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A NEW CONVERGENCE ANALYSIS FOR MINIMUM NORM SOLUTION OF SPLIT SYSTEM OF NONSMOOTH MINIMIZATION PROBLEMS

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Abstract This work presents a novel self-adaptive steepest-descent type algorithm for solving the split system of minimization problem (SSMP) related to convex nonsmooth functions. The algorithm includes a self-adaptive step size mechanism, which uses a step size that does not need prior information about the operator norm. Under certain weakened assumptions of parameters, a strong convergence theorem is established and proved for the algorithm. Specifically, the sequence generated by this new algorithm strongly converges towards the minimum norm element of the SSPM. To assess the implementation of our algorithm, a numerical example is provided. According to the numerical results, our algorithm showcases effectiveness and simplicity in its implementation. Furthermore, the primary numerical experiment results indicate that our proposed algorithm surpasses some existing results in the literature in terms of CPU time and iteration count. Our result represents an extension and enhancement of recent findings in this area.

MSC: 9J53; 49J52; 90C25

Keywords: Nonsmooth function; Minimum norm; Proximal point; Self-adaptive step size

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1. Introduction

One of the major problems in optimization is to find $\bar{x} \in H$ such that

$$f(\bar{x}) = \min_{x \in H} f(x), \tag{1.1}$$

where H is a real Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ is a proper and convex function. Many authors have proposed some efficient and implementable algorithms and obtained some convergence theorems for solving (1.1) and its generalizations, see for example, [9, 26] and the reference therein.

Definition 1.1. [3, 14, 19] Let H be a real Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semi-continuous function. Then,

(i) $\arg\min_{x \in H} f = \{\bar{x} \in H : f(\bar{x}) \leq f(x), \forall x \in H\}$,

(ii) Moreau-Yosida approximate of the function f of parameter λ is given by

$$f_\lambda(y) = \min_{u \in H}\{f(u) + \frac{1}{2\lambda}\|y - u\|^2\},$$

(iii) the proximal operator of the function f with scaling parameter λ is a mapping $\text{prox}_{\lambda f} : H \to H$ given by

$$\text{prox}_{\lambda f}(x) = \arg\min_{y \in H}\{f(y) + \frac{1}{2\lambda}\|x - y\|^2\}.$$

For a proper, convex, and lower semicontinuous function f, an effective approach to solve (1.1) involves utilizing the proximal operator method, commonly referred to as the proximal point algorithm (PPA), which was introduced by Martinet [15] in 1970. In 1976, Rockafellar [18, 19] examined the PPA and its convergence to a solution of the convex minimization problem within the context of Hilbert spaces, also see in [17]. The PPA is defined by: $x_1 \in H$ and $x_{n+1} = \text{prox}_{\lambda_n f}(x_n)$ where $\lambda_n > 0$, $n \in \mathbb{N}$.

In this paper, we study the *split system of minimization problem* (SSMP), see Gebrie and Wangkeeree in [6, 8]. To be precise, the SSMP is finding a point $\bar{x} \in H_1$ with the property

$$\bar{x} \in \bigcap_{i=1}^{N}(\arg\min f_i) \text{ such that } A\bar{x} \in \bigcap_{j=1}^{M}(\arg\min g_j), \tag{1.2}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, $f_i : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g_j : H_2 \to \mathbb{R} \cup \{+\infty\}$ are proper, lower semicontinuous convex functions for all $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, M\}$.

Let Ω be the solution set of SSMP (1.2). Note that if $f_i = f$ for all $i \in \{1, \ldots, N\}$ and $g_j = g$ for all $j \in \{1, \ldots, M\}$, then the SSMP (1.2) reduces to the split minimization problem, which is to find a point $\bar{x} \in H_1$ with the property

$$\bar{x} \in \arg\min f \text{ such that } A\bar{x} \in \arg\min g. \tag{1.3}$$

Based on the concept of the step size selection method in the work of López et al. [12], Moudafi and Thakur [16] presented a novel approach to selecting step sizes and put forward weak convergence outcomes for solving the optimization problem given by

$$\min_{x \in H_1}\{f(x) + g_\lambda(Ax)\}, \tag{1.4}$$

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where, $f : H_1 \to \mathbb{R} \cup \{+\infty\}$, $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex, and lower-semicontinuous functions, and g_λ is the Moreau-Yosida approximation [19] of the function g with parameter λ, defined as $g_\lambda(y) = \min_{u\in H_2}\{g(u) + \frac{1}{2\lambda}\|y-u\|^2\}$.

It should be noticed that (1.4) is equivalent to the problem (1.3). Later, after Moudafi and Thakur [16], several iterative methods have been proposed for solving the split minimization problem (1.3), see for example [1, 16, 20–25] and the references therein. Inspired by the results in [1, 16, 20–25], Gebrie and Wangkeeree in [6–8], considered a SSPM and its general case, and they extended the way of selecting step sizes used by Moudafi and Thakur [16] for solving split minimization problem to the framework of SSMP and its generalization problem, so that the implementation of the proposed algorithm does not need any prior information about the operator norm.

In this paper, we use the following settings. For $\lambda>0$ and $x \in H_1$:

(A1) For each $i \in \{1,\dots,N\}$, define

$$l_i(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f_i})x\|^2 \text{ and } \nabla l_i(x) = (I - \mathrm{prox}_{\lambda f_i})x.$$

(A2) $l(x)$ and $\nabla l(x)$ are defined as $l(x) = l_{i_x}(x)$ and so $\nabla l(x) = \nabla l_{i_x}(x)$ where $i_x \in \arg\max\{\|(I - \mathrm{prox}_{\lambda f_i})x\| : i \in \{1,\dots,N\}\}$.

(A3) For each $j \in \{1,\dots,M\}$, define

$$h_j(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g_j})Ax\|^2 \text{ and } \nabla h_j(x) = A^*(I - \mathrm{prox}_{\lambda g_j})Ax.$$

(A4) For each $j \in \{1,\dots,M\}$, define $\theta_j(x) = \|\nabla h_j(x) + \nabla l(x)\|$.

Note that in [6] and [8], the definition of θ_j is given by $\theta_j(x) = \max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\}$ and $\theta_j(x) = \sqrt{\|\nabla h_j(x)\|^2 + \|\nabla l(x)\|^2}$, respectively. However, in this paper, θ_j is defined as in **(A4)**.

In [23], an accelerated hybrid steepest-descent algorithm has been proposed for proximal split feasibility problems. However, a strong convergence result is obtained assuming that $\{(I - \mathrm{prox}_{\lambda g})x_n\}$ is bounded, which is a strong assumption.

Question: Can we adapt and extend the findings in [6, 8, 23] to introduce a new steepest-descent type algorithm for solving the SSMP associated with convex nonsmooth functions in such a way that (1) the algorithm utilizes a self-adaptive step size mechanism, eliminating the need for prior information about the operator norm and (2) we establish and prove a strong convergence theorem under certain weakened assumptions for the proposed algorithm?

This paper aims to present a steepest-descent algorithm with a modified approach and simplified parameter restrictions to address the SSMP. Specifically, drawing inspiration from previous studies, a new proximal-type algorithm is introduced in this paper to determine the minimum norm solution of the SSPM for nonsmooth functions. The iterative algorithm outlined in this paper offers a fresh perspective on solving the aforementioned problem, building upon and enhancing existing findings. In contrast to the methods outlined in [6, 8], the newly proposed approach incorporates a self-adaptive step size mechanism, representing an advancement and broadening of the step sizes discussed in [6, 8]. Additionally, the inclusion of appropriate assumptions enables the generation of a sequence that converges strongly to the minimum norm solution of SSMP.

The structure of this paper is as follows: Section 2 covers essential preliminaries. Section 3 introduces our novel proximal-type algorithm utilizing the the settings l_i, ∇l_i, l, ∇l, h_j, ∇h_j, and θ_j under the conditions **(A1)-(A4)**, along with the proof of its strong

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convergence. Additionally, we provide some insights on the theoretical perspective and structural framework of our proposed algorithm in contrast to some existing findings. Finally, in Section 4, we evaluate the numerical performance of our new algorithm in comparison to the algorithms discussed in [6] and [8].

2. Preliminary

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 2.1. [3] *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \ \ \forall y \in C$.*

Let $T : H \to H$. Then,

(i) T is L-Lipschitz if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \ \forall x, y \in H.$$

If $L \in (0, 1)$, then we call T a contraction. If $L = 1$, then T is called a nonexpansive mapping.

(ii) T is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in H,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \ \ \forall x, y \in H.$$

If T is firmly nonexpansive, $I - T$ is also firmly nonexpansive.

Lemma 2.2. [3] *Let H be a real Hilbert space. Then,*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \ \ \forall x, y \in H.$$

Lemma 2.3. [10] *Let $M \in \mathbb{N}$ and $\{a_j\}_{j=k}^{M} \subset \mathbb{R}$, where k is a fixed nonnegative integer with $k + 1 \leq M$. Then the following holds:*

$$a_k + \sum_{j=k+1}^{M} a_j \prod_{t=k}^{j-1}(1 - a_t) + \prod_{t=k}^{M}(1 - a_t) = 1.$$

Lemma 2.4. [10] *Let w be arbitrary element of a real Hilbert space H. Let $M \in \mathbb{N}$ and k is a fixed nonnegative integer such that $k+1 \leq M$. Let $\{v_j\}_{j=k}^{M} \subset H$ and $\{a_j\}_{j=k}^{M} \subset [0, 1]$. Define*

$$z = a_k w + \sum_{j=k+1}^{M} a_j \prod_{t=k}^{j-1}(1 - a_t)v_{j-1} + \prod_{t=k}^{M}(1 - a_t)v_M.$$

Then, for any $u \in H$, we have

$$\begin{aligned}\|z - u\|^2 \leq{} & a_k\|w - u\|^2 + \sum_{j=k+1}^{M} a_j \prod_{t=k}^{j-1}(1 - a_t)\|v_{j-1} - \bar{x}\|^2 \\ & + \prod_{t=k}^{M}(1 - a_t)\|v_M - u\|^2 \\ & - a_k\Big[\sum_{j=k+1}^{M} a_j \prod_{t=k}^{j-1}(1 - a_t)\|v_{j-1} - w\|^2 + \prod_{t=k}^{M}(1 - a_t)\|v_M - w\|^2\Big].\end{aligned}$$

Lemma 2.5. [13] *Let* $\{d_n\}$ *be the sequence of nonnegative numbers such that*

$$d_{n+1} \le (1-\alpha_n)d_n + \alpha_n \vartheta_n,$$

where $\{\vartheta_n\}$ *is a sequence of real numbers bounded from above and* $0 \le \alpha_n \le 1$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then it holds that*

$$\limsup_{n\to\infty} d_n \le \limsup_{n\to\infty} \vartheta_n.$$

3. Main result

In this section, we introduce our suggested algorithm to solve SSPM for $M \ge 2$ and demonstrate its strong convergence theorem under the assumptions in Assumption 1.

Assumption 1: Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\eta_n^{(j)}\}_{n=1}^{\infty}$ ($j \in \{1,\dots,M\}$) be real sequences satisfying the following conditions:

- **(C1)** $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- **(C2)** $0 < \beta \le \beta_n \le \delta < 1$;
- **(C3)** $0 < \eta_n^{(j)} < 1$ for each $j \in \{1,\dots,M\}$ with
  - **(C3a)** $\liminf_{n\to\infty} \eta_n^{(j)} \prod_{t=1}^{j-1}(1-\eta_n^{(t)}) > 0$ for each $j \in \{1,\dots,M-1\}$,
  - **(C3b)** $\liminf_{n\to\infty} \prod_{t=1}^{M}(1-\eta_n^{(t)}) > 0$.

With the help of l_i, ∇l_i, l, ∇l, h_j, ∇h_j, and θ_j under the conditions **(A1)-(A4)** and the mild parameter assumptions outlined in Assumption 1, we are ready to introduce our new proximal-type algorithm, along with the proof of its strong convergence theorem.

Algorithm 1

Initialization: Choose $x_1 \in H_1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\eta_n^{(j)}\}_{n=1}^{\infty}$ ($j \in \{1,\dots,M\}$) be real sequences satisfying Assumption 1. Compute the following iterative steps for $n = 1,2,3,\dots$.

Step 1: Evaluate $y_n = (1-\alpha_n)x_n$.

Let $\Psi_n = \{j \in \{1,\dots,M\} : h_j(y_n) + l(y_n) \neq 0\}$.

Step 2: For a small $\epsilon^{(j)} > 0$ ($j \in \{1,\dots,M\}$), choose $\mu_n^{(j)}$ such that if $j \in \Psi_n$ and $\theta_j(y_n) \neq 0$

$$\mu_n^{(j)} \in \Big(\epsilon^{(j)}, \frac{h_j(y_n)+l(y_n)}{\theta_j^2(y_n)} - \epsilon^{(j)}\Big),$$

otherwise $\mu_n^{(j)} = \mu^{(j)}$, where $\mu^{(j)}$ is a nonnegative real number.

Step 3: Find $w_n^j = y_n - \mu_n^{(j)}(\nabla h_j(y_n) + \nabla l(y_n))$ for $j \in \{1,\dots,M\}$.

Step 4: Find

$$z_n = \eta_n^{(1)} y_n + \sum_{j=2}^{M} \eta_n^{(j)} \prod_{t=1}^{j-1}(1-\eta_n^{(t)}) w_n^{(j-1)} + \prod_{t=1}^{M}(1-\eta_n^{(t)}) w_n^{(M)}.$$

Step 5: Evaluate $x_{n+1} = (1-\beta_n)y_n + \beta_n z_n$.

Step 6: If $\nabla h_j(y_n) = 0 = \nabla l(y_n)$ for all $j \in \{1,\dots,M\}$ and $x_{n+1} = x_n$, then the iteration process stops, otherwise

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Step 7: Set $n := n+1$ and go to Step 1.

Remark 3.1. Note that Algorithm 1 uses a self-adaptive step size $\mu_n^{(j)}$ which does not need any prior information of the operator norm.

Theorem 3.2. *Assume $\Omega \neq \emptyset$ and the conditions in Assumption 1 are satisfied. The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\hat{x} \in \Omega$ which is also the minimum-norm solution of SSMP, i.e., $\hat{x} = P_\Omega(0)$.*

Proof. Let $\bar{x} \in \Omega$. Since $I - \mathrm{prox}_{\lambda f_i}$ and $I - \mathrm{prox}_{\lambda g_j}$ are firmly nonexpansive, we have for all $x \in H_1$ that

$$\langle \nabla l(x), x - \bar{x} \rangle = \langle (I - \mathrm{prox}_{\lambda f_i})x, x - \bar{x} \rangle \geq \|(I - \mathrm{prox}_{\lambda f_i})x\|^2 = 2l(x), \tag{3.1}$$

and

$$\begin{aligned} \langle \nabla h_j(x), x - \bar{x} \rangle &= \langle A^*(I - \mathrm{prox}_{\lambda g_j})Ax, x - \bar{x} \rangle \\ &= \langle (I - \mathrm{prox}_{\lambda g_j})A(x), A(x) - A(\bar{x}) \rangle \\ &\geq \|(I - \mathrm{prox}_{\lambda g_j})A(x)\|^2 = 2h_j(x). \end{aligned} \tag{3.2}$$

The definition of y_n and Lemma 2.2 yields

$$\begin{aligned} \|y_n - \bar{x}\|^2 &= \|(1-\alpha_n)x_n - \bar{x}\|^2 = \|(1-\alpha_n)(x_n - \bar{x}) - \alpha_n \bar{x}\|^2 \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\|^2 - 2\alpha_n(1-\alpha_n)\langle x_n - \bar{x}, \bar{x} \rangle + \alpha_n^2 \|\bar{x}\|^2. \end{aligned} \tag{3.3}$$

Using the definition of $w_n^{(j)}$, (3.1), (3.2), and Lemma 2.2, we have

$$\begin{aligned} &\|w_n^{(j)} - \bar{x}\|^2 = \|y_n - \mu_n^{(j)}(\nabla h_j(y_n) + \nabla l(y_n)) - \bar{x}\|^2 \\ &= \|y_n - \bar{x}\|^2 - 2\mu_n^{(j)} \langle y_n - \bar{x}, \nabla h_j(y_n) + \nabla l(y_n) \rangle \\ &\qquad + (\mu_n^{(j)})^2 \|\nabla h_j(y_n) + \nabla l(y_n)\|^2 \\ &\leq \|y_n - \bar{x}\|^2 - 4\mu_n^{(j)}(h_j(y_n) + l(y_n)) \\ &\qquad + (\mu_n^{(j)})^2 \|\nabla h_j(y_n) + \nabla l(y_n)\|^2 \qquad (3.4) \\ &= \|y_n - \bar{x}\|^2 - \mu_n^{(j)}\big(4(h_j(y_n) + l(y_n)) - \mu_n^{(j)} \|\nabla h_j(y_n) + \nabla l(y_n)\|^2\big). \qquad (3.5) \end{aligned}$$

From (3.5) and the condition of $\mu_n^{(j)}$, we get

$$\|w_n^{(j)} - \bar{x}\| \leq \|y_n - \bar{x}\|, \quad \forall j \in \{1, \ldots, M\}. \tag{3.6}$$

Using the definition of y_n, (3.6) and Lemma 2.3 and 2.4 (for $k=1$), we have

$$\begin{aligned}
\|z_n-\bar{x}\|^2 &= \|\eta_n^{(1)}y_n+\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})w_n^{(j-1)} \\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})w_n^{(M)}-\bar{x}\|^2 \\
&\le \eta_n^{(1)}\|y_n-\bar{x}\|^2+\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})\|w_n^{(j-1)}-\bar{x}\|^2 \\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})\|w_n^{(M)}-\bar{x}\|^2 \\
&\quad -\eta_n^{(1)}\Big[\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})\|w_n^{(j-1)}-y_n\|^2 \\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})\|w_n^{(M)}-y_n\|^2\Big] \\
&\le \eta_n^{(1)}\|y_n-\bar{x}\|^2+\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})\|y_n-\bar{x}\|^2 \\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})\|y_n-\bar{x}\|^2 \\
&\quad -\eta_n^{(1)}\Big[\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})\|w_n^{(j-1)}-y_n\|^2 \\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})\|w_n^{(M)}-y_n\|^2\Big] \\
&\le \|y_n-\bar{x}\|^2-\eta_n^{(1)}\Big[\sum_{j=2}^{M}\eta_n^{(j)}\prod_{t=1}^{j-1}(1-\eta_n^{(t)})\|w_n^{(j-1)}-y_n\|^2 \qquad (3.7)\\
&\qquad +\prod_{t=1}^{M}(1-\eta_n^{(t)})\|w_n^{(M)}-y_n\|^2\Big].
\end{aligned}$$

From (3.7), we have

$$\|z_n-\bar{x}\|\le\|y_n-\bar{x}\|. \tag{3.8}$$

The definition of x_{n+1} and (3.8) implies

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|(1-\beta_n)y_n+\beta_n z_n-\bar{x}\|^2 \\
&= (1-\beta_n)\|y_n-\bar{x}\|^2+\beta_n\|z_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|y_n-z_n\|^2 \\
&\le \|y_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|y_n-z_n\|^2. \qquad (3.9)
\end{aligned}$$

From the definition of x_{n+1}, we obtain

$$z_n-y_n=\tfrac{1}{\beta_n}(x_{n+1}-y_n),$$

consequently,

$$\|z_n - y_n\|^2 = \frac{\alpha_n}{\beta_n}\Big(\frac{\|x_{n+1}-y_n\|^2}{\alpha_n\beta_n}\Big). \tag{3.10}$$

By (3.9) and (3.10), we obtain

$$\|x_{n+1} - \bar{x}\|^2 \leq \|y_n - \bar{x}\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - y_n\|^2. \tag{3.11}$$

In view of (3.11) and definition of y_n, we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\| \leq \|y_n - \bar{x}\| &= \|(1-\alpha_n)x_n - \bar{x}\| \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n\|\bar{x}\| \\ &\leq \max\{\|x_n - \bar{x}\|, \|\bar{x}\|\} \\ &\ \vdots \\ &\leq \max\{\|x_1 - \bar{x}\|, \|\bar{x}\|\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ and $\{z_n\}$ are also bounded.
Thus (3.3) and (3.11) implies

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \|y_n - \bar{x}\|^2 - \frac{1-\beta_n}{\beta_n}\|x_{n+1} - y_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\|^2 - \alpha_n\Big(2(1-\alpha_n)\langle x_n - \bar{x}, \bar{x}\rangle \\ &\qquad -\alpha_n\|\bar{x}\|^2 + \frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1} - y_n\|^2\Big) \\ &= (1-\alpha_n)\|x_n - \bar{x}\|^2 - \alpha_n\Gamma_n\end{aligned}$$

where

$$\Gamma_n = 2(1-\alpha_n)\langle x_n - \bar{x}, \bar{x}\rangle - \alpha_n\|\bar{x}\|^2 + \frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1} - y_n\|^2.$$

Since $\{x_n\}$ is bounded and so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.5, we have

$$\limsup_{n\to\infty}\|x_n - \bar{x}\|^2 \leq \limsup_{n\to\infty}(-\Gamma_n) = -\liminf_{n\to\infty}\Gamma_n. \tag{3.12}$$

Therefore, $\liminf_{n\to\infty}\Gamma_n$ is finite. Now, using (C1) of Assumption 1, we get

$$\begin{aligned}\liminf_{n\to\infty}\Gamma_n &= \liminf_{n\to\infty}\Big(2(1-\alpha_n)\langle x_n - \bar{x}, \bar{x}\rangle - \alpha_n\|\bar{x}\|^2 + \frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1} - y_n\|^2\Big) \\ &= \liminf_{n\to\infty}\Big(2\langle x_n - \bar{x}, \bar{x}\rangle + \frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1} - y_n\|^2\Big).\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ in H_1 and

$$\liminf_{n\to\infty}\Gamma_n = \liminf_{k\to\infty}\Big(2\langle x_{n_k} - \bar{x}, \bar{x}\rangle + \frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - y_{n_k}\|^2\Big). \tag{3.13}$$

Since $\{x_n\}$ is bounded and $\liminf_{n\to\infty}\Gamma_n$ is finite, we have that $\frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - y_{n_k}\|^2$ is bounded. Also, by (C2) of Assumption 1, we have $\frac{1-\beta_n}{\alpha_n\beta_n} \geq \frac{1-\delta}{\alpha_n\beta_n} > 0$ and so we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - y_{n_k}\|^2$ is bounded. Observe from (C1) and (C2) of Assumption 1, we have

$$0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{\beta} \to 0, \quad k \to \infty.$$

Therefore, we obtain from (3.10) and $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0, \quad k \to \infty$ that

$$\|y_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty. \tag{3.14}$$

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From the definition of x_{n+1} and (3.14), we have

$$\|x_{n_k+1} - y_{n_k}\| = \beta_{n_k}\|y_{n_k} - z_{n_k}\| \to 0, \;\; k \to \infty, \tag{3.15}$$

and using the definition of y_n, we obtain

$$\|y_{n_k} - x_{n_k}\| = \alpha_{n_k}\|x_{n_k}\| \to 0, \;\; k \to \infty. \tag{3.16}$$

Hence, (3.15) and (3.16) gives

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0, \;\; k \to \infty.$$

Now, using (3.7), we obtain

$$\begin{aligned}
\eta_{n_k}^{(1)}\Big[\sum_{j=2}^{M}\eta_{n_k}^{(j)}\prod_{t=1}^{j-1}(1-\eta_{n_k}^{(t)})\|w_{n_k}^{(j-1)} - y_{n_k}\|^2 + \prod_{t=1}^{M}(1-\eta_{n-k}^{(t)})\|w_{n_k}^{(M)} - y_{n_k}\|^2\Big]& \\
\leq \|y_{n_k} - \bar{x}\|^2 - \|z_{n_k} - \bar{x}\|^2 & \\
\leq (\|y_{n_k} - \bar{x}\| - \|z_{n_k} - \bar{x}\|)(\|y_{n_k} - \bar{x}\| + \|z_{n_k} - \bar{x}\|) & \\
\leq \|y_{n_k} - z_{n_k}\|(\|y_{n_k} - \bar{x}\| + \|z_{n_k} - \bar{x}\|). &
\end{aligned} \tag{3.17}$$

Therefore, (3.14), (3.17) and (C3) of Assumption 1 gives

$$\eta_{n_k}^{(1)}\Big[\sum_{j=2}^{M}\eta_{n_k}^{(j)}\prod_{t=1}^{j-1}(1-\eta_{n_k}^{(t)})\|w_{n_k}^{(j-1)} - y_{n_k}\|^2 + \prod_{t=1}^{M}(1-\eta_{n_k}^{(t)})\|w_{n_k}^{(M)} - y_{n_k}\|^2\Big] \to 0, \tag{3.18}$$

as $k \to \infty$. Hence, (C3) of Assumption 1 together with (3.18) yields

$$\|w_{n_k}^{(j)} - y_{n_k}\| \to 0, \;\; k \to \infty \tag{3.19}$$

for all $j \in \{1, \ldots, M\}$. Using the definition of $w_n^{(j)}$ and (3.19), we have

$$(\mu_{n_k}^{(j)})^2\|\nabla h_j(y_{n_k}) + \nabla l(y_{n_k})\|^2 = \|w_{n_k}^{(j)} - y_{n_k}\|^2 \to 0, \;\; k \to \infty. \tag{3.20}$$

Now from (3.4), we have

$$\begin{aligned}
&4\mu_{n_k}^{(j)}(h_j(y_{n_k}) + l(y_{n_k})) \\
&\leq \|y_{n_k} - \bar{x}\|^2 - \|w_{n_k}^{(j)} - \bar{x}\|^2 + \mu_{n_k}^{(j)}\|\nabla h_j(y_{n_k}) + \nabla l(y_{n_k})\|^2 \\
&\leq \|y_{n_k} - w_{n_k}^{(j)}\|(\|y_{n_k} - \bar{x}\| - \|w_{n_k}^{(j)} - \bar{x}\|) + \mu_{n_k}^{(j)}\|\nabla h_j(y_{n_k}) + \nabla l(y_{n_k})\|^2.
\end{aligned} \tag{3.21}$$

Therefore, from (3.19), (3.20) and (3.21), we obtain

$$4\mu_{n_k}^{(j)}(h_j(y_{n_k}) + l(y_{n_k})) \to 0, \;\; k \to \infty. \tag{3.22}$$

Note that if $j \notin \Psi_{n_k}$, then $h_j(y_{n_k}) + l(y_{n_k}) = 0$ implying that $h_j(y_{n_k}) = 0$ and $l(y_{n_k}) = 0$. Now for $j \in \Psi_{n_k}$ (i.e., $h_j(y_{n_k}) + l(y_{n_k}) \neq 0$), using the choice on $\mu_{n_k}^{(j)}$ given by

$$\mu_{n_k}^{(j)} \in \Big(\epsilon^{(j)}, \frac{h_j(y_{n_k}) + l(y_{n_k})}{\theta_j^2(y_{n_k})} - \epsilon^{(j)}\Big),$$

and from (3.22), we have

$$0 < 4\epsilon^{(j)}\big(h_j(y_{n_k}) + l(y_{n_k})\big) \leq 4\mu_{n_k}^{(j)}(h_j(y_{n_k}) + l(y_{n_k})) \to 0, \;\; k \to \infty. \tag{3.23}$$

Consequently, (3.23) gives

$$\lim_{k\to\infty}(h_j(y_{n_k}) + l(y_{n_k})) = 0 \Leftrightarrow \lim_{k\to\infty} h_j(y_{n_k}) = \lim_{k\to\infty} l(y_{n_k}) = 0$$

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for all $j \in \{1, \ldots, M\}$. Thus, from the definition of $l(y_{n_k})$, we can have

$$\lim_{k\to\infty} h_j(y_{n_k}) = \lim_{k\to\infty} l_i(y_{n_k}) = 0, \quad \forall i \in \{1, \ldots, N\}, \forall j \in \{1, \ldots, M\}.$$

Using CauchySchwarz inequality, we have

$$\langle y_{n_k}, y\rangle = \langle y_{n_k} - x_{n_k}, y\rangle + \langle x_{n_k}, y\rangle \leq \|y_{n_k} - x_{n_k}\|\|y\| + \langle x_{n_k}, y\rangle \tag{3.24}$$

for all $y \in H$. Now, from $x_{n_k} \rightharpoonup p$, (3.16), and (3.24), we have $y_{n_k} \rightharpoonup p$. The lower-semicontinuity of $h_j(.)$ implies that

$$0 \leq h_j(p) \leq \liminf_{k\to\infty} h_j(y_{n_k}) = \lim_{k\to\infty} h_j(y_{n_k}) = 0.$$

That is, $h_j(p) = \frac{1}{2}\|(I - \text{prox}_{\lambda g_j})Ap\|^2 = 0$ for all $j \in \{1, \ldots, M\}$, i.e., $0 \in \partial g_j(Ap)$ for all $j \in \{1, \ldots, M\}$. In other words, Ap is a minimizer of each g_j for all $j \in \{1, \ldots, M\}$. Likewise, the lower-semicontinuity of $l_i(.)$ implies that

$$0 \leq l_i(p) \leq \liminf_{k\to\infty} l_i(y_{n_k}) = \lim_{k\to\infty} l_i(y_{n_k}) = 0.$$

That is, $l_i(p) = \frac{1}{2}\|(I - \text{prox}_{\lambda f_i})p\|^2 = 0$ for all $i \in \{1, \ldots, N\}$, i.e., $0 \in \partial f_i(p)$ for all $i \in \{1, \ldots, N\}$. In other words, p is a minimizer of each f_i for all $i \in \{1, \ldots, N\}$. Therefore, $p \in \Omega$.
Take $\hat{x} = P_\Omega(0)$, i.e., $\hat{x} \in \Omega$ and $\|\hat{x}\| \leq \|y\|$ for all $y \in \Omega$. Thus, from (3.13), we obtain that

$$\begin{aligned} \liminf_{n\to\infty} \Gamma_n &= \liminf_{k\to\infty} \left(2\langle x_{n_k} - \hat{x}, \hat{x}\rangle + \frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - y_{n_k}\|^2\right) \\ &\geq 2\liminf_{k\to\infty} \langle x_{n_k} - \hat{x}, \hat{x}\rangle \\ &\geq 2\langle p - \hat{x}, \hat{x}\rangle = 2\langle p - \hat{x}, \hat{x} - 0\rangle \geq 0. \end{aligned} \tag{3.25}$$

Hence, we have from (3.12) and (3.25) that

$$\limsup_{n\to\infty} \|x_n - \hat{x}\|^2 \leq \limsup_{n\to\infty}(-\Gamma_n) = -\liminf_{n\to\infty} \Gamma_n \leq 0.$$

Therefore, $\|x_n - \hat{x}\| \to 0$ and this implies that $\{x_n\}$ converges strongly to $\hat{x}$. This completes the proof. ∎

Remark 3.3. **(i)** Our algorithm, Algorithm 1, works only for $M \geq 2$. Moreover, Algorithm 1 can be taken as an improved and extended version of accelerated hybrid steepest-descent algorithm in [4, 11, 23].

(ii) Compared to the algorithms in [2, 5–8], our algorithm uses different scheme and it does not require control sequence $\{\rho_n\}$ with the condition $0 < \rho_n < \sigma$ and $\liminf_{n\to\infty} \rho_n(\sigma - \rho_n) > 0$ (for some $\sigma > 0$).

(iii) Our iterative method provides a convenient method to solve split type problems that can be studied as a fixed point of firmly nonexpansive mappings; for example, split system of inclusion problem, multiple-set split feasibility problem and split system of equilibrium problem.

4. Numerical experiments

In this section, we provide a numerical example to validate our proposed iterative method, Algorithm 1 (Alg-1). On top of that, we compare Alg-1 with two existing results, namely, the parallel proximal algorithms [6] (PPA-[6]) and [8] (PPA-[8]) for SSMP.

Example 4.1. Consider the problem (1.2) for $H_1 = \mathbb{R}^p$, $H_2 = \mathbb{R}^q$, $A : H_1 \to H_2$ and

$$f_i(x) = \frac{1}{2}x^T B_i x + x^T D_i, \quad i \in \{1, \dots, N\},$$

$$g_1(u) = \|u\| \text{ and } g_2(u) = \sum_{k=1}^{q} h(u_k) \text{ and } A = G_{q\times p},$$

where $G_{q\times p}$ is $q \times p$ matrix, each B_i is invertible symmetric positive semidefinite $p \times p$ matrix and each D_i is vector in $\mathbb{R}^p$ for all $i \in \{1, \dots, N\}$, $u = (u_1, u_2, \dots, u_q) \in \mathbb{R}^q$, $\|.\|$ is the Euclidean norm in $\mathbb{R}^q$ and $h(u_k) = \max\{|u_k| - 1, 0\}$ for $k = 1, \dots, q$.

In this example, we examine the numerical performance of Alg-1 compare to PPA-[6] and PPA-[8]. For this purpose, we used the following data:

- $G_{q\times p}$ is randomly generated $q \times p$ matrix,
- $M = 3$, B_i is randomly generated invertible symmetric positive semidefinite $p \times p$ matrix and D_i is zero vector in $\mathbb{R}^p$,
- proximal of f_i and g_j with scaling parameter $\lambda = 1$,
- parameter restrictions:
  Alg-1: $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{2n+1}{3n+6}$, $\eta_n^{(j)} = \frac{1}{j+1}$,
  PPA-[6]: $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{2n+1}{3n+6}$, $\delta_n^i = \frac{i}{6}$, $\xi_n^j = \frac{j}{3}$, $\rho_n = \frac{1}{10}$,
  PPA-[8]: $F : H_1 \to H_1$, $V : H_1 \to H_1$ where $V = F = I$, $\mu = 1$, $\gamma = 0.5$, $\alpha_n = \frac{1}{n+1}$, $\rho_n = 0.1$, $\delta_n^i = \frac{i}{6}$, $\xi_n^j = \frac{j}{10}$,
- Stopping Criterion: $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} \leq \text{TOL} = 10^{-3}$.

For a randomly generated starting point x_1 and different choices of p and q, the numerical results of Alg-1, PPA-[6], and PPA-[8] is reported in Table 1 in terms of number of iterations (iter(n)) and cpu time of excursion in seconds (Cpu(s)).

Table 1. Numerical results for randomly generated starting point x_1.

		$p = q = 2$	$p = 3$, $q = 5$	$p = 15$, $q = 10$
Alg-1	Iter(n)	13	15	16
	Cpu(s)	0.00354	0.00654	0.08034
PPA-[6]	Iter(n)	12	17	16
	Cpu(s)	0.00293	0.00597	0.07429
PPA-[8]	Iter(n)	21	27	29
	Cpu(s)	0.01069	0.01638	0.096425

Also, for different choices of x_1, p, and q, the numerical results of Alg-1, PPA-[6], and PPA-[8] is reported Figures 1 and 2.

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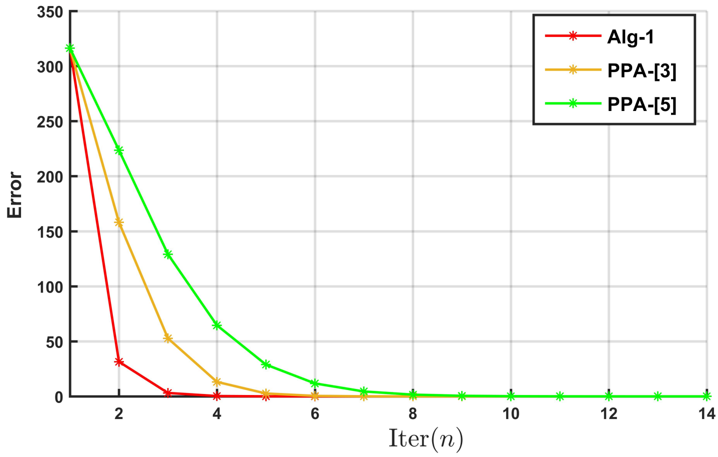

FIGURE 1. For $p = q = 15$ and starting point $x_1 = (10, \ldots, 10) \in \mathbb{R}^p$.

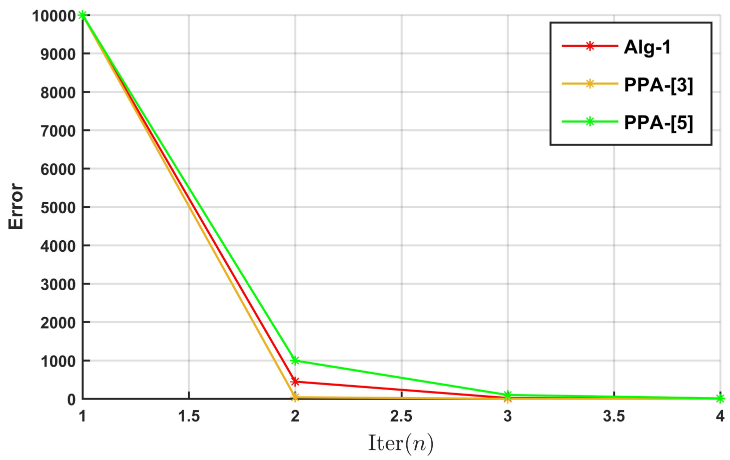

FIGURE 2. For $p = q = 50$ and starting point $x_1 = (100, \ldots, 100) \in \mathbb{R}^p$.

Remark 4.2. **(i)** Based on the findings in Example 4.1, it is evident that our algorithm demonstrates effectiveness and simplicity in implementation.

(ii) The data presented in Table 1 illustrates that our proposed algorithm outperforms the one used in PPA-[8] in terms of CPU time and iteration count. Additionally, the preliminary numerical experiment outcomes indicate that our algorithm performs competitively compared to PPA-[6]. And it is important to underline that our new method extends and generalizes the findings in [6] in such a way that the algorithm utilizes a self-adaptive step size mechanism, eliminating the need for prior information about the operator norm, which enables us to establish and prove a strong convergence theorem under certain weakened assumptions to solve the SSPMs related to convex nonsmooth functions. This shows that our new method is easier to implement.

Conclusions

A new self-adaptive steepest-descent type algorithm, Algorithm 1, is proposed in this paper to solve the SSPM for convex nonsmooth functions. The algorithm includes a self-adaptive step size mechanism and, under certain assumptions, a strong convergence result (Theorem 3.2) is established and proven for the proposed algorithm. Specifically, the sequence generated by Algorithm 1 strongly converges to the minimum norm element of the SSPM. To validate the performance and implementation of the algorithm, a numerical example (Example 4.1) is provided. The results of Example 4.1 demonstrate the effectiveness and simplicity of the algorithm's implementation. Furthermore, preliminary numerical experiment outcomes suggest that the algorithm performs competitively when compared to PPA-[6]. Additionally, the data presented in Table 1 shows that the proposed algorithm outperforms the one used in PPA-[8] in terms of CPU time and iteration count. The main result of this paper extends and improves the results [23]. Furthermore, our result complements recent results in this area.

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Data Availability Statement

Not applicable.

Conflicts of Interest

The authors declare no conflict of interest.

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