation (4.1). Hence, for some \(\mu_0, TB_{\mu_0} \subseteq B_{\mu_0}\).

Let \(C := B_{\mu_0}\) and let \(\tilde{T} := T|_c\), i.e., \(T : C \rightarrow C\) with \(\tilde{T}\xi = T\xi\). By Arzelà-Ascoli thus, for any \((\xi_n)_n \subseteq C\), since \(C\) is bounded \((\xi_n)_n\) is bounded and by \((\tilde{T}\xi_n)_n \equiv (T\xi_n)_n\) is
5. Example

Example 1 [49]. Consider the fractional integro-differential equation of Fredholm type

\[ D^{\frac{1}{2}} \xi (x) = x^{0.25} + (x \cos x - \sin x) \xi (x) + \int_{0}^{1} x \sin \tau \xi (\tau) d\tau. \]  

Subject to \( \xi (0) = 0 \) with exact solution \( \xi (x) = x \).

Solution: Equation (5.1) can be written as

\[
\| (T\xi_2) (x) - (T\xi_1) (x) \| = \left| \frac{1}{\Gamma \left( \frac{3}{4} \right)} \int_{0}^{x} (x - \sigma)^{-\frac{1}{4}} \left( \frac{\sigma \cos \sigma - \sin \sigma}{\sigma \cos \sigma} \right) (\xi_2 (\sigma) - \xi_1 (\sigma)) d\sigma \right|
\]

\[
= \left| \frac{1}{\Gamma \left( \frac{3}{4} \right)} \int_{0}^{x} (x - \sigma)^{-\frac{1}{4}} (\sigma - \sin \sigma) (\xi_2 (\sigma) - \xi_1 (\sigma)) d\sigma \right|
\]

\[
\leq \frac{1}{\Gamma \left( \frac{3}{4} \right)} \int_{0}^{x} (x - \sigma)^{-\frac{1}{4}} |\sigma - \sin \sigma| |\xi_2 (\sigma) - \xi_1 (\sigma)| d\sigma
\]

\[
\leq \frac{\|\xi_2 - \xi_1\|_{\infty}}{\Gamma \left( \frac{3}{4} \right)} \int_{0}^{x} (x - \sigma)^{-\frac{1}{4}} \left( \sigma - \frac{(-1)^{n} \sigma^{2n+1}}{(2n+1)!} \right) d\sigma.
\]

By Property (iii) we have,

\[
\| T\xi_2 - T\xi_1 \|_{\infty} \leq \left( \frac{\Gamma (2)}{\Gamma (2.75)} - \frac{(-1)^{n} \Gamma (2n + 2)}{(2n + 1)! \Gamma (2n + \frac{11}{4})} \right) \|\xi_2 - \xi_1\|_{\infty}, \quad x \in [0,1], \quad n \in \mathbb{N}.
\]

Thus,

\[
\| T\xi_2 - T\xi_1 \|_{\infty} \leq (0.62175) \|\xi_2 - \xi_1\|_{\infty}.
\]

(5.2)

Since 0.62175 < 1, we say that the problem satisfies the condition of Theorem 4.1.

Example 2 [21]. Consider the multi-term fractional order integro-differential equation

\[
a D^{2} \xi (x) + b (x) D^{\delta_1} \xi (x) + c (x) D \xi (x) + e (x) D^{\delta_2} \xi (x)
\]

\[
+ k (x) \xi (x) + \lambda \int_{0}^{1} K (\sigma, \tau) \xi (\tau) d\tau = h (x). \]  

Subject to \( \xi (0) = 2, \; \xi' (0) = 0 \) with exact solution \( \xi (x) = 2 - \frac{x^2}{2} \),

where \( h (x) = -a - \frac{b(x)}{\Gamma (3 - \delta_1)} x^{2-\delta_1} - c (x) x - \frac{e(x)}{\Gamma (3-\delta_2)} x^{2-\delta_2} + k (x) \left( 2 - \frac{x^2}{2} \right), \; \lambda = 0 \)

\( a = 1, \; b (x) = x^2, \; c (x) = x^3, \; e (x) = x^4, \; k (x) = x^5, \; \delta_2 = 0.333 \) and \( \delta_1 = 1.234. \)
Solution: Equation (5.3) can be written as
\[
\begin{align*}
| (T\xi_2)(x) - (T\xi_1)(x) | & \leq \frac{1}{\Gamma(2)\Gamma(1.766)} \int_0^x (x - \sigma)^{2-1} |\sigma^{1.266}||\xi_2(\sigma) - \xi_1(\sigma)| d\sigma \\
& + \frac{1}{\Gamma(2)} \int_0^x (x - \sigma)^{2-1} |\sigma^{1.333}||\xi_2(\sigma) - \xi_1(\sigma)| d\sigma \\
& + \frac{1}{\Gamma(2)\Gamma(1.667)} \int_0^x (x - \sigma)^{2-1} |\sigma^{0.917}||\xi_2(\sigma) - \xi_1(\sigma)| d\sigma \\
& + \frac{1}{\Gamma(2)} \int_0^x (x - \sigma)^{2-1} |\sigma^{0.2}||\xi_2(\sigma) - \xi_1(\sigma)| d\sigma \\
& \leq \frac{\|\xi_2 - \xi_1\|_\infty}{\Gamma(2)\Gamma(1.766)} \int_0^x (x - \sigma)^{1.266} d\sigma \\
& + \frac{\|\xi_2 - \xi_1\|_\infty}{\Gamma(2)} \int_0^x (x - \sigma)^{1.333} d\sigma \\
& + \frac{\|\xi_2 - \xi_1\|_\infty}{\Gamma(2)\Gamma(1.667)} \int_0^x (x - \sigma)^{0.917} d\sigma \\
& + \frac{\|\xi_2 - \xi_1\|_\infty}{\Gamma(2)} \int_0^x (x - \sigma)^{0.2} d\sigma.
\end{align*}
\]

By Property (iii), we have
\[
| (T\xi_2)(x) - (T\xi_1)(x) | \leq \left( \frac{\Gamma(2.266) x^{3.266}}{\Gamma(1.766) \Gamma(4.266)} + \frac{\Gamma\left(\frac{5}{3}\right) x^{\frac{10}{3}}}{\Gamma\left(\frac{16}{3}\right)} \right) \|\xi_2 - \xi_1\|_\infty.
\]

Thus,
\[
\|T\xi_2 - T\xi_1\|_\infty \leq (0.85187) \|\xi_2 - \xi_1\|_\infty.
\]

Since 0.85187 < 1, we say that the problem satisfies the condition of the Theorem 4.1.

6. Conclusion

This research focused on multi-term fractional order Fredholm integro-differential equation, whereby the multi-term fractional order Fredholm integro-differential equation was converted to its corresponding integral equation using Riemann-Liouville fractional integral. Schauder’s fixed point theorem is further used in the study of the existence of the solution for the multi-term fractional order Fredholm integro-differential equation. Furthermore, examples were considered to demonstrate the validity of the proposed existence of solution theorem for the solution of multi-term fractional order Fredholm integro-differential equations.

References


