BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
ISSN: 2408-154X (Print)
ISSN: 3057-0557 (Online)
https://bangmodjmcs.com
Volume 11 (2025)
Pages 421–449

A New Parallel Inertial Splitting Algorithm for Inclusion and Fixed Point Problems in Banach Spaces

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Received: 16 June 2025 / Accepted: 19 October 2025

Abstract This article proposes a new parallel method with an inertial extrapolation term for solving monotone inclusion and fixed-point problems involving a finite family of maximal monotone operators and Bregman strongly nonexpansive mappings in the setting of a reflexive Banach space. We establish a strong convergence result for the proposed iterative method without requiring knowledge of the Lipschitz constant of the underlying functions. Additionally, we provide a numerical comparison of our method's performance with existing methods in the literature. Numerical illustrations suggest that the proposed method is competitive and promising. Furthermore, we demonstrate an application of the algorithm in restoring test images degraded by motion blur and random noise. The results indicate that our method achieves superior restoration quality compared to existing methods.

MSC: 47H06, 47H09, 47J05, 47J25

Keywords: Inertia; Monotone inclusion problem; Bregman distance; Image restoration

Published online: 31 October 2025

Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: H.A. Abass et al., A New Parallel Inertial Splitting Algorithm for Inclusion and Fixed Point Problems in Banach Spaces, Bangmod Int. J. Math. & Comp. Sci., Vol. 11 (2025), 421–449. https://doi.org/10.58715/bangmodjmcs.2025.11.19

1. Introduction

Let $\mathcal{E}$ to be a real Banach space with dual space, $\mathcal{E}^*$. The problem of finding a point $u^* \in \mathcal{E}$ that satisfies the inclusion:

$$0 \in (\mathcal{A} + \mathcal{B})u^*, \tag{1.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are respectively, single valued and multi-valued monotone operators is popularly known as monotone inclusion problem (MIP). In consistency with what has been frequently used in the literature, we shall denote the set of solution to the MIP (1.1) by $(\mathcal{A} + \mathcal{B})^{-1}(0^*)$. The MIP allows an elegant formulation to a wide range of problems which involves finding an optimal solution for optimization related problems such as mathematical programming, optimal control and variational inequalities, to mention a few, (see, e.g., [4, 17, 40, 44] and the references therein). The MIP have found its various applications in diverse areas of mathematics such as image processing, statistical regression and signal recovery (see, e.g., [6, 18–20, 22, 35] and the references therein).

Due to the variety of applicability and importance of the monotone inclusion problem, over the years researchers in this direction have proposed different iterative methods for solving (1.1) (see [1, 3, 7, 8, 10, 15, 21, 26, 29, 30, 43]). One of such method is the forward-backward splitting method introduced by Lions and Mercier [26] in the settings of real Hilbert space $\mathcal{H}$. This method is implemented in the following manner: Given an initial guess $q_0 \in \mathcal{H}$ and setting $k = 0$, the next point of the sequence is generated by:

$$q_{k+1} = J^{\mathcal{B}}_{\lambda_k}(q_k - \lambda_k \mathcal{A} q_k), \; k \geq 0, \tag{1.2}$$

where $J^{\mathcal{B}}_{\lambda_k} := (I + \lambda_k \mathcal{B})^{-1}$ is the resolvent operator of the maximal monotone operator $\mathcal{B}$, I is the identity function and $\{\lambda_k\}$ is a sequence with non-negative points on $\mathbb{R}$. Lions and Mercier [26] proved that the sequence $\{q_k\}$ generated by (1.2) converges weakly to a solution of the MIP (1.1) under the restrictive condition that the operator $\mathcal{A}$ is α-inverse strongly monotone. However, as rightly pointed out by Tseng [41], this condition on A limits the applicability of the algorithm.

The Tseng's iterative splitting method is another iterative method for solving MIP which does not require inverse strongly monotonicity assumption on any of the functions. It was introduced by Tseng [41] in 2000. The Tseng iterative method is computed using the following procedures, let $q_1 \in \mathcal{H}$:

$$\begin{cases} y_k = J^{\mathcal{B}}_{\lambda_k}(q_k - \lambda_k \mathcal{A}), \\ q_{k+1} = y_k - \lambda_k(\mathcal{A} y_k - \mathcal{A} q_k), \; \forall \; k \geq \; 1, \end{cases}$$

where $\mathcal{A}$ is monotone and L-Lipschitz operator, $\mathcal{B}$ is a multi-valued operator and $\{\lambda_k\}$ is a sequence in $(0, \frac{1}{L})$.

Remark 1.1. *Although, Tseng [41] was able to dispense with the restriction on the operator A in algorithm (1.2), a drawback of his method is that it requires the prior knowledge of the Lipschitz constant to evaluate the step-size $\{\lambda_k\}$. However, from practical point of view, the Lipschitz constant in this case is very difficult to approximate.*

In 2019, Shehu [37] extended the Tseng's [41] iterative method to the setting of 2-uniformly convex Banach space $\mathcal{E}$ as follows:

$$\begin{cases} q_1 \in \mathcal{E}, \\ y_k = J^{\mathcal{B}}_{\lambda_n} \circ J^{-1}(Jq_k - \lambda_k \mathcal{A}q_k), \\ q_{k+1} = J^{-1}(Jy_k - \lambda_k(\mathcal{A}y_k - \mathcal{A}q_k)), \end{cases} \tag{1.3}$$

where $\mathcal{A} : \mathcal{E} \to \mathcal{E}^*$ is monotone and L-lipschitz continuous, $\mathcal{B} : \mathcal{E} \to 2^{\mathcal{E}^*}$ is a maximal monotone operator, $J^{\mathcal{B}}_{\lambda_n} := (J + \lambda_n \mathcal{A})^{-1} J$ is the resolvent of $\mathcal{B}$ and J denotes the duality mapping from $\mathcal{E}$ into $\mathcal{E}^*$. A weak convergence result was obtained. It is worth-mentioning that the λ_n in (1.3) depends on the Lipschitz constant.

Very recently, the results of Tseng [41] and Shehu [37] was extended by Sunthrayuth *et al.* [36] to the setting of a reflexive Banach spaces. The authors [36] introduced two iterative algorithms for solving MIP and fixed point problem for a Bregman relatively nonexpansive mapping. One of these iterative method is defined below:

Algorithm 1 Mann splitting algorithm for solving MIP.

Initialization: Choose $\lambda^1 > 0$, $\mu, \theta \in (0, \sigma)$, where σ is a constant defined in (2.3). Let $q^1 \in \mathcal{E}$ be arbitrary starting points.

Iterative step:

Step 1: Compute

$$w^k = J^{\mathcal{B}}_{\lambda^k} \nabla g^*(\nabla g(q^k) - \lambda^k \mathcal{A}(q^k)).$$

Step 2: Compute

$$z^k = \nabla g^*(\nabla g(w^k) - \lambda^k(\mathcal{A}(w^k) - \mathcal{A}(q^k)),$$

where λ^{k+1} is updated as follows:

$$\lambda^{k+1} = \begin{cases} \min\left\{ \dfrac{\mu\|q^k - w^k\|}{\|\mathcal{A}q^k - \mathcal{A}w^k\|}, \ \lambda^k \right\} & \text{if } \mathcal{A}q^k \neq \mathcal{A}w^k, \\ \lambda^k, & \text{otherwise.} \end{cases} \tag{1.4}$$

Step 3: Compute

$$q^{k+1} = \nabla g^*((1 - \alpha^k)\nabla g(z^k) + \alpha^k \nabla g(Tz^k)).$$

Stopping criterion: If $q^{k+1} = z^k$ for some positive k then stop. Otherwise set $k := k + 1$ and return to **Iterative step** .

A weak convergence result was obtained using their iterative algorithm without any prior knowledge of the Lipschitz constant of the underlying operator. The other iterative method employed an armijo linesearch to eradicate the existence of Lipschitz constant. It is well-known that a linesearch approach would necessitate numerous additional computations and further increase the computational cost of Algorithm 1 of [36]. Also, the self-adaptive stepsize employed in Algorithm 2 of [36] reduces to the one used in this article if $\eta_n \equiv 0$.

Inspired by the heavy ball methods of a two-order time dynamical system, Polyak [34] and Nestrov [32] proposed the following inertial method:

$$\begin{cases} u^k = q^k + \theta^k(q^k - q^{k-1}), \\ q^{k+1} = u^k - \lambda^k \bigtriangledown f(u^k), \ \forall\ k \geq 1, \end{cases} \tag{1.5}$$

where $\theta^k \in [0,1)$ is simply the inertial and λ^k is a non-negative (see, e.g., [5, 9, 38, 39]). This idea is now known as inertial extrapolation method. In 2008, the concept of inertial extrapolation was developed with the Mann iterative algorithm by Maingé [28], they expressed their method as follows:

$$\begin{cases} y^k = q^k + \theta^k(q^k - q^{k-1}), \\ q^{k+1} = (1-\lambda^k)y^k + \lambda^k T y^k, \end{cases}$$

for each $k \geq 1$. Result from this iterative scheme shows that the sequence $\{q^k\}$ is weakly convergent to the fixed point problem of the underlying mapping using the conditions stated below:

(i) $\theta^k \in [0,\nu)$ for each $k \geq 1$, where $\nu \in [0,1)$,

(ii) $\sum\limits_{k=1}^{\infty} \theta^k \|q^k - q^{k-1}\| < \infty$,

(iii) $0 < \liminf\limits_{k\to\infty} \lambda^k \leq \limsup\limits_{k\to\infty} \lambda^k < 1$.

Spurred by the results in [41], [37], [36] and results from related literature in this direction, we develop a new parallel method of iteration equipped with an inertial extrapolation method for solving a common solution of a finite family of MIP and fixed point problems for Bregman strongly nonexpansive mapping in the framework of a reflexive Banach space. We establish a strong convergence result for solving the finite families of the discussed problems without the knowledge of the Lipschitz constant of the underlying operator. We selected our stepsize to be self-adaptive, therefore it does not require the knowledge of the Lipschitz constant as well as the sequential weak continuity of the operator. In summary, we illustrate few numerical experiments to show that our proposed method is implementable. Our result is a further contribution to related results in the literature. We highlights of some of the contributions in this study:

(i) Results from [2, 37, 41] were extended to a more general Banach spaces.

(ii) Introduction of a self-adaptive procedure which increase from iteration to iteration and is independent of the Lipschitz constant of underlying operator is studied. This differs from the methods of Shehu [37] and Tseng [41] where the knowledge of Lipschitz constant is known.

(iii) A strong convergence result desirable to weak convergence result was established (see [36]).

(iv) We were able to obtain a strong convergence result without the assumption $\sum\limits_{k=1}^{\infty} \theta^k \|q^k - q^{k-1}\| < \infty$ and $\sum\limits_{k=1}^{\infty} \|q^k - q^{k-1}\| < \infty$, where θ^k is the inertial factor.

2. Preliminaries

In this section, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Let C be a nonempty closed and convex subset of $\mathcal{E}$. Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of g denoted by $g^* : \mathcal{E}^* \to (-\infty, +\infty]$ is defined by

$$g^*(u^*) = \sup\{\langle u^*, u\rangle - g(u) : u \in \mathcal{E}\}, \; u^* \in \mathcal{E}^*.$$

The domain of g be denoted as $dom(g) = \{u \in \mathcal{E} : g(u) < +\infty\}$, thus for any $u \in intdomg$ and $v \in \mathcal{E}$, the right-hand derivative of g at x in the direction of v is defined by

$$g^0(u, v) = \lim_{t \to 0^+} \frac{g(u + tv) - g(u)}{t}.$$

The function g is said to be

(i) Gâteaux differentiable at u if $\lim_{t\to 0^+} \frac{g(u+tv)-g(u)}{t}$ exists for any v. In this case, $g^0(u, v)$ coincides with $\nabla g(u)$;
(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $u \in intdomg$;
(iii) Fréchet differentiable at u, if its limit is attained uniformly in $||v|| = 1$;
(iv) Uniformly Fréchet differentiable on a subset C of $\mathcal{E}$, if the above limit is attained uniformly for $u \in C$ and $||v|| = 1$.

Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a mapping, then g is said to be:

(i) essentially smooth, if the subdifferential of g denoted as ∂g is both locally bounded and single-valued on its domain, where $\partial g(u) = \{w \in \mathcal{E} : g(u) - g(v) \geq \langle w, v - u\rangle,\ v \in \mathcal{E}\}$;
(ii) essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $dom\ \partial g$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [12, 13] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:

(i) The $intdomg$ is nonempty, g is Gâteaux differentiable on $intdomg$ and $dom\nabla g = intdomg$;
(ii) The $intdomg^*$ is nonempty, g^* is Gâteaux differentiable on $intdomg^*$ and $dom\nabla g^* = intdomg$.

Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of g at $x \in domg$ is the function $v_g(x, .) : [0, +\infty) \to [0, +\infty)$ defined by

$$v_g(x, t) := inf\{D_g(y, x) : y \in domg, ||y - x||\}.$$

If $B_s := \{z \in \mathcal{E} : ||z|| \leq s\}$ for all $s > 0$. Then, a function $g : \mathcal{E} \to \mathbb{R}$ is called uniformly convex on bounded subsets of $\mathcal{E}$, [see pp. 203 and 221] [45] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, ||x-y||=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of E, [see pp. 221] [45], if $\lim_{t\downarrow 0} \frac{\sigma_s}{t}$, for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)ty) + (1-\alpha)g(x - \alpha ty) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function g is said to be uniformly convex if the function $\delta g : [0, +\infty) \to [0, +\infty)$ defined by

$$\delta g(t) := \sup\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g(\frac{x+y}{2}) : ||y - x|| = t\},$$

satisfies $\lim_{t\downarrow 0} \frac{\delta g(t)}{t} = 0$.

Definition 2.1. [14] *Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then, the function $D_g : \mathcal{E} \times \mathcal{E} \to [0, +\infty)$ defined by*

$$D_g(u, v) := g(u) - g(v) - \langle \nabla g(v), u - v \rangle \tag{2.1}$$

is called the Bregman distance with respect to g, where $u, v \in \mathcal{E}$.

However, the Bregman distance satisfies the following three point identity: for any $u \in domg$ and $v, z \in intdomg$,

$$D_g(u, v) + D_g(v, z) - D_g(u, z) = \langle \nabla g(z) - \nabla g(v), u - v \rangle. \tag{2.2}$$

Also, the relationship between D_g and $||.||$ with strong convexity constant $\sigma > 0$ i.e.,

$$D_g(x, y) \geq \frac{\sigma}{2} ||x - y||^2, \ \forall \ x \in domg, \ \ y \in int(domg). \tag{2.3}$$

Let $\mathcal{U} : C \to int(domg)$ be a nonlinear operator. An element $p \in C$ is said to be a fixed point of $\mathcal{U}$ if $\mathcal{U}p = p$. We denote by $F(U)$ fixed point set of the operator $\mathcal{U}$. In addition, a point $p \in F(\mathcal{U})$ is said to be an asymptotic fixed point of $\mathcal{U}$ if C contains a sequence $\{x^k\}$ such that $\{x^k\} \rightharpoonup p$ and $\lim_{k\to\infty} ||\mathcal{U}x^k - x^k|| = 0$. We denote by $\hat{F}(\mathcal{U})$ the asymptotic fixed point set of $\mathcal{U}$. The mapping $\mathcal{U}$ is called

(i) Bregman nonexpansive, if

$$D_g(\mathcal{U}q, \mathcal{U}r) \leq D_g(q, r), \forall \ q, r \in C,$$

(ii) Bregman relatively nonexpansive, if $F(\mathcal{U}) \neq \emptyset$, and

$$D_g(z, \mathcal{U}q) \leq D_g(z, q), \ \forall \ z \in F(\mathcal{U}), \ \ q \in C \text{ and } F(\hat{\mathcal{U}}) = F(\mathcal{U}), \tag{2.4}$$

(iii) Bregman firmly nonexpansive (BFNE) if

$$\langle \nabla g(\mathcal{U}q) - \nabla g(\mathcal{U}r), \mathcal{U}q - \mathcal{U}r \rangle \leq \langle \nabla g(q) - \nabla g(r), \mathcal{U}q - \mathcal{U}r \rangle, \forall \ q, r \in C.$$

(iv) Bregman strongly nonexpansive (BSNE) with $\hat{F}(\mathcal{U}) \neq \emptyset$ if

$$D_g(z, \mathcal{U}q) \leq D_g(z, q), \forall \ r \in C, z \in \hat{F}(\mathcal{U}),$$

for any bounded sequence $\{q^k\}_{k\geq 1} \subset C$,

$$\lim_{k\to\infty} (D_g(z, q^k) - D_g(z, \mathcal{U}q^k)) = 0$$

implies

$$\lim_{n\to\infty} D_g(\mathcal{U}q^k, q^k) = 0.$$

(v) quasi-Bregman nonexpansive if $F(\mathcal{U}) \neq \emptyset$ and for all $q \in C, \ z \in F(T)$

$$D_g(z, \mathcal{U}q) \leq D_g(z, \mathcal{U}).$$

Definition 2.2. *A function $g : \mathcal{E} \to \mathbb{R}$ is called strongly coercive if*

$$\lim_{||q^k||\to\infty} \frac{g(q^k)}{||q^k||} = \infty.$$

Lemma 2.3. [16] *Let $\mathcal{E}$ be a reflexive Banach space, $g : \mathcal{E} \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(u, u^*) = g(u) - \langle u, u^* \rangle + g^*(u^*), \ \ x \in \mathcal{E}, \ \ u^* \in \mathcal{E}^*.$$

Then the following holds:

$$D_g(u, \nabla g^*(u^*)) = V(u, u^*), \text{ for all } x \in \mathcal{E}^* \text{ and } u^* \in \mathcal{E}^*.$$

$$V(u, u^*) + \langle \nabla g^*(u^*) - u, v^* \rangle \leq V(u, u^* + v^*) \text{ for all } u \in \mathcal{E} \text{ and } u^*, v^* \in \mathcal{E}^*.$$

Lemma 2.4. [16] *Let $\mathcal{E}$ be a Banach space and $g : \mathcal{E} \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of $\mathcal{E}$. Suppose $\{x^k\}_{k\in\mathbb{N}}$ and $\{y^k\}_{k\in\mathbb{N}}$ are bounded sequences in $\mathcal{E}$. Then,*

$$\lim_{k\to\infty} D_g(y^k, x^k) = 0 \Rightarrow \lim_{k\to\infty} ||y^k - x^k|| = 0.$$

Lemma 2.5. [24] *Suppose $g : \mathcal{E} \to \mathbb{R}$ is a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. If $u_0 \in \mathcal{E}$ and the sequence $\{D_g(u^k, u_0)\}$ is bounded, then the sequence $\{u^k\}$ is also bounded.*

Definition 2.6. *Let $\mathcal{E}$ be a reflexive Banach space and C be a nonempty closed and convex subset of $\mathcal{E}$. A Bregman projection of $x \in int(domg)$ onto $C \subset int(domg)$ is the unique vector $Proj^g_C(x) \in C$ satisfying*

$$D_g(Proj^g_C(u), u) = inf\{D_g(v, u) : v \in C\}.$$

Lemma 2.7. [11] *Let $\mathcal{E}$ be a real Banach space and $\mathcal{A} : \mathcal{E} \to \mathcal{E}^*$ be a monotone, hemicontinuous and bounded operator. Suppose $B : \mathcal{E} \to \mathcal{E}^*$ is a maximal monotone operator. Then $\mathcal{A} + \mathcal{B}$ is maximal monotone.*

Lemma 2.8. [31] *Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a continuous uniformly convex function on bounded subsets of X and $\tau > 0$ be a constant. Then*

$$g\Big(\sum_{k=0}^{n} \delta_k x_k\Big) \leq \sum_{k=0}^{n} \delta_k g(x_k) - \delta_i \alpha_j \rho_\tau(\|x_i - x_j\|),$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r, \delta_k \in (0,1)$ and $k \in \mathbb{N} \cup 0$ with $\sum_{k=0}^{n} \delta_k = 1$, where ρ_s is the gauge of uniform convexity of g.

Lemma 2.9. [42] *Let $\{u_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\{v_n\}$ be a sequence of real numbers. Assume that*

$$u_{n+1} \leq (1 - \alpha_n)u_n + \alpha_n v_n \ \forall \ n \geq 1.$$

If $\limsup\limits_{k\to\infty} v_{n_k} \leq 0$ for every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying the condition

$$\liminf_{k\to\infty} (u_{n_k+1} - u_{n_k}) \geq 0,$$

then $\lim\limits_{n\to\infty} u_n = 0$.

3. Main Result

In this section, we introduce a parallel iterative method for approximating finite families of MIP and fixed point problem of Bregman strongly nonexpansive mapping which is based on Tseng's method. Below are some important assumptions:

Assumption 3.1.

(B1) *Suppose E is a real Banach space, the mapping $A^j : \mathcal{E} \to \mathcal{E}^*$, $j = 1, 2, \dots N$ be a monotone and L_j-Lipschitz continuous.*

(B2) *For $j = 1, 2, \cdots, N$, let $B^j : \mathcal{E} \to 2^{\mathcal{E}^*}$ be a maximal monotone mapping and $T^j : \mathcal{E} \to \mathcal{E}$ be a finite family of Bregman strongly nonexpansive mapping.*

(B3) *Suppose $g : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ is a function that is Legendre, uniformly Fréchet differentiable, ρ-strongly convex, and bounded on bounded subsets of $\mathcal{E}$.*

(B4) *The solution set $\Delta := \bigcap_{j=1}^{\infty} (F(T^j) \bigcap (A^j + B^j)^{-1}(0))$ is nonempty.*

Assumption 3.2.

(D1) *$\alpha^k \in (0, 1)$ such that $\lim_{k\to\infty} \alpha^k = 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$.*

(D2) *$\circ(\alpha^k) = \frac{1}{k^2}$, i.e. $\lim_{k\to\infty} \frac{1}{k^2\alpha^k} = 0$,*

(D3) *$\beta^{k,j} \in (0, 1)$ such that $\sum_{j=0}^{N} \beta^{k,j} = 1$ and $\liminf_{k\to\infty} \beta^{k,0}\beta^{k,j} > 0$, $j = 1, 2, \cdots, N$.*

(D4) *$\{\eta^k\}$ is a nonnegative real numbers sequence such that $\sum_{k=1}^{\infty} \eta^k < \infty$.*

Algorithm 2 A parallel inertial self adaptive method for solving MIP and fixed point problems.

Initialization: Choose $\lambda^1 > 0$, $\mu, \theta \in (0, \sigma)$, such that σ is a constant defined in (2.3) and $\theta \in (0, 1)$. Let $x^0, x^1 \in E$ be arbitrary starting points.

Iterative step: Given x^{k+1} and λ^{k+1} for each $k \geq 1$ as follows:

Step 1: Given x^{k-1}, x^k and λ^k, choose $\theta^k \in [0, \overline{\theta_k}]$ where

$$\bar{\theta^k} = \begin{cases} \min\left\{\dfrac{1}{k^2\|x^k - x^{k-1}\|}, \theta\right\}, & \text{if } x^k \neq x^{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Calculate

$$\begin{cases} z^k = \nabla g^*(\nabla g(x^k) + \theta^k(\nabla g(x^{k-1}) - \nabla g(x^k))) \\ w^{k,j} = J_{\lambda^k}^{B^j} \nabla g^*(\nabla g(z^k) - \lambda^k A^j(z^k)) \end{cases} \tag{3.2}$$

Step 3: Calculate

$$y^{k,j} = \nabla g^*(\nabla g(w^{k,j}) - \lambda^k(A^j(w^{k,j}) - A^j(z^k)), \tag{3.3}$$

and λ^{k+1} is updated as stated below:

$$\lambda^{k+1} = \begin{cases} \min\left\{\min_{1\leq j\leq N}\left\{\dfrac{\mu\|z^k - w^{k,j}\|}{\|A^j z^k - A^j w^{k,j}\|}\right\}, \lambda^k + \eta^k\right\} & \text{if } A^j z^k \neq A^j w^{k,j}, \\ \lambda^k + \eta^k, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 4: Calculate

$$u^k = \nabla g^*(\beta^{k,0}\nabla g(z^k) + \sum_{j=1}^{N} \beta^{k,j}\nabla g(T^j y^{k,j})) \tag{3.5}$$

Step 5: Calculate x^{k+1} by

$$x^{k+1} = \nabla g^*(\alpha^k \nabla g(x^0) + (1-\alpha^k)\nabla g(u^k)) \tag{3.6}$$

Stopping criterion: If $x^{k+1} = z^k$ for $k \geq 1$ then stop. Otherwise set $k := k+1$ and return to **Iterative step** .

Remark 3.3. *Our step size is allowed to increase from iteration to iteration and hence the dependence on the initial step size λ^0 is being reduced by our method. From (D4) of Algorithm 2, it can be seen that $\{\eta^k\}$ is summable, then we get that $\lim_{k\to\infty} \eta^k = 0$. So the stepsize λ^k may be non-increasing if k is large. If $\eta^k \equiv 0$, then the stepsize in Algorithm 2 reduces to the ones in [25, 34, 36].*

Remark 3.4. *Note that the property of $A^j, j = 1, 2, \cdots, N$ in (B1) is weaker than the inverse strongly monotone imposed on the operators in [23, 33].*

Lemma 3.5. *Let $\{\lambda^k\}$ be defined as in (3.4). Thus we obtain $\lim_{k\to\infty} \lambda^k = \lambda$ and $\lambda \in \min\big[\min\{\frac{\mu}{L_j}, \lambda^0\}, \lambda^0 + \eta\big]$, where $\eta = \sum_{k=0}^{\infty} \eta^k$.*

Proof. It can be seen from Assumption 3.1 that A^j is Lipschitz continuous with constant $L_j > 0$. Now, when $A^j(z^k) - A^j(w^{k,j}) \neq 0$, we have

$$\frac{\mu\|z^k - w^{k,j}\|}{\|A^j z^k - A^j w^{k,j}\|} \geq \frac{\mu\|z^k - w^{k,j}\|}{L_j\|z^k - w^{k,j}\|} = \frac{\mu}{L_j}, \ j = 1, 2, \cdots, N.$$

Thus, by the definition of λ^{k+1} in (3.4) and applying mathematical induction, then the sequence $\{\lambda^k\}$ has a lower bound of $\min\big\{\min_{1\leq j\leq N}\{\frac{\mu}{L_j}\}, \lambda^0 + \eta^0\big\}$. The remaining part of the proof follows from Lemma 3.1 in [27], so we exclude it.

Lemma 3.6. *Let $\{x_n\}$ be generated iteratively by Algorithm 2 and $a \in \Delta$, then the following inequality holds:*

$$D_g(a, y^{k,j}) \leq D_g(a, z^k) - \big(1 - \frac{\mu\lambda^k}{\rho\lambda^{k+1}}\big)\Big[D_g(w^{k,j}, z^k) + D_g(y^{k,j}, w^{k,j})\Big].$$

Proof. Let $a \in \Delta$, then using (2.1), we have

$$\begin{aligned} D_g(a, y^{k,j}) &= D_g(a, \nabla g^*(\nabla g(w^{k,j}) - \lambda^k(A^j(w^{k,j}) - A^j(z^k))) \\ &= g(a) - g(y^{k,j}) - \langle a - y^{k,j}, \nabla g(w^{k,j}) - \lambda^k(A^j(w^{k,j}) - A^j(z^k))\rangle \end{aligned}$$

$$
\begin{aligned}
&= g(a) - g(y^{k,j}) - \langle a - y^{k,j}, \nabla g(w^{k,j})\rangle \\
&+ \lambda^k \langle a - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&= g(a) - g(w^{k,j}) - \langle a - w^{k,j}, \nabla g(w^{k,j})\rangle + \langle a - y^{k,j}, \nabla g(w^{k,j})\rangle \\
&+ g(w^{k,j}) - g(y^{k,j}) - \langle a - y^{k,j}, \nabla g(w^{k,j})\rangle \\
&+ \lambda^k \langle a - y^{k,j}, A^j(w^{k,j}) - A^(z^k)\rangle \\
&= g(a) - g(w^{k,j}) - \langle a - w^{k,j}, \nabla g(w^{k,j})\rangle - g(y^{k,j}) + g(w^{k,j}) \\
&+ \langle y^{k,j} - w^{k,j}, \nabla g(w^{k,j})\rangle + \lambda^k \langle a - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&= D_g(a, w^{k,j}) - D_g(y^{k,j}, w^{k,j}) + \lambda^k \langle a - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle.
\end{aligned} \tag{3.7}
$$

Using (2.2), we obtain

$$
\begin{aligned}
D_g(a, w^{k,j}) &= D_g(a, z^k) - D_g(w^{k,j}, z^k) \\
&+ \langle a - w^{k,j}, \nabla g(z^k) - \nabla g(w^{k,j})\rangle.
\end{aligned} \tag{3.8}
$$

By combining (3.7) and (3.8), we have

$$
\begin{aligned}
D_g(a, y^{k,j}) &= D_g(a, z^k) - D_g(w^{k,j}, z^k) + \langle a - w^{k,j}, \nabla g(z^k) - \nabla g(w^{k,j})\rangle \\
&- D_g(y^{k,j}, w^{k,j}) + \lambda^k \langle a - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&= D_g(a, z^k) - D_g(w^{k,j}, z^k) + \langle a - w^{k,j}, \nabla g(z^k) - \nabla g(w^{k,j})\rangle \\
&- D_g(y^{k,j}, w^{k,j}) + \lambda^k \langle w^{k,j} - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&- \lambda^k \langle w^{k,j} - a, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&= D_g(a, z^k) - D_g(w^{k,j}, z^k) - D_g(y^{k,j}, w^{k,j}) \\
&+ \lambda^k \langle w^{k,j} - y^{k,j}, A^j(w^{k,j}) - A^j(z^k)\rangle \\
&- \langle w^{k,j} - a, \nabla g(z^k) - \nabla g(w^{k,j}) - \lambda^k (A^j(z^k) - A^j(w^{k,j})).
\end{aligned} \tag{3.9}
$$

By applying step 2 of Algorithm 2, it can be seen that $\nabla g(z^k) - \lambda^k A^j(z^k) \in \nabla g(w^{k,j}) + \lambda^k B^j(w^{k,j})$. From Assumption $(B2)$, we have that $B^j, j = 1, 2, \cdots, N$ is maximal monotone, then there exists $d^j \in B^j(w^{k,j})$ such that $\nabla g(z^k) - \lambda^k A^j(z^k) = \nabla g(w^{k,j}) + \lambda^k d^j$. Hence, it follows that

$$
d^j = \frac{1}{\lambda^k}(\nabla g(z^k) - \nabla g(w^{k,j}) - \lambda^k A^j(z^k)). \tag{3.10}
$$

Since $0 \in (A^j + B^j)a + A^j(w^{k,j}) + d^j \in (A^j + B^{k,j})w^{k,j}$, we obtain from Lemma 2.7 that $A + B$ is maximal monotone. Hence

$$
\langle w^{k,j} - a, A^j(w^{k,j}) + d^j\rangle \geq 0. \tag{3.11}
$$

On substituting (3.10) into (3.11), we obtain

$$
\frac{1}{\lambda^k}\langle w^{k,j} - a, \nabla g(z^k) - \nabla g(w^{k,j}) - \lambda^k A^j(z^k) + \lambda^k A^j(w^{k,j})\rangle \geq 0. \tag{3.12}
$$

That is

$$
\langle w^{k,j} - a, \nabla g(z^k) - \nabla g(w^{k,j}) - \lambda^k (A^j(z^k) - A^j(w^{k,j}))\rangle \geq 0. \tag{3.13}
$$

It is obvious from (3.9) and (3.13) that

$$\begin{aligned} D_g(a, y^{k,j}) \leq D_g(a, z^k) &- D_g(w^{k,j}, z^k) - D_g(y^{k,j}, w^{k,j}) \\ &+ \lambda^k \langle w^{k,j} - y^{k,j}, A^j(w^{k,j}) - A^j(z^k) \rangle, \end{aligned} \tag{3.14}$$

but

$$\begin{aligned} \lambda^k \langle w^{k,j} - y^{k,j}, A^j(w^{k,j}) - A^j(z^k) \rangle &\leq \lambda^k \|w^{k,j} - y^{k,j}\| \, \|A^j(w^{k,j}) - A^j(z^k)\| \\ &\mu \frac{\lambda^k}{\lambda^{k+1}} \|w^{k,j} - y^{k,j}\| \, \|w^{k,j} - z^k\| \\ &\leq \frac{\mu}{2} \frac{\lambda^j}{\lambda^{j+1}} \big(\|w^{k,j} - y^{k,j}\|^2 + \|w^{k,j} - z^k\|^2 \big). \end{aligned} \tag{3.15}$$

On substituting (3.15) into (3.14) and applying (2.3), we have

$$\begin{aligned} D_g(a, y^{k,j}) \leq D_g(a, z^k) &- D_g(w^{k,j}, w^{k,j}) - D_g(y^{k,j}, w^{k,j}) \\ &+ \frac{\mu}{\rho} \frac{\lambda^k}{\lambda^{k+1}} \Big[D_g(w^{k,j}, y^{k,j}) + D_g(w^{k,j}, z^k) \Big] \\ &= D_g(a, z^k) - \left(1 - \frac{\mu}{\rho} \frac{\lambda^k}{\lambda^{k+1}} \right) D_g(w^{k,j}, z^k) \\ &- \left(1 - \frac{\mu}{\rho} \frac{\lambda^k}{\lambda^{k+1}} \right) D_g(w^{k,j}, y^{k,j}). \end{aligned} \tag{3.16}$$

Since $\lim_{k \to \infty} \lambda^k$ exists and $\mu \in (0, \rho)$, we have that $\lim_{k \to \infty} (1 - \frac{\mu}{\rho} \frac{\lambda^k}{\lambda^{k+1}}) = 1 - \frac{\mu}{\rho} > 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that $1 - \frac{\mu}{\rho} \frac{\lambda^k}{\lambda^{k+1}} > 0, \ \forall \ n \geq n_0$.
Hence,

$$D_g(a, y^{k,j}) \leq D_g(a, z^k). \tag{3.17}$$

Hence, the proof completes.

Lemma 3.7. *Let $\{x^k\}$ be a sequence generated by Algorithm 2, then $\{x^k\}$, $\{w^{k,j}\}$, $\{y^{k,j}\}$ and $\{u^k\}$ are bounded.*

Proof. Let $a \in \Delta$, then from step 1 of Algorithm 2, we get

$$\begin{aligned} D_g(a, z^k) &= D_g(a, \nabla g^*(\nabla g(x^k) + \theta^k (\nabla g(x^{k-1}) - \nabla g(x^k)))) \\ &\leq (1 - \theta^k) D_g(a, x^k) + \theta^k D_g(a, x^{k-1}). \end{aligned} \tag{3.18}$$

Also, from Algorithm 2, (3.17) and Lemma 2.8, we have

$$
\begin{aligned}
D_g(a,u^k) &= D_g(a,\nabla g^*(\beta^{k,0}\nabla g(z^k)+\sum_{j=1}^{N}\beta^{k,j}\nabla g(T^jy^{k,j})))\\
&= V_g(a,\beta^{k,0}\nabla g(z^k)+\sum_{j=1}^{N}\beta^{k,j}\nabla g(T^jy^{k,j}))\\
&= g(a)-\langle a,\beta^{k,0}\nabla g(z^k)+\sum_{j=1}^{N}\beta^{k,j}\nabla g(T^jy^{k,j})\rangle\\
&+ g^*(\beta^{k,0}\nabla g(z^k)+\sum_{j=1}^{N}\beta^{k,j}\nabla g(T^jy^{k,j}))\\
\\
&= g(a)-\beta^{k,0}\langle a,\nabla g(z^k)\rangle+\sum_{j=1}^{N}\beta^{k,j}\langle a,\nabla g(T^jy^{k,j})\\
&+\beta^{k.0}(\nabla g(z^k))+\sum_{j=1}^{N}\beta^{k,j}g^*(\nabla g(T^jy^{k,j}))\\
&-\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|)\\
&= \beta^{k,0}V_g(a,z^k)+\sum_{j=1}^{N}\beta^{k,j}V_g(a,T^jy^{k,j})\\
&-\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|)\\
&= \beta^{k,0}D_g(a,z^k)+\sum_{j=1}^{N}\beta^{k,j}D_g(a,T^jy^{k,j})-\\
&\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|)\\
&\le D_g(a,z^k)\beta^{k,0}V_g(a,z^k)+\sum_{j=1}^{N}\beta^{k,j}V_g(a,T^jy^{k,j})\\
&-\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|) \qquad (3.19)\\
&\le D_g(a,z^k). \qquad (3.20)
\end{aligned}
$$

Thus, we obtain from (3.18) and (3.20) that

$$D_g(a,u^k)\le(1-\theta^k)D_g(a,x^k)+\theta^kD_g(a,x^{k-1}). \qquad (3.21)$$

Finally, we obtain from step 5 of Algorithm 2, (3.17) and (3.21) that

$$\begin{aligned}
D_g(a, x^{k+1}) &= D_g(a, \nabla g^*(\alpha^k \nabla g(x^0) + (1-\alpha^k)\nabla g(u^k))) \\
&\leq \alpha^k D_g(a, x^0) + (1-\alpha^k) D_g(a, u^k) \\
&= \alpha^k D_g(a, x^0) + (1-\alpha^k)\Big[(1-\theta^k) D_g(a, x^k) + \theta^k D_g(a, x^{k-1})\Big] \\
&\leq \max\Big\{D_g(a, x^0),\ \max\{D_g(a, x^k), D_g(a, x^{k-1})\}\Big\} \\
&\ \ \vdots \\
&\leq \max\{D_g(a, x^0),\ D_g(a, x^1)\}.
\end{aligned}$$

Hence, $\{D_g(a, x^k)\}$. By Lemma 2.5, $\{x^k\}$ is bounded. Consequently, other sequences defined in Algorithm 2 are bounded.

Theorem 3.8. *Let Assumptions $(B1)-(B4)$ and $(D1)-(D3)$ holds. Then the sequence $\{x^k\}$ generated iteratively by Algorithm 2 strongly converges to an element $a \in \Delta$, where $a = Proj_\Delta(x^0)$.*

Proof. Let $a \in \Delta$, then using (3.16), (3.19) and (3.21)

$$\begin{aligned}
D_g(a, x^{k+1}) &= D_g(a, \nabla g^*(\alpha^k)\nabla g(x^0) + (1-\alpha^k)\nabla g(u^k)) \\
&\leq \alpha^k D_g(a, a) + (1-\alpha^k) D_g(a, u^k) + \alpha^k \langle \nabla g(x^0) - \nabla g(a), x^{k+1} - a\rangle \\
&\leq (1-\alpha^k)\big[\beta^{k,0} D_g(a, z^k) + \sum_{j=1}^{N} \beta^{k,j} D_g(a, T^j y^{k,j}) \\
&- \sum_{j=1}^{N} \beta^{k,0}\beta^{k,j} \nu_s^*\big(\|\nabla g(z^k) - \nabla g(T^j y^{k,j})\|\big)\big] \\
&+ \alpha^k \langle \nabla g(x^0) - \nabla g(a), x^{k+1} - a\rangle \\
&\leq (1-\alpha^k)\big[\beta^{k,0} D_g(a, z^k) + \sum_{j=1}^{N} \beta^{k,j} + \sum_{j=1}^{N} \beta^{k,j} D_g(a, y^{k,j}) \\
&- \sum_{j=1}^{N} \beta^{k,0}\beta^{k,j} \nu_s^*\big(\|\nabla g(z^k) - \nabla g(T^j y^{k,j})\|\big)\big] \\
&+ \alpha^k \langle \nabla g(x^0) - \nabla g(a), x^{k+1} - a\rangle \\
&\leq (1-\alpha^k)\Bigg[\beta^{k,0} D_g(a, z^k) + \sum_{j=1}^{N} \beta^{k,j} D_g(a, z^k) \\
&- \sum_{j=1}^{N} \beta^{k,j}\left(1 - \frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right) D_g(w^{k,j}, z^k) \\
&- \sum_{j=1}^{N} \beta^{k,j}\left(1 - \frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right) D_g(w^{k,j}, y^{k,j})
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*\big(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|\big)\Bigg]\\
&+\alpha^k\langle\nabla g(x^0)-\nabla g(a),x^{k+1}-a\rangle\\
&=(1-\alpha^k)D_g(a,z^k)-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,j}\left(1-\frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right)D_g(w^{k,j},z^k)\\
&-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,j}\left(1-\frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right)D_g(w^{k,j},y^{k,j})\\
&-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*\big(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|\big)\\
&+\alpha^k\langle\nabla g(x^0)-\nabla g(a),x^{k+1}-a\rangle\\
&\le(1-\alpha^k)D_g(a,x^k)+\alpha^k\Bigg[\frac{\theta^k}{\alpha^k}\big(D_g(a,x^{k-1})-D_g(a,x^k)\big)\\
&+\langle\nabla g(x^0)-\nabla g(a),x^{k+1}-a\rangle\Bigg]\\
&-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,j}\left(1-\frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right)D_g(w^{k,j},z^k)\\
&-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,j}\left(1-\frac{\mu}{\rho}\frac{\lambda^k}{\lambda^{k+1}}\right)D_g(w^{k,j},y^{k,j})\\
&-(1-\alpha^k)\sum_{j=1}^{N}\beta^{k,0}\beta^{k,j}\nu_s^*\big(\|\nabla g(z^k)-\nabla g(T^jy^{k,j})\|\big) \qquad (3.22)\\
&\le(1-\alpha^k)D_g(a,x^k)+\alpha^kH^k, \qquad (3.23)
\end{aligned}
$$

where $H^k := \left[\frac{\theta^k}{\alpha^k}\big(D_g(a,x^{k-1})-D_g(a,x^k)\big)+\langle\nabla g(x^0)-\nabla g(a),x^{k+1}-a\rangle\right]$. We prove that $\{x^k\}$ converges strongly to $a\in\Delta$. To establish this, it follows from (3.23) that

$$
b^{k+1}\le(1-\delta^k)b^k+\delta^kH^k,
$$

where $b^k := D_g(a,x^k)$. By applying Lemma 2.9, we only need to show that $\limsup_{l\to\infty}H^{k_l}\le 0$, where the subsequence $\{b^{k_l}\}$ of $\{b^k\}$ satisfies

$$
\liminf_{l\to\infty}(b^{k_l+1}-b^{k_l})\ge 0. \qquad (3.24)
$$

Let $\{b^{k_l}\}$ be a subsequence of $\{b^k\}$ satisfies (3.24). Hence, from (3.22), $(D1)$ and $(D3)$, we obtain that

$$\begin{aligned}
&\limsup_{l\to\infty}(1-\alpha^{k_l})\sum_{j=1}^{N}\beta^{k_l,j}\big(1-\frac{\mu\lambda^{k_l,j}}{\rho\lambda^{k_{l+1},j}}\big)\big[D_g(w^{k_l,j},z^{k_l})+D_g(w^{k_l,j},y^{k_l,j})\big]\\
&\le \limsup_{l\to\infty}\big[(1-\alpha^{k_l})b^{k_l,j}-b^{k_{l+1},j}+\alpha^{k_l}N_1\big]\\
&\le \limsup_{l\to\infty}(b^{k_l}-b^{k_{l+1}})+N_1\limsup_{l\to\infty}\delta^{k_l}\\
&=-\liminf_{l\to\infty}(b^{k_{l+1}}-b^{k_l})\\
&\le 0,
\end{aligned}\tag{3.25}$$

where $N_1=\sup\{H^k:k\in\mathbb{N}\}$. Hence, we obtain

$$\lim_{l\to\infty}D_g(w^{k_l,j},z^{k_l})=0=\lim_{l\to\infty}D_g(w^{k_l,j},y^{k_l,j}).\tag{3.26}$$

Thus, from Lemma 2.4, we obtain

$$\lim_{l\to\infty}\|w^{k_l,j}-z^{k_l}\|=0=\lim_{l\to\infty}\|w^{k_l,j}-y^{k_l,j}\|.\tag{3.27}$$

Following the same argument as in (3.25) and applying (3.22), we get

$$\lim_{l\to\infty}\sum_{j=1}^{N}\beta^{k_l,0}\beta^{k_l,j}\nu_s^*\big(\|\nabla g(z^{k_l})-\nabla g(T^jy^{k_l,j})\|\big)=0, j=1,2,\cdots,N.\tag{3.28}$$

From Lemma 2.8 and (D3), we get

$$\lim_{l\to\infty}\big(\|\nabla g(z^{k_l})-\nabla g(T^jy^{k_l,j})\|\big)=0,\ j=1,2,\cdots,N.\tag{3.29}$$

Since ∇g is uniformly continuous on bounded subset of $\mathcal{E}^*$, we have

$$\lim_{l\to\infty}\|z^{k_l}-T^jy^{kl,j}\|=0,\ j=1,2,\cdots,N.\tag{3.30}$$

It is clear from (3.27) and (3.30) that

$$\begin{cases}\lim\limits_{l\to\infty}\|z^{k_l}-y^{k_l,j}\|=0, j=1,2,\cdots,N\\ \lim\limits_{l\to\infty}\|y^{k_l,j}-T^jy^{k_l,j}\|=0, j=1,2,\cdots,N.\end{cases}\tag{3.31}$$

From step 5 of Algorithm 2 and $(D1)$, we have

$$D_g(u^{k_l},x^{k_l+1})\le\alpha^kD_g(u^{k_l},x^0)\to 0,\ l\to\infty,\tag{3.32}$$

which implies from Lemma 2.4 that

$$\lim_{l\to\infty}\|u^{k_l}-x^{k_l+1}\|=0.\tag{3.33}$$

Also, from step 4 of Algorithm 2 and (3.29), we get

$$\begin{aligned}
\|\nabla g(u^{k_l})-\nabla g(z^{k_l})\|&=\beta^{k_l,0}\|\nabla g(z^{k_l})-\nabla g(z^{k_l})\|\\
&\quad+\sum_{j=1}^{N}\beta^{k_l,j}\|\nabla g(T^jy^{k_l,j})-\nabla g(z^{k_l})\|\to 0,\ l\to\infty.
\end{aligned}\tag{3.34}$$

Thus, since ∇g is uniformly continuous on bounded subset of $\mathcal{E}^*$, we get

$$\lim_{l\to\infty} \|u^{k_l} - z^{k_l}\| = 0. \tag{3.35}$$

Observe from step 2 of Algorithm 2 that

$$\begin{aligned}\|\nabla g(z^{k_l}) - \nabla g(x^{k_l})\| &= \alpha^{k_l} \cdot \frac{\theta^{k_l}}{\alpha^{k_l}} \|\nabla g(x^{k_l-1}) - \nabla g(x^{k_l})\| \\ &= \alpha^{k_l} \cdot \frac{\theta^{k_l}}{\alpha^{k_l}} \|x^{k_l-1} - x^{k_l}\| \to 0,\ l \to \infty.\end{aligned} \tag{3.36}$$

Since ∇g is uniformly continuous on bounded subset of $\mathcal{E}^*$, we get

$$\lim_{l\to\infty} \|z^{k_l} - x^{k_l}\| = 0. \tag{3.37}$$

Now, from (3.27), (3.33), (3.35) and (3.37), we get

$$\begin{cases}\lim\limits_{l\to\infty} \|u^{k_l} - x^{k_l}\| = 0,\\ \lim\limits_{l\to\infty} \|w^{k_l,j} - x^{k_l}\| = 0,\\ \lim\limits_{l\to\infty} \|x^{k_l} - x^{k_l+1}\| = 0.\end{cases} \tag{3.38}$$

By the boundedness of $\{x^{k_l}\}$, its subsequence $\{x^{k_{l_m}}\} \rightharpoonup x^* \in \mathcal{E}$. Using (3.38), it is easy to see that $u^{k_{l_m}}$ which is a subsequence of $\{u^{k_l}\} \rightharpoonup x^*$. Using (3.31), it is obvious that $x^* \in \hat{F}(T^j) = F(T^j) = \bigcap\limits_{j=1}^{N} F(T^j)$. We now establish that $x^* \in (A^j + B^j)^{-1}(0), j = 1, 2, \cdots, N$. Suppose $(g, h) \in Gra(A^j + B^j)$, then $h - A^j g \in B^j g$. Using the definition of $w^{k_l,j}$, we observe that

$$\nabla g(z^{k_{l_m},j}) - \lambda^{k_{l_m}} A^j z^{k_{l_m},j} \in \nabla g(w^{k_{l_m},j}) + \lambda^{k_{l_m}} B^j w^{k_{l_m},j},\ j = 1, 2, \cdots, N,$$

which implies

$$\frac{1}{\lambda^{k_{l_m}}}(\nabla g(z^{k_{l_m},j}) - \nabla g(w^{k_{l_m},j}) - \lambda^{k_{l_m}} A^j z^{k_{l_m}}) \in B^j w^{k_{l_m},j}.$$

By applying the condition $B^j, j = 1, 2, \cdots, N$, in Assumption 3.2, we get

$$\langle g - w^{k_{l_m},j}, h - A^j g + \frac{1}{\lambda^{k_{l_m}}}(\nabla g(z^{k_{l_m},j}) - \nabla g(w^{k_{l_m},j}) - \lambda^{k_{l_m}} A^j z^{k_{l_m}})\rangle \geq 0.$$

By applying the monotonicity of $A^j, j = 1, 2, \cdots, N$ yields

$$\begin{aligned}\langle g - w^{k_{l_m},j}, h\rangle &\geq \langle g - w^{k_{l_m},j}, A^j g + \frac{1}{\lambda^{k_{l_m}}}(\nabla g(z^{k_{l_m},j})\\ &- \nabla g(w^{k_{l_m},j}) - \lambda^{k_{l_m}} A^j z^{k_{l_m}})\rangle\\ &= \langle g - w^{k_{l_m},j}, A^j g - A^j z^{k_{l_m}}\rangle\\ &+ \frac{1}{\lambda^{k_{l_m}}}\langle g - w^{k_{l_m},j}, \nabla g(z^{k_{l_m}}) - \nabla g(w^{k_{l_m},j})\rangle\\ &= \langle g - w^{k_{l_m},j}, A^j g - A^j w^{k_{l_m},j}\rangle\\ &+ \langle g - w^{k_{l_m},j}, A^j w^{k_{l_m},j} - A^j z^{k_{l_m}}\rangle\\ &+ \frac{1}{\lambda^{k_{l_m}}}\langle g - w^{k_{l_m},j}, \nabla g(z^{k_{l_m}}) - \nabla g(w^{k_{l_m},j})\rangle\\ &\geq \langle g - w^{k_{l_m},j}, A^j w^{k_{l_m},j} - A^j z^{k_{l_m}}\rangle\\ &+ \frac{1}{\lambda^{k_{l_m}}}\langle g - w^{k_{l_m},j}, \nabla g(z^{k_{l_m}}) - \nabla g(w^{k_{l_m},j})\rangle. \end{aligned} \tag{3.39}$$

Since $A^j, j = 1, 2, \cdots, N$ is Lipschitz continuous and $w^{k_{l_m},j} \rightharpoonup x^*$. It follows from (3.27) that

$$\langle g - x^*, h\rangle \geq 0.$$

By the monotonicity of $A^j + B^j, j = 1, 2, \cdots, N$, we get that $0 \in (A^j + B^j)x^*$, that is $x^* \in (A^j+B^j)^{-1}0$. Therefore, we conclude that $x^* \in \bigcap_{j=1}^{N}(F(T^j)\bigcap(A^j+B^j)^{-1}(0))$. Lastly, we show that $\limsup_{l\to\infty} H^{k_l} \leq 0$. Indeed, we only need to show that $\limsup\limits_{l\to\infty}\langle\nabla g(x^0) - \nabla g(a), x^{k_l+1} - a\rangle \leq 0$. Now, since $\{x^{k_l}\}$ is bounded, there exists a subsequence $\{x^{k_{l_m}}\} \rightharpoonup x^*$. Thus,

$$\begin{aligned}\limsup_{k\to\infty}\langle\nabla g(x^0) - \nabla g(a), x^{k+1} - a\rangle &= \lim_{l\to\infty}\langle\nabla g(x^0) - \nabla g(a), x^{k_{l_m}+1} - a\rangle\\ &= \langle\nabla g(x^0) - \nabla g(a), x^* - a\rangle\\ &\leq 0. \end{aligned} \tag{3.40}$$

Hence, $\limsup\limits_{l\to\infty} H^{k_l} \leq 0$. By applying (3.40) and Lemma 2.9 in (3.23), we conclude that $\{x^k\}$ converges to $a \in \Delta$.

We briefly state some consequences of our result.

(i) Let $N = 1,\ x^{k+1} = I$ and $\eta^k \equiv 0$, then our result reduces to the result of Sunthrayuth *et al.* [36].

(ii) Let $N = 1, x^{k+1} = I, \eta^k \equiv 0$ and $\mathcal{E} = \mathcal{H}$, then our result reduces to the result of Gibali and Thong [25].

4. Numerical Illustrations

In Example 4.1, we give a concrete example to show that our proposed algorithm is implementable. In the subsequent examples, we will set $N = 1$ and $T^j = I$, the identity map to enable us compare the performance of our proposed Algorithm 2 with some existing algorithms in the literature.

Example 4.1. *Suppose $\mathcal{E} = \mathbb{R}$. Suppose $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \frac{x^4}{4}$, then $\nabla g(x) = x^3$ and $g(x^*) = \frac{3}{4}x^{*\frac{4}{3}}$ and $\nabla g^*(x^*) = x^{*\frac{1}{3}}$. Let $A^j : \mathbb{R} \to \mathbb{R}$ and $B^j : \mathbb{R} \to \mathbb{R}$ be defined respectively for all $j = 1, 2, \cdots, N$ by $A^j(x) = 2jx$ and $B^j(x) = \frac{3jx}{j+1}$. Define $T^j : \mathbb{R} \to \mathbb{R}$ be defined by $T^j(x) = \frac{j-1}{2j+3}x$, $\beta^{k,0} = \beta^{k,j} = \frac{1}{4}$, $j = 1, 2, 3$. Choose $\alpha^k = \frac{1}{10^4(k+1)}$, $\eta^k = \frac{1}{(k+1)^4}$. Also, set $\theta = 0.1$ and $\lambda^1 = 0.1$, $\mu = 0.9$. Using the stopping criterion $E^k = \|x^{k+1} - x^k\| < \epsilon$ where $\epsilon = 10^{-6}$ and test our proposed Algorithm 2 for the following initial values of z_0 and z_1, we consider the following cases Case 1: $z_0 = 2$ and $z_1 = 4$; Case 2: $z_0 = -3$ and $z_1 = 1$; Case 3: $z_0 = 0.5$; and $z_1 = 1.5$; Case 4: $z_0 = -0.1$ and $z_1 = -0.5$;. The report of this numerical experiment can be found in Table 1 and, Figures 1.*

TABLE 1. Numerical performance of all algorithms in Example 4.1.

Algorithms	*Case 1*		*Case 2*		*Case 3*		*Case 4*	
	Iter.	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*
Algorithm 2	*95*	*0.0059*	*78*	*0.0035*	*81*	*0.0041*	*65*	*0.0033*

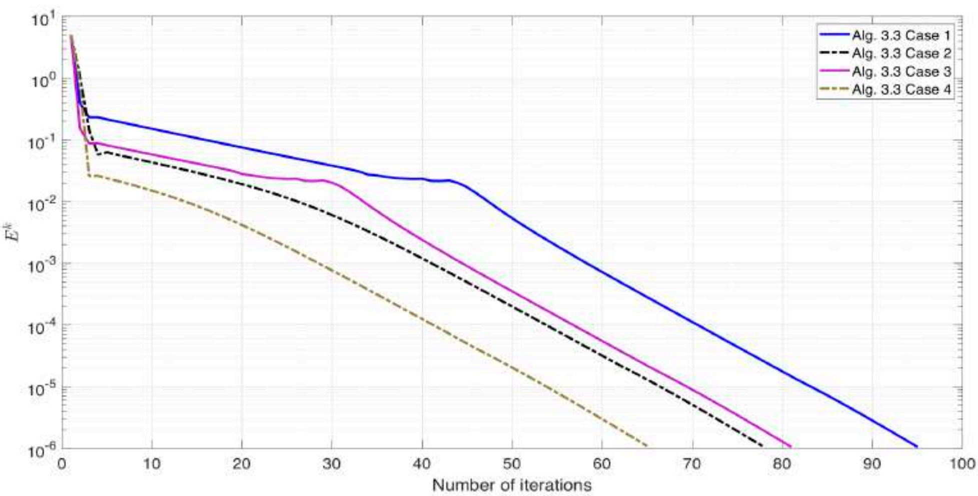

FIGURE 1. Graphical presentation of the error iterates for Cases 1–4 in Example 4.1.

Discussion of Results. *The results obtained here shows that our proposed Algorithm 2 is implementable. Furthermore, we saw that for initial points closer to zero, our proposed algorithm satisfies the stopping criteria faster than when the initial points are far away from zero.*

In the subsequent examples, we will compare the performance of our proposed algorithm with that of Shehu [37], Sunthrayuth *et al.* [36] and Tseng [41], using two examples.

Example 4.2. *Let* $\mathcal{E} = \mathbb{R}^4$. *Let* $N = 1$, $g(x) = \frac{1}{2}\|x\|^2$. *Then,* $\nabla g = \nabla g^* = x$. *Let* $A, B : \mathbb{R}^4 \to \mathbb{R}^4$ *be defined by*

$$Ax = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0.6 & 0.2 \\ 0 & 0 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad Bx = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Let T *be the identity map on* $\mathbb{R}^4$. *We choose the following set of parameters for the implementation of our proposed algorithm:*

- *In Algorithm 3.3, we choose* $\lambda^1 = 0.1$, $\mu = 0.9$, $\theta = 0.1$, $\beta^k = 0.1$, $\eta^k = \frac{1}{(k+1)^4}$ *and* $\alpha^k = \frac{1}{(k+1)}$.
- *In Shehu [37] (S Alg. 3.11), we choose* $\lambda = 0.1$ *and* $\alpha^k = \frac{1}{(k+1)}$.
- *In Sunthrayuth et al. [36] (SPC Alg. 2), we choose* $T = I$*, the identity map,* $\lambda^1 = 0.1$, $\mu = 0.9$ *and* $\alpha^k = \frac{1}{(k+1)}$.
- *In Tseng [41] (Tseng Alg.), we choose* $\lambda = 0.1$.

Using the stopping criterion $E^k = \|x^{k+1} - x^k\| < \epsilon$ *where* $\epsilon = 10^{-6}$ *and compare the performance of our proposed Algorithm 2 with these algorithms using the following initial points: Case 1:* $z_0 = (1, 2, 3, 1)^T$, *Case 2:* $z_0 = (-0.1, -0.2, 0.5, -1)^T$, *Case 3:* $z_0 = (2, -2, -1, 3)^T$ *and Case 4:* $z_0 = (-0.25, -0.125, -1.5, -2)^T$. *Next, we set* $z_1 = z_0$ *and run the codes. The report of this numerical experiment can be found in Table 2 and Figures 2.*

Table 2. Numerical performance of all algorithms in Example 4.1.

Algorithms	*Case 1*		*Case 2*		*Case 3*		*Case 4*	
	Iter.	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*
Algorithm 2	*83*	*0.0180*	*87*	*0.0102*	*92*	*0.0094*	*75*	*0.0117*
S Alg. 3.11	*101*	*0.0127*	*106*	*0.0091*	*113*	*0.0079*	*90*	*0.0076*
SPC Alg. 2	*180*	*0.0172*	*186*	*0.0140*	*194*	*0.0136*	*167*	*0.0143*
Tseng Alg.	*180*	*0.0173*	*186*	*0.0165*	*194*	*0.0124*	*167*	*0.0175*

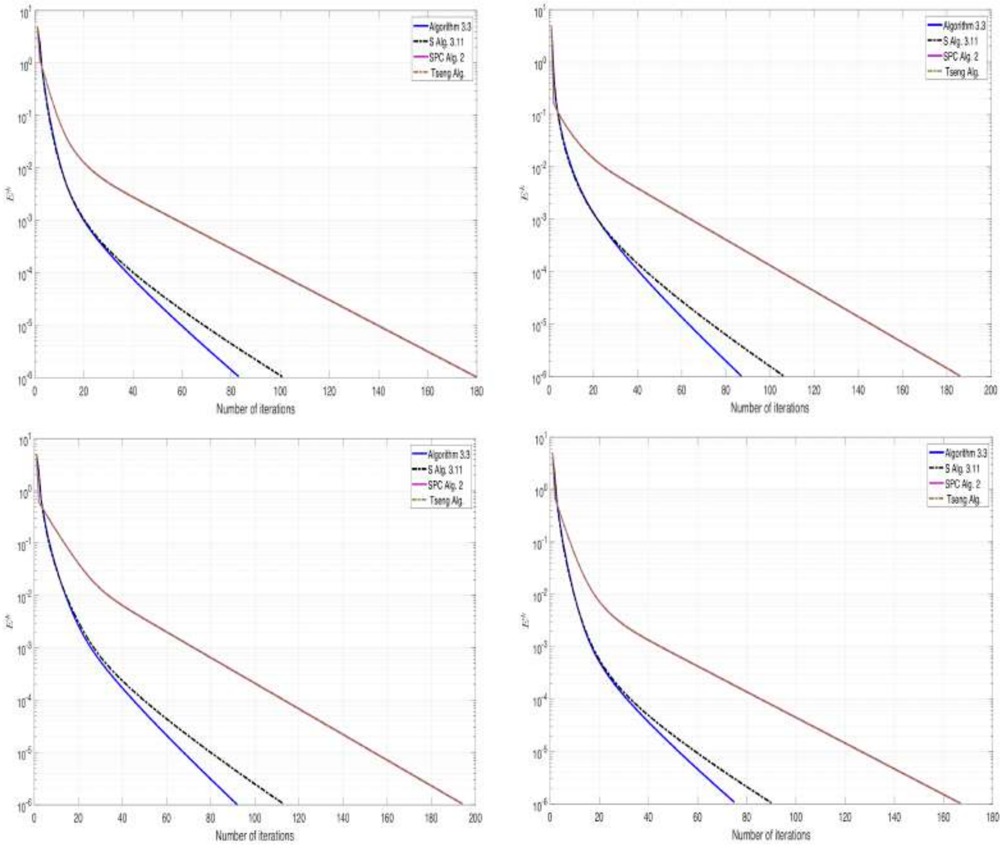

FIGURE 2. Graphical presentation of the error iterates for Cases 1–4 in Example 4.2.

Example 4.3. *Let* $\mathcal{E} = L_3([0,1])$ *with dual space* $E^* = L_{\frac{3}{2}}([0,1])$ *and, for* $p > 1$ *let* $g(x) = \frac{1}{p}\|x\|^p$. *Then,* $\nabla g = \nabla g^* = J_p x := \{x^* \in \mathcal{E} : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|^{p-1}\}$. *For* $x \in \mathcal{E}$ *and* $y \in \mathcal{E}^*$ *the duality pair between* x *and* y *is given by* $\langle x, y \rangle = \int_0^1 x(t)y(t)dt$ *and induced norm* $\|x\|_{L_3} = \left(\int_0^1 |x(t)|^3 dt\right)^{1/3}$, *for all* $x \in L_3([0,1])$. *Let* $A, B : L_3([0,1]) \to L_{\frac{3}{2}}([0,1])$ *be defined by*

$$(Ax)(t) := x(t) - \int_0^1 tx(s)ds \text{ and } (Bx)(t) := 5J_3(x(t)),\ \forall x \in L_3([0,1]).$$

Since for $0 < p < q < \infty$, $L_q \subset L_p$, A *and* B *are well-defined. Clearly,* A *is 2-Lipchitz and* B *is maximal monotone. Let* T *be the identity map. We choose the following set of parameters for the implementation of our proposed algorithm:*

- *In Algorithm 3.3, we choose* $\lambda^1 = 0.1,\ \mu = 0.9,\ \theta = 0.01,\ \beta^k = 0.01,\ \eta^k = \frac{1}{(k+1)^4}$ *and* $\alpha^k = \frac{1}{(k+1)}$.
- *In Shehu [37] (S Alg. 3.11), we choose* $\lambda = 0.1$ *and* $\alpha^k = \frac{1}{(k+1)}$.

- *In Sunthrayuth et al. [36] (SPC Alg. 2), we choose* $T = I$, *the identity map,* $\lambda^1 = 0.1,\ \mu = 0.9$ *and* $\alpha^k = \frac{1}{(k+1)}$.
- *In Tseng [41] (Tseng Alg.), we choose* $\lambda = 0.1$.

Using the stopping criterion $E^k = \|x^{k+1} - x^k\| < \epsilon$ *where* $\epsilon = 10^{-6}$ *and compare the performance of our proposed Algorithm 2 with these algorithms using the following initial points: Case 1:* $z_0(t) = t^2 + 1$, *Case 2:* $z_0(t) = t^3 + 2t - 1$ *Case 3:* $z_0(t) = \exp(5t^2 + 3)$ *and Case 4:* $z_0(t) = \cos(t) + 1$. *Next, we set* $z_1(t) = z_0(t)$ *and run the codes. The report of this numerical experiment can be found in Table 3 and Figures 3 and 4.*

TABLE 3. Numerical performance of all algorithms in Example 4.3.

Algorithms	*Case 1*		*Case 2*		*Case 3*		*Case 4*	
	Iter.	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*	*Iter.*	*Time (s)*
Algorithm 2	*21*	*0.1286*	*23*	*0.8903*	*34*	*1.2159*	*22*	*0.1351*
S Alg. 3.11	*28*	*0.1659*	*33*	*0.9199*	*52*	*1.3067*	*31*	*0.1669*
SPC Alg. 2	*36*	*0.3135*	*42*	*1.0575*	*62*	*1.3813*	*40*	*0.2345*
Tseng Alg.	*36*	*0.2570*	*42*	*1.0390*	*62*	*1.3583*	*40*	*0.2160*

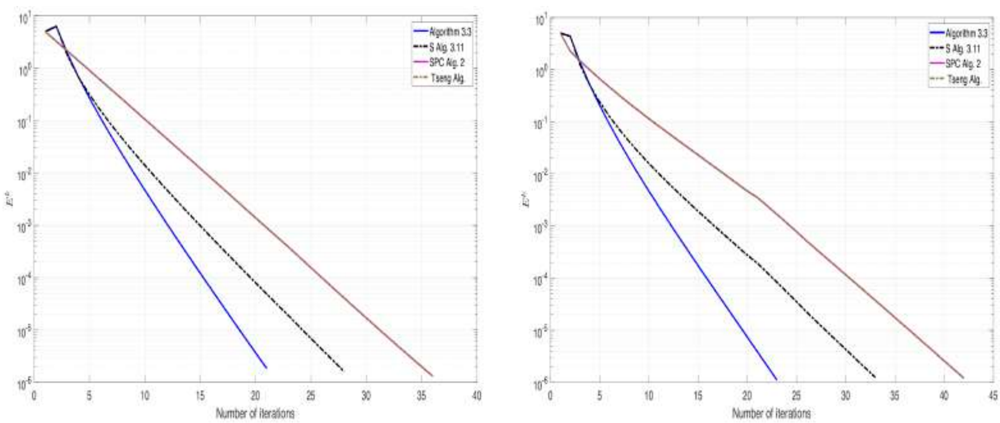

FIGURE 3. Graphical presentation of the error iterates for Cases 1 and 2 in Example 4.3.

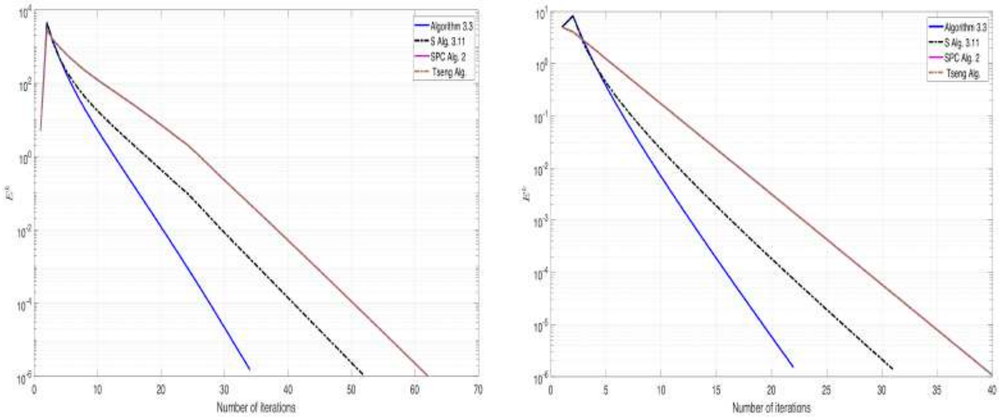

FIGURE 4. Graphical presentation of the error iterates for Cases 3 and 4 in Example 4.3.

Discussion of Results Obtained in Examples 4.2 and 4.3. From the numerical experiments, we observe that SPC Alg. 2 and Tseng Alg. required the same number of iterations to satisfy the stopping criteria. This is expected, since when T is set to the identity map both algorithms coincide; the only distinction lies in the step-size λ, which, as seen in these examples, does not significantly affect performance. Although S Alg. 11 was competitive, our proposed Algorithm 2 outperformed all other methods, requiring fewer iterations and less computational time to meet the stopping criteria.

4.1. Application to image processing

The general image processing problem can be formulated by the inversion of the following observation model

$$y = Dx + b,$$

where x is the unknown image, y is the observed image, b is the added noise and D is a linear operator depending on the concerned image processing problem. It is well known that regularization methods finds great application in solving image processing problems. One of such methods is the ℓ_1 regularization method. The restoration process is given by

$$\min_x \frac{1}{2}\|Dx - b\|^2 + \lambda\|x\|_{\ell_1},$$

where $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_{\ell_1}$ is the regularization term and $\lambda > 0$ is a regularization parameter. To apply our method, we set $N = 1$, $A = \nabla h_1$ and $B = \partial h_2$ where $h_1(\cdot) = \frac{1}{2}\|D(\cdot) - b\|^2$ and $h_2(\cdot) = \lambda_n\|\cdot\|_{\ell_1}$. By these adaptations, we use our method to solve the convex minimization problem:

$$\text{Find } \arg\min_x \left\{\frac{1}{2}\|Dx - b\|^2 + \lambda_n\|x\|_{\ell_1}\right\},$$

which can be cast easily as the problem of finding

$$0 \in \nabla h_1(x) + \partial h_2(x).$$

Now, let T be the identity mapping. Let $g(x) = \frac{1}{2}\|x\|^2$. Then, ∇g and ∇g^* are the identity map on $\mathbb{R}^n$. We choose the same set of parameters used in Example 4.3 for each algorithm. The test images we use for the experiment are images of: Barbra, Duangkamon, Yodjai and Pepper. In MATLAB, these images were degraded using $P = fspecial('motion', 20, 30)$ and we added random noise with a scaling factor of 0.001. Taking the initial points $x_0 = zeros(size(x))$ and $x_1 = Dx + b$, where x is the test image under consideration and setting maximum number of iterations to 200, the results of the image restoration problem is presented in Figures 5 and 6 and Table 4.

(A) original test images

(B) test images degraded by motion blur and random noise

(C) restored images with Algorithm 2

(D) restored images with S Alg. 3.11

FIGURE 5. Test images and their restorations via Algorithm 2 and S Alg. 3.11.

(A) restored images with SPC Alg. 2

(B) restored images with Tseng Alg.

FIGURE 6. Restored images via S SPC Alg. 2 and Tseng Alg. 2.

To evaluate the performance of the algorithms in the restoration process of the test images, we use three (3) performance evaluation metrics: signal to noise ratio (SNR), improvement in signal to noise ratio (ISNR) and structural similarity index measure (SSIM) to measure the quality of the restored images via our algorithm. These metrics are defined as:

$$\text{SNR} := 10 \log \frac{\|x\|^2}{\|x - x_n\|} \quad \text{and} \quad \text{ISNR} := 10 \log \frac{\|x - b\|^2}{\|x - x_n\|},$$

where x, b and x_n are the original, observed and estimated image at iteration n, respectively.

$$\text{SSIM}(x, y) = \frac{(2\mu_x \mu_y + c_1)(2\sigma_{xy} + c_2)}{(\mu_x^2 + \mu_y^2 + c_1)(\sigma_x^2 + \sigma_y^2 + c_2)},$$

where x and y are the original and restored images, μ_x and μ_y are are the mean values of x and y, σ_x and σ_y are the standard deviations of x and y, σ_{xy} is the covariance between x and y c_1 and c_2 are small constants added to avoid division by zero.

Using this performance metrics, for SNR and ISNR higher values indicate better restoration and SSIM value ranges from 0 to 1, and 1 means perfect recovery. The performance of all the algorithms using these metrics are presented in Table 4.

TABLE 4. Performance Metrics (SSIM, ISNR, SNR, and Time).

Algorithm	**Barbra**				**Duangkamon**			
	SSIM	ISNR	SNR	Time (s):	SSIM	ISNR	SNR	Time (s):
Algorithm 2	0.9718	7.64	45.91	4.4812	0.9124	7.34	45.03	5.6907
S Alg. 3.11	0.9617	5.99	42.62	3.6271	0.8897	5.68	41.70	4.5469
SPC Alg. 2	0.9706	7.07	44.77	3.6453	0.9109	7.03	44.41	4.5350
Tseng Alg.	0.9706	7.07	44.77	3.6453	0.9109	7.03	44.41	4.5053
Algorithm	**Yodjai**				**Peppers**			
	SSIM	ISNR	SNR	Time (s):	SSIM	ISNR	SNR	Time (s):
Algorithm 2	0.8910	8.13	49.47	15.4710	0.9763	8.36	49.46	27.7849
S Alg. 3.11	0.8812	6.38	45.98	12.85	0.9680	6.63	46.01	22.7219
SPC Alg. 2	0.8861	7.65	48.52	12.8790	0.9775	8.28	49.30	22.7501
Tseng Alg.	0.8861	7.65	48.52	12.4599	0.9775	8.28	49.30	22.4224

Discussion of Results. From the restored images in Figures 5 and 6, together with the quality metrics reported in Table 4, our proposed algorithm demonstrates promising performance. Although it required the longest computational time, the quality of the restored images was better to that obtained by the other algorithms.

5. Conclusion

In this article, we studied the approximation of solutions for finite families of monotone inclusion problems and fixed points of Bregman strongly nonexpansive mappings within the framework of a reflexive Banach space. We established a strong convergence theorem for the sequences generated by the proposed method. Furthermore, we applied our algorithm to the restoration of test images degraded by motion blur and random noise. Numerical experiments, including comparisons with existing algorithms from the literature, demonstrate that our method achieves superior restoration quality and provides faster convergence in several cases, thereby confirming its effectiveness and practical relevance.

Acknowledgment

The authors acknowledge their individual universities for making their facilities available for the research.

Declaration

Funding
Not applicable.

Availability of data and material
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors worked equally on the results and approved the final manuscript.

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