



Stability and Convergence Theorems for Enriched Kannan Mappings in $CAT_p(0)$ Spaces with Applications



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Abstract This paper establishes convergence results for enriched Kannan mappings in $CAT_p(0)$ spaces, emphasizing both Δ -convergence and strong convergence of CR-type iterative schemes. A Krasnoselskii-type assignment is shown to meet the contractive criteria of classical Kannan mappings. Stability under perturbations is also analyzed. An application to the Split Feasibility Problem is presented, including a numerical example in image reconstruction. Additional numerical experiments illustrate the theoretical findings.

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Keywords: Fixed Point Theory; Enriched Kannan Mappings; $CAT_p(0)$ Spaces; CR Iteration; Strong Convergence; Δ -Convergence; Asymptotic Regularity; Stability Analysis

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1. INTRODUCTION

The study of fixed point theorems in metric spaces has long been a cornerstone of research in functional analysis and optimization theory. Among the various classes of contractive-type operators, Kannan-type mappings have emerged as a particularly significant generalization, garnering increasing attention due to their versatility in iteration theory and their applicability to diverse mathematical models, including those arising in differential equations, dynamical systems and optimization problems formulated in general metric structures.

Introduced by Kannan in 1969, these mappings relax the conditions of Banach's classical contraction principle by requiring that the distance between the images of two points is controlled not by the distance between the points themselves, but rather by a function of the sum of their distances to their respective images. Specifically, a self-mapping T on a metric space (X, d) satisfies the Kannan condition if there exists a constant $a \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq a [d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X.$$

This condition, though weaker than Banach's contraction, still ensures the existence and uniqueness of fixed points under suitable completeness conditions. As such, Kannan-type mappings have proven particularly useful in scenarios where standard contraction conditions fail to hold.

In recent years, the scope of fixed point theory has been considerably broadened through its extension to more general geometric frameworks, notably $\text{CAT}(0)$ spaces. Metric spaces of non-positive curvature in the sense of Alexandrov. These spaces, which generalize Riemannian manifolds with non-positive sectional curvature, arise naturally in differential geometry and group theory and they provide a powerful setting for fixed point analysis where conventional tools from linear or normed spaces may no longer apply.

Within this geometric context, several novel classes of mappings have been introduced, including enriched Kannan mappings, which extend the original definition by incorporating curvature-sensitive parameters. These mappings allow a finer adaptation to the intrinsic geometry of the space and have been studied using tools such as asymptotic centers, Δ -convergence and iterative fixed point algorithms.

Definition 1.1 ([1]). Let $(X, \|\cdot\|)$ be a normed linear space. A mapping $T : X \rightarrow X$ is said to be an *enriched contraction* if there exist $b \in [0, +\infty)$ and $\theta \in [0, b + 1)$ such that

$$\|b(x - y) + Tx - Ty\| \leq \theta \|x - y\|, \quad \forall x, y \in X. \quad (2.1)$$

Definition 1.2 ([2]). Let $(X, \|\cdot\|)$ be a normed linear space and let K be a nonempty subset of X . A mapping $T : K \rightarrow X$ is said to be an *enriched Kannan mapping* if there exist $a \in [0, 1/2)$ and $b \in [0, +\infty)$ such that

$$\|b(x - y) + Tx - Ty\| \leq a (\|x - Tx\| + \|y - Ty\|) \quad \forall x, y \in K.$$

Recent developments in fixed point theory have demonstrated that enriched mappings retain essential convergence properties and that their fixed points can be effectively approximated through iterative procedures, notably those of Krasnoselskii-type [1–4]. These mappings represent a significant generalization of classical contractive operators, allowing the extension of fixed point results to more general settings, particularly in cases where the mappings involved are neither continuous nor contractive in the standard sense. In this



context, the contributions in [5] and [6] exemplify two distinct yet complementary directions of such generalizations. The former introduces the notion of interpolative enriched cyclic Kannan contractions, which combine cyclic representations with interpolative inequalities to establish fixed point results applicable to nonlinear integral equations. The latter proposes an implicit midpoint-type iterative scheme tailored for enriched nonexpansive mappings, achieving robust convergence even in the presence of stiffness, a property often encountered in differential equations. Further extending this framework, the recent work in [7] introduces the concept of enriched nonexpansive semigroups, where each operator in the semigroup satisfies a Berinde-type inequality and acts in a uniformly convex Banach space. Using the Mann iteration process, the authors establish both weak and strong convergence theorems for common fixed points of such semigroups, supported by a constructive numerical example. These results underscore the flexibility and applicability of enriched mappings, wherein additional parameters such as α , b , or averaging constants enhance the functional framework of fixed point theory, enabling its application to a wider range of problems in nonlinear analysis, numerical approximation and optimization.

This work focuses on the study of convergence theorems for enriched Kannan mappings in $\text{CAT}_p(0)$ spaces, a generalization of $\text{CAT}(0)$ spaces in which the metric is modified through a parameter $p \geq 2$. In this setting, we show that a Krasnoselskii-type iterative scheme associated with an enriched Kannan operator satisfies the criteria of a standard Kannan mapping under the modified metric. Motivated by prior developments on convergence of iterative processes in non-positively curved geometries particularly the modified S-iteration and proximal point-type algorithms studied in [8, 9] and [10] we investigate both Δ -convergence and strong convergence of the sequences generated by our proposed scheme, and we establish sufficient conditions under which these sequences converge to a fixed point of the enriched operator.

A key contribution of this study lies in the analysis of the stability of the enriched Kannan assignment. Following ideas related to robustness under perturbations, as explored in [10], we demonstrate that the iterative process remains stable under small deviations either in the initial point or during computation. This is crucial in practical applications where round-off errors or uncertainties in data may occur. The robustness of the scheme guarantees reliability and convergence, thus enhancing its effectiveness in numerical computation and large-scale optimization.

Overall, by leveraging the geometry of $\text{CAT}_p(0)$ spaces and incorporating tools from nonlinear analysis such as demiclosedness principles and asymptotic regularity this work provides new convergence theorems applicable to a wide range of theoretical and practical problems in mathematics, computer science, and optimization.

2. PRELIMINARIES

In this section, we present several properties of geodesic metric spaces that will be useful for formulating and proving our results. In particular, we focus on certain properties of $\text{CAT}(0)$ spaces and their generalization, the so-called $\text{CAT}_p(0)$ spaces. For a more comprehensive treatment of these spaces, we refer the reader to, e.g., [11–16], and the references therein.

We recall that in a metric space (X, d) , a geodesic path between two points $x, y \in X$ is a continuous map $\gamma_{x,y} : [0, 1] \rightarrow X$ such that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and



$d(\gamma_{x,y}(t), \gamma_{x,y}(s)) = |t - s| \cdot d(x, y)$ for all $s, t \in [0, 1]$. In particular, $\gamma_{x,y}$ is a constant-speed parameterization of the segment joining x and y and the image $\gamma_{x,y}([0, 1])$ is called the geodesic segment between x and y , denoted by $[x, y]$.

The space (X, d) is called a geodesic space if every pair of points $x, y \in X$ can be joined by a geodesic path. If, in addition, such a path is unique for each pair of points, then (X, d) is said to be uniquely geodesic.

A subset $Y \subset X$ is said to be convex if, for every $x, y \in Y$, the geodesic segment $[x, y] = \gamma_{x,y}([0, 1])$ is contained in Y .

Note that strictly convex Banach spaces are uniquely geodesic (see [12]).

A geodesic triangle in a metric space (X, d) , denoted by $\Delta(x_1, x_2, x_3)$, consists of three points $x_1, x_2, x_3 \in X$, called the vertices of the triangle, together with geodesic segments joining each pair of vertices, called the edges of the triangle Δ . The triangle is said to be degenerate if the three vertices lie on a common geodesic.

For a real number κ , let M_κ^2 denote the unique, simply connected, complete, 2-dimensional Riemannian manifold of constant curvature κ . Recall that the diameter of M_κ^2 is $+\infty$ if $\kappa \leq 0$ and $\pi/\sqrt{\kappa}$ if $\kappa > 0$. In particular, $M_0^2 = \mathbb{R}^2$.

A comparison triangle for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that

$$d_{M_\kappa^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j) \quad \text{for all } i, j \in \{1, 2, 3\}.$$

Given a point $x \in [x_i, x_j]$, a point $\bar{x} \in [\bar{x}_i, \bar{x}_j]$ is called a comparison point for x if $d(x_i, x) = d_{M_\kappa^2}(\bar{x}_i, \bar{x})$.

Let $\kappa \in \mathbb{R}$ and let X be a geodesic metric space. For $\kappa \leq 0$, the space X is said to be a $\text{CAT}(\kappa)$ space if every geodesic triangle in X satisfies the following comparison axiom. Let $\Delta = \Delta(x_1, x_2, x_3)$ be any geodesic triangle in X , and let $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a comparison triangle in M_κ^2 . Then Δ satisfies the $\text{CAT}(\kappa)$ inequality if for all points $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{M_\kappa^2}(\bar{x}, \bar{y}). \quad (2.1)$$

If $\kappa > 0$, then X is said to be a $\text{CAT}(\kappa)$ space if every geodesic triangle in X with perimeter less than $2 \cdot \text{diam}(M_\kappa^2)$ satisfies the $\text{CAT}(\kappa)$ inequality.

In metric fixed point theory, $\text{CAT}(0)$ spaces are of special interest since the appearance of Kirk's seminal [17], where the existence of fixed points for nonexpansive mappings was established in this setting an analog of the Leray-Schauder condition in Banach spaces.

A geodesic metric space (X, d) is said to be a $\text{CAT}(0)$ space if it satisfies the following Bruhat-Tits convexity inequality: for all $x, y_1, y_2 \in X$, if $y_0 := \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$ denotes the midpoint of the geodesic segment $[y_1, y_2]$, then

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This inequality is often referred to as the (CN) inequality (cf. [18], see also [12, p. 163]) and it characterizes $\text{CAT}(0)$ spaces among geodesic spaces.

Some fundamental properties of $\text{CAT}(0)$ spaces (X, d) include:

- (i) The space is uniquely geodesic.
- (ii) For all $p, x, y \in X$ and for all $\alpha \in [0, 1]$, define

$$m_1 := (1 - \alpha)p \oplus \alpha x, \quad m_2 := (1 - \alpha)p \oplus \alpha y.$$

Then,

$$d(m_1, m_2) \leq \alpha d(x, y).$$



- (iii) If $x \neq y$ and $z, w \in [x, y]$ with $d(x, z) = d(x, w)$, then $z = w$.
- (iv) For all $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \quad d(y, z) = (1 - t) d(x, y). \quad (2.2)$$

We denote this unique point z by $(1 - t)x \oplus ty$. More generally, for $x_1, \dots, x_n \in X$ and $t_1, \dots, t_n \in [0, 1]$ with $\sum_{i=1}^n t_i = 1$, we define the convex combination recursively by:

$$\bigoplus_{i=1}^n t_i x_i := (1 - t_n) \left(\bigoplus_{i=1}^{n-1} \frac{t_i}{1 - t_n} x_i \right) \oplus t_n x_n.$$

Now, recall that in the definition of a CAT(0) space, the comparison triangle lies in the Euclidean plane $\mathbb{R}^2$ and furthermore, by Proposition 1.14 in [12], any real linear normed space is CAT(0) if and only if it is a pre-Hilbert space. Motivated by this, the authors in [19] proposed a generalization of CAT(0) spaces by comparing triangles in a geodesic space X to triangles in a general normed space $(E, \|\cdot\|_E)$. Specifically, for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq \|\bar{x} - \bar{y}\|_E.$$

With this formulation, the space $(E, \|\cdot\|_E)$ is itself a generalized CAT(0) space regardless of whether it is a pre-Hilbert space.

Definition 2.1 ([19]). Let (X, d) be a geodesic metric space. We say that X is a $\text{CAT}_p(0)$ space, for some $p > 2$, if for every geodesic triangle Δ in X , there exists a comparison triangle $\bar{\Delta}$ in ℓ_p such that for all $x, y \in \Delta$ and all corresponding comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, we have

$$d(x, y) \leq \|\bar{x} - \bar{y}\|_{\ell_p}.$$

In a different direction, Naor and Silberman [20] introduced a notion of uniform convexity for metric spaces, extending the classical p -uniform convexity from Banach space theory.

Definition 2.2 ([20]). Let $1 < p < \infty$. A geodesic metric space (X, d) is said to be p -uniformly convex with parameter $c > 0$ if for all $x, y, z \in X$ and all $t \in [0, 1]$,

$$d^p((1 - t)x \oplus ty, z) \leq (1 - t) d^p(x, z) + t d^p(y, z) - \frac{c}{2} t(1 - t) d^p(x, y).$$

This inequality ensures that X is uniquely geodesic (see [21, Lemma 2.2] for the case $p = 2$). Moreover, any closed convex subset of a p -uniformly convex space remains p -uniformly convex with the same parameter.

Note that CAT(0) spaces are precisely the 2-uniformly convex spaces with parameter $c = 2$ and $\text{CAT}(\kappa)$ spaces (for $\kappa > 0$) with diameter less than $\frac{\pi}{2\sqrt{\kappa}}$ are 2-uniformly convex with parameter $c = (\pi - 2\sqrt{\kappa}\epsilon) \tan(\sqrt{\kappa}\epsilon)$, for any $\epsilon \leq \frac{\pi}{2\sqrt{\kappa}} - \text{diam}(X)$ (see [22]).

Within the framework of $\text{CAT}_p(0)$ spaces, the following result establishes the p -uniform convexity and a corresponding inequality (CN_p) for these generalized spaces.

Lemma 2.3 ([23], Lemma 1.3). Let X be a $\text{CAT}_p(0)$ space, with $p \geq 2$, with $x, x_i \in X$ and $t_i \in [0, 1]$ for $i = 1, 2, \dots, n$ ($n \geq 2$) such that $\sum_{i=1}^n t_i = 1$. Then

- (i) $d(\bigoplus_{i=1}^n t_i x_i, x) \leq \sum_{i=1}^n t_i d(x_i, x);$
- (ii) $d^p(\bigoplus_{i=1}^n t_i x_i, x) \leq \sum_{i=1}^n t_i d^p(x_i, x) - \frac{1}{2^{p-1}} t_i t_j d^p(x_i, x_j) \text{ for } i, j \in \{1, 2, \dots, n\}.$



It is important to recall that a bounded sequence $\{x_n\}$ in a metric space X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$, where $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ is the so-called asymptotic radius, with $r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x_n, x)$.

It is known that every bounded sequence in a Banach space admits a regular subsequence (see [24], p. 166). The proof of this fact is purely metric in nature and therefore extends naturally to the current setting. Since every regular sequence is Δ -convergent, it follows immediately that every bounded sequence in X has a Δ -convergent subsequence.

Furthermore, suppose $\{x_n\} \subset X$ Δ -converges to some point $x \in X$ and let $y \in X$ with $y \neq x$. Then the following inequality holds:

$$\limsup_{n \rightarrow \infty} d(x_n, x) < \limsup_{n \rightarrow \infty} d(x_n, y). \quad (2.3)$$

Which is known in Banach space theory as the Opial property. For more details, see [25]. It is also easy to verify that in a complete p -uniformly convex space, any bounded sequence admits a Δ -convergent subsequence.

In the setting of $\text{CAT}(0)$ spaces, Δ -convergence is equivalent to another notion of weak convergence based on projections onto geodesic segments (see [26]). A similar argument to that in [26] shows that this equivalence also holds in p -uniformly convex metric spaces.

Lemma 2.4 ([27], Lemma 3.2). *Let (X, d) be a complete $\text{CAT}(0)$ space and let K be a nonempty closed convex subset. Then,*

- (i) *every bounded sequence in (X, d) has Δ -convergent subsequence;*
- (ii) *the asymptotic center of any bounded sequence in K is contained in K .*

Lemma 2.5 ([27], Lemma 3.1). *The asymptotic center of any bounded sequence in a $\text{CAT}_p(0)$ spaces X has exactly one element.*

Definition 2.6 ([28], also [16]). A sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$ and we call x the $\Delta\text{-}\lim_{n \rightarrow \infty} x_n = x$.

Also in [27] they establish that a map T on X is said to have demiclosedness-type property if for any sequence $\{x_n\} \subseteq X$,

$$\left. \begin{array}{l} \Delta\text{-}\lim_{n \rightarrow \infty} x_n = x \\ \lim_{n \rightarrow \infty} d(Tx_n, x_n) = 0 \end{array} \right\} \implies Tx = x. \quad (2.4)$$

From now on, X is a complete $\text{CAT}_p(0)$ space, K is a nonempty convex subset of X and $T : X \rightarrow K$ is a mapping. The mapping T is called Enriched Kannan mapping if for each $x, y \in K$, $b \in [0, \infty)$ and $a \in [0, \frac{1}{2})$,

$$\begin{aligned} & d\left(\frac{b}{b+1}x \oplus \frac{1}{b+1}Tx, \frac{b}{b+1}y \oplus \frac{1}{b+1}Ty\right) \\ & \leq a \left(d\left(x, \frac{b}{b+1}x \oplus \frac{1}{b+1}Tx\right) + d\left(y, \frac{b}{b+1}y \oplus \frac{1}{b+1}Ty\right) \right). \end{aligned}$$

A point $x \in K$ is called a fixed point of T if $x = Tx$. We shall denote with $F(T)$ the set of fixed points of T .



Lemma 2.7 ([4], Lemma 3.5). *Let (H, d) be a Hadamard space and let $T : H \rightarrow H$ be a mapping. For $\lambda \in (0, 1]$, define the mapping*

$$T_\lambda x := (1 - \lambda)x \oplus \lambda Tx, \quad \forall x \in H.$$

Then the following statements hold:

- (i) $F(T) = F(T_\lambda)$;
- (ii) $d^2(T_\lambda x, T_\lambda y) \leq (1 - \lambda)^2 d^2(x, y) + \lambda^2 d^2(Tx, Ty) + 2\lambda(1 - \lambda) \langle \overrightarrow{xy}, \overrightarrow{T_x T_y} \rangle$;
- (iii) *If T is an enriched contraction, then there exists $\lambda_0 \in (0, 1]$ such that T_{λ_0} is a k -contraction on H .*

Theorem 2.8 ([2], Theorem 3.2). *Let (H, d) be a Hadamard space and let $T : H \rightarrow H$ be a enriched Kannan mapping. Then there exists $p \in H$ such that:*

- (i) $\text{Fix}(T) = \{p\}$;
- (ii) *there exists $\delta \in (0, 1]$ such that the sequence $\{x_n\}$ defined iteratively by*

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1 - \delta)x_n \oplus \delta Tx_n, \quad n \geq 1, \end{cases}$$

converges to p ;

- (iii) *for $q \in [0, 1)$, we have the following estimate:*

$$d(x_{n+j-1}, p) \leq \frac{q^j}{1 - q} d(x_n, x_{n-1}), \quad n > 1, j \geq 1.$$

In the framework of nonlinear analysis and fixed point theory, the study of iterative processes in spaces with curvature bounded above has attracted considerable interest. One important class of such spaces is given by $\text{CAT}_p(0)$ spaces, which generalize Hilbert and Hadamard spaces by allowing a notion of convexity based on comparison triangles in a model space of nonpositive curvature.

Given a nonexpansive-type mapping $T_\lambda : X \rightarrow X$ acting on a $\text{CAT}_p(0)$ space (X, d) , the *CR iteration process* [29] is defined recursively by the system:

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)y_n \oplus \alpha_n T_\lambda y_n, \\ y_n &= (1 - \beta_n)T_\lambda x_n \oplus \beta_n T_\lambda z_n, \\ z_n &= (1 - \gamma_n)x_n \oplus \gamma_n T_\lambda x_n, \end{cases} \quad n = 1, 2, 3, \dots \quad (2.5)$$

Here, the operation $\oplus$ denotes the geodesic convex combination between two points in the $\text{CAT}_p(0)$ space, which plays the role of convex combinations in linear spaces. The sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real-valued control sequences contained in the interval $[0, 1]$, which determine the weight of the geodesic interpolation at each step of the iteration.

Under suitable conditions on the operator T_λ and appropriate choices for the control sequences $\alpha_n, \beta_n, \gamma_n$, one can prove the strong convergence of the sequence $\{x_n\}$ to a fixed point of T_λ , provided such a point exists. This iterative method has proven effective in non-linear metric settings, especially when standard tools from Banach space theory are no longer available. The structure of $\text{CAT}_p(0)$ spaces, including the convexity of the distance function and the uniqueness of geodesics, plays a crucial role in establishing convergence results.



To prove some of our main theorems we are going to use the following properties of sequences of real numbers.

Lemma 2.9. (See [30, Lemma 1.7]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. If $\sum_{k=0}^{\infty} b_k < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.*

Lemma 2.10. (See [30, Lemma 1.6]). *Let $\{a_n\}$, $\{b_n\}$ be sequences of nonnegative real numbers and $0 \leq q < 1$ so that $a_{n+1} \leq qa_n + b_n$, for all $n \geq 0$. If $\lim_{n \rightarrow \infty} b_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.*

3. MAIN RESULTS

Now, we state and prove first result of this section as follows.

Theorem 3.1. *Let X be a complete $\text{CAT}_p(0)$ space and $T : X \rightarrow X$ be a Enriched Kannan mapping. Then λ -Krasnoselskii mapping $T_\lambda : X \rightarrow X$ is Kannan mapping.*

Proof. Since T is an Enriched Kanna mapping, we have for all $x, y \in X$.

$$\begin{aligned} & d\left(\frac{b}{b+1}x \oplus \frac{1}{b+1}Tx, \frac{b}{b+1}y \oplus \frac{1}{b+1}Ty\right) \\ & \leq a\left(d(x, \frac{b}{b+1}x \oplus \frac{1}{b+1}Tx) + d(y, \frac{b}{b+1}y \oplus \frac{1}{b+1}Ty)\right) \end{aligned}$$

set $\lambda = \frac{1}{b+1}$, we have

$$\begin{aligned} & d\left(\lambda\left(\frac{1}{\lambda} - 1\right)x \oplus \lambda Tx, \lambda\left(\frac{1}{\lambda} - 1\right)y \oplus \lambda Ty\right) \\ & \leq a\left[d(x, \lambda\left(\frac{1}{\lambda} - 1\right)x \oplus \lambda Tx) + d(y, \lambda\left(\frac{1}{\lambda} - 1\right)y \oplus \lambda Ty)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} & d(\lambda - 1)x \oplus \lambda Tx, (\lambda - 1)y \oplus \lambda Ty) \\ & \leq a[d(x, (1 - \lambda)x \oplus \lambda Tx) + d(y, (1 - \lambda)y \oplus \lambda Ty)]. \end{aligned}$$

This gives

$$d(T_\lambda x, T_\lambda y) \leq a[d(x, T_\lambda x) + d(y, T_\lambda y)]. \quad (3.1)$$

Hence T_λ is a Kannan mapping.

Lemma 3.2. *Let K be a convex subset of X and let $T : X \rightarrow K$ be a Enriched Kannan mapping. Then the conditions $\{x_n\}$ Δ -converges to $x \in K$ and $d(x_n, Tx_n) \rightarrow 0$, imply $x \in F(T)$.*

Proof. From Theorem 3.1, we know that T_λ is Kannan mapping for $\lambda = \frac{1}{b+1}$. Let $x \in K$ such that the sequence $\{x_n\}$ Δ -converges to x . Assume that $Tx \neq x$. Then, the triangle inequality we get

$$\begin{aligned} d(T_\lambda x, T_\lambda x_n) & \leq a[d(x, T_\lambda x) + d(x_n, T_\lambda x_n)] \\ & \leq a[d(x, x_n) + d(x_n, T_\lambda x) + d(x_n, T_\lambda x_n)]. \end{aligned}$$



Thus we have,

$$\begin{aligned} d(T_\lambda x, x_n) &\leq d(T_\lambda x, T_\lambda x_n) + d(x_n, T_\lambda x_n) \\ &\leq a [d(x, x_n) + d(x_n, T_\lambda x) + d(x_n, T_\lambda x_n)] + d(x_n, T_\lambda x_n) \\ &= \frac{a}{1-a} d(x, x_n) + \frac{a+1}{1-a} d(x_n, T_\lambda x_n). \end{aligned}$$

Taking $\limsup$ and using the hypothesis $d(x_n, T_\lambda x_n) \rightarrow 0$ we conclude that

$$\begin{aligned} \limsup_n d(T_\lambda x, x_n) &\leq \limsup_n \frac{a}{1-a} d(x, x_n) \\ &< \limsup_n d(x, x_n). \end{aligned}$$

Since the sequence $\{x_n\}$ Δ -converges to x , the Opial-type property implies that last inequality cannot hold, therefore it must be the case that $T_\lambda x = x$. It can be easily seen from Lemma 2.5, $F(T_\lambda) = F(T)$. Hence, $T(x) = x$.

To prove the Δ -convergence of process (2.5), first we prove that the sequence $\{x_n\}$ generated by this process is an approximate fixed point sequence of T .

Lemma 3.3. *Let (X, d) be a complete $\text{CAT}_p(0)$ space with $p \geq 2$. Let K be a nonempty closed, bounded and convex subset of X and let $T_\lambda : K \rightarrow K$ be a Enriched Kannan mapping. Let $\{x_n\}$ be the sequence of CR iteration defined by (2.5) with sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in $[0, 1]$ such that,*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \liminf_{n \rightarrow \infty} \beta_n > 0, \quad \liminf_{n \rightarrow \infty} \gamma_n > 0 \text{ and } \limsup_{n \rightarrow \infty} \gamma_n < 1.$$

Then,

- (i) $\lim_{n \rightarrow \infty} d(x_n, x^*)$ exists for any fixed point $x^* \in F(T_\lambda)$;
- (ii) $\lim_{n \rightarrow \infty} d(T_\lambda x_n, x_n) = 0$.

Proof. Now in (3.1), taking $y = x_n$ and $x = x^*$, we have

$$\begin{aligned} d(T_\lambda x_n, T_\lambda x^*) &\leq a [d(x_n, T_\lambda x_n) + d(x^*, T_\lambda x^*)] \\ d(T_\lambda x_n, x^*) &\leq a d(x_n, T_\lambda x_n) \\ d(T_\lambda x_n, x^*) &\leq a [d(x_n, x^*) + d(T_\lambda x_n, x^*)] \\ d(T_\lambda x_n, x^*) &\leq \frac{a}{1-a} d(x_n, x^*). \end{aligned} \tag{3.2}$$

We denote $\delta = \frac{a}{1-a}$. Notice that $\delta \in [0, 1)$, then equation (3.2) becomes,

$$d(T_\lambda x_n, x^*) \leq \delta d(x_n, x^*) \tag{3.3}$$

similarly, we have

$$d(T_\lambda y_n, x^*) \leq \delta d(y_n, x^*) \tag{3.4}$$

and

$$d(T_\lambda z_n, x^*) \leq \delta d(z_n, x^*). \tag{3.5}$$



From (2.5), (3.3), (3.4), (3.5) and Lemma 2.3, we get

$$\begin{aligned} d(x_{n+1}, x^*) &= d((1 - \alpha_n)y_n \oplus \alpha_n T_\lambda y_n, x^*) \\ &\leq (1 - \alpha_n)d(y_n, x^*) + \alpha_n d(T_\lambda y_n, x^*) \\ &\leq (1 - \alpha_n)d(y_n, x^*) + \delta \alpha_n d(y_n, x^*) \\ &= (1 - \alpha_n(1 - \delta))d(y_n, x^*). \end{aligned} \quad (3.6)$$

Now, we have the following estimates:

$$\begin{aligned} d(y_n, x^*) &= d((1 - \beta_n)T_\lambda x_n \oplus \beta_n T_\lambda z_n, x^*) \\ &\leq (1 - \beta_n)d(T_\lambda x_n, x^*) + \beta_n d(T_\lambda z_n, x^*) \\ &= (1 - \beta_n)\delta d(x_n, x^*) + \beta_n \delta d(z_n, x^*). \end{aligned}$$

Also,

$$\begin{aligned} d(z_n, x^*) &= d((1 - \gamma_n)x_n \oplus \gamma_n T_\lambda x_n, x^*) \\ &\leq (1 - \gamma_n)d(x_n, x^*) + \gamma_n d(T_\lambda x_n, x^*) \\ &\leq (1 - \gamma_n)d(x_n, x^*) + \gamma_n \delta d(x_n, x^*) \\ &\leq (1 - \gamma_n(1 - \delta))d(x_n, x^*). \end{aligned} \quad (3.7)$$

It follows from (3.7) that

$$d(y_n, x^*) \leq (1 - \beta_n)\delta d(x_n, x^*) + \beta_n \delta (1 - \gamma(1 - \delta))d(x_n, x^*). \quad (3.8)$$

Using the estimates $(1 - \beta_n)\delta \leq (1 - \beta_n)$ and $\beta_n \delta (1 - \gamma(1 - \delta)) \leq \beta_n \delta$, inequality (3.8) yields

$$d(y_n, x^*) \leq (1 - \beta_n(1 - \delta))d(x_n, x^*). \quad (3.9)$$

Therefore, from (3.9) and (3.6) it follows that

$$\begin{aligned} d(x_{n+1}, x^*) &\leq (1 - \alpha_n(1 - \delta))(1 - \beta_n(1 - \delta))d(x_n, x^*) \\ &\leq (1 - \alpha_n(1 - \delta))d(x_n, x^*). \end{aligned} \quad (3.10)$$

To prove (i) notice from the fact $1 - \alpha_n(1 - \delta) \leq 1$ and from the inequality above that

$$d(x_{n+1}, x^*) \leq d(x_n, x^*), \quad (3.11)$$

we have,

$$d(x_{n+1}, x^*) \leq d(x_n, x^*) \leq d(x_n, x^*) + b_n$$

for any sequence of nonnegative real numbers satisfying that $\sum_{k=0}^{\infty} b_k < \infty$. Lemma 2.9 implies (i). Let us prove (ii). From Lemma 2.3 and inequality (3.4), we have:

$$\begin{aligned} d^p(x_{n+1}, x^*) &= d^p((1 - \alpha_n)y_n \oplus \alpha_n T_\lambda y_n, x^*) \\ &\leq (1 - \alpha_n)d^p(y_n, x^*) + \alpha_n d^p(T_\lambda y_n, x^*) - \frac{\alpha_n(1 - \alpha_n)}{2^{p-1}}d^p(T_\lambda y_n, y_n) \\ &\leq (1 - \alpha_n)d^p(y_n, x^*) + \delta^p \alpha_n d^p(y_n, x^*) - \frac{\alpha_n(1 - \alpha_n)}{2^{p-1}}d^p(T_\lambda y_n, y_n) \\ &= (1 - \alpha_n(1 - \delta^p))d^p(y_n, x^*) - \frac{\alpha_n(1 - \alpha_n)}{2^{p-1}}d^p(T_\lambda y_n, y_n) \\ &\leq (1 - \alpha_n(1 - \delta^p))d^p(y_n, x^*). \end{aligned} \quad (3.12)$$



Now, from Lemma 2.3 and inequalities (3.3) and (3.5) we obtain

$$\begin{aligned} d^p(y_n, x^*) &= d^p((1 - \beta_n)T_\lambda x_n \oplus \beta_n T_\lambda z_n, x^*) \\ &\leq (1 - \beta_n)d^p(T_\lambda x_n, x^*) + \beta_n d^p(T_\lambda z_n, x^*) - \frac{\beta_n(1 - \beta_n)}{2^{p-1}}d^p(T_\lambda x_n, T_\lambda z_n) \\ &\leq \delta^p(1 - \beta_n)d^p(x_n, x^*) + \delta^p \beta_n d^p(z_n, x^*). \end{aligned} \quad (3.13)$$

Again, from Lemma 2.3 and inequality (3.3) we conclude that

$$\begin{aligned} d^p(z_n, x^*) &= d^p((1 - \gamma_n)x_n \oplus \gamma_n T_\lambda x_n, x^*) \\ &\leq (1 - \gamma_n)d^p(x_n, x^*) + \gamma_n d^p(T_\lambda x_n, x^*) - \frac{\gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n) \\ &\leq (1 - \gamma_n)d^p(x_n, x^*) + \delta^p \gamma_n d^p(x_n, x^*) - \frac{\gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n) \\ &\leq (1 - \gamma_n(1 - \delta^p))d^p(x_n, x^*) - \frac{\gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n). \end{aligned} \quad (3.14)$$

Using (3.13) and (3.14) we obtain the following estimate:

$$\begin{aligned} d^p(y_n, x^*) &\leq \delta^p(1 - \beta_n)d^p(x_n, x^*) + \delta^p \beta_n [(1 - \gamma_n(1 - \delta^p))d^p(x_n, x^*) \\ &\quad - \frac{\gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n)] . \end{aligned} \quad (3.15)$$

From the facts $\delta \in (0, 1]$ and $\beta_n \delta^p (1 - \gamma_n(1 - \delta^p)) \leq \beta_n$, inequality (3.15) yields

$$\begin{aligned} d^p(y_n, x^*) &\leq (1 - \beta_n)d^p(x_n, x^*) + \beta_n d^p(x_n, x^*) - \frac{\beta_n \gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n) \\ &= d^p(x_n, x^*) - \frac{\beta_n \gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n). \end{aligned} \quad (3.16)$$

Substituting (3.16) in (3.12), we have

$$d^p(x_{n+1}, x^*) \leq (1 - \alpha_n(1 - \delta^p)) \left[d^p(x_n, x^*) - \frac{\beta_n \gamma_n(1 - \gamma_n)}{2^{p-1}}d^p(T_\lambda x_n, x_n) \right]$$

which implies that

$$\begin{aligned} d^p(T_\lambda x_n, x_n) &\leq \frac{2^{p-1}}{\beta_n \gamma_n(1 - \gamma_n)}d^p(x_n, x^*) \\ &\quad - \frac{2^{p-1}}{(1 - \alpha_n(1 - \delta^p))(\beta_n \gamma_n(1 - \gamma_n))}d^p(x_{n+1}, x^*). \end{aligned}$$

Since $-\frac{1}{1 - \alpha_n(1 - \delta^p)} \leq 1$, we obtain the estimate

$$d^p(T_\lambda x_n, x_n) \leq \frac{2^{p-1}}{\beta_n \gamma_n(1 - \gamma_n)} [d^p(x_n, x^*) + d^p(x_{n+1}, x^*)] .$$



Due to the facts that

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \liminf_{n \rightarrow \infty} \beta_n > 0, \quad \liminf_{n \rightarrow \infty} \gamma_n > 0 \text{ and } \limsup_{n \rightarrow \infty} \gamma_n < 1,$$

since $\lim_{n \rightarrow \infty} d(x_n, x^*)$ exists, we conclude $\lim_{n \rightarrow \infty} d(T_\lambda x_n, x_n) = 0$.

Theorem 3.4. *Let (X, d) be a complete $\text{CAT}_p(0)$ space, with $p \geq 2$. Let K be a nonempty closed, bounded and convex subset of X and let $T_\lambda : K \rightarrow K$ be a Enriched Kannan mapping. Let $\{x_n\}$ be the sequence of CR iteration defined by (2.5) with sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ Δ -converges to a fixed point of T .*

Proof. We first show that $w_\Delta(x_n) \subseteq F(T_\lambda)$, where we denote $w_\Delta := \bigcup \{A(\{u_n\})\}$, with the union taken over all subsequence $\{u_n\}$ of $\{x_n\}$. Let $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. By Lemma 3.2, $v \in F(T_\lambda)$. Now, from Lemma 3.3, $\lim d(x_n, v)$ exists. We now claim that $u = v$. Assume on the contrary, that $u \neq v$. Then, by the uniqueness of asymptotic centers we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(v_n, v) &< \limsup_{n \rightarrow \infty} d(v_n, u) \\ &\leq \limsup_{n \rightarrow \infty} d(u_n, u) \\ &< \limsup_{n \rightarrow \infty} d(u_n, v) \\ &= \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(v_n, v). \end{aligned}$$

Which is a contradiction. Thus, $u = v \in F(T_\lambda)$ and hence $w_\Delta(x_n) \subseteq F(T_\lambda)$. To show that $\{x_n\}$ Δ -converges to a fixed point of T , we show that $w_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.4, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T_\lambda)$. Finally, we claim that $x = v$. If not, then the existence of $\lim_{n \rightarrow \infty} d(x_n, v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(v_n, v) &< \limsup_{n \rightarrow \infty} d(v_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &< \limsup_{n \rightarrow \infty} d(x_n, v) \\ &= \limsup_{n \rightarrow \infty} d(v_n, v). \end{aligned}$$

Again, it is a contradiction, hence $x = v \in F(T_\lambda)$. Therefore, $w_\Delta(x_n) = \{x\}$.

Theorem 3.5. *Let (X, d) be a complete $\text{CAT}_p(0)$ space with $p \geq 2$. Let K be a nonempty closed, bounded and convex subset of X and let $T_\lambda : K \rightarrow K$ be a Enriched Kannan mapping. Let $\{x_n\}$ be the sequence of CR iteration defined by (2.5) with sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ in $[0, 1]$. The sequence $\{x_n\}$ defined by (2.5) converges strongly to a fixed point of T_λ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T_\lambda)) = 0$, where $d(x, F(T_\lambda)) = \inf\{d(x, x^*) : x^* \in F(T_\lambda)\}$.*



Proof. The necessity is obvious. To prove the converse, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F(T_\lambda)) = 0$. Is proved in Lemma (3.3) (i), $\lim_{n \rightarrow \infty} d(x_n, F(T_\lambda))$ exists for all $x^* \in F(T_\lambda)$. Thus by hypothesis $\lim_{n \rightarrow \infty} d(x_n, F(T_\lambda)) = 0$. Next, we show that $\{x_n\}$ is Cauchy sequence in K . Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n \rightarrow \infty} d(x_n, F(T_\lambda)) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F(T_\lambda)) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0}, x^*) : x^* \in F(T_\lambda)\} < \frac{\epsilon}{4}$. Thus there must exist $x^{**} \in F(T_\lambda)$ such that,

$$d(x_{n_0}, x^{**}) < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x^{**}) + d(x^{**}, x_n) \\ &\leq 2d(x_{n_0}, x^{**}) \\ &\leq 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K . Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n \rightarrow \infty} x_n = q$. Since K is closed, therefore $q \in K$. Next, we show that $q \in F(T_\lambda)$. Since $\lim_{n \rightarrow \infty} d(x_n, F(T_\lambda)) = 0$, we get $d(q, F(T_\lambda)) = 0$, closedness of $F(T_\lambda)$ gives that $q \in F(T_\lambda)$. Thus $\{x_n\}$ converges strongly to a point in $F(T_\lambda)$. This completes the proof.

Now we will prove the strong convergence under the conditions demiclosedness-type property and assuming the compactness of the subset K .

Theorem 3.6. *Let K be a nonempty subset of the $\text{CAT}_p(0)$ space (X, d) and T_λ be a Enriched Kannan mapping. Then T_λ has the demiclosedness-type property (2.4).*

Proof. The proof is immediate from Theorem 3.4 and Lemma 3.3(ii).

Theorem 3.7. *Let (X, d) be a complete $\text{CAT}_p(0)$ space with $p \leq 2$ and K be a nonempty closed convex subset. Let $T_\lambda : X \rightarrow K$ be a Enriched Kannan mapping the satisfying demiclosedness-type property (2.4). Suppose $\{x_n\}$ is the sequence generated by (2.5). If K is compact, then the convergence is strong.*

Proof. Suppose K is compact, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges strongly to some point $x \in K$. Thus $\{x_{n_k}\}$ Δ -converges to $x \in K$. By Lemma 3.3(ii) we have that $x \in F(T_\lambda)$. Consequently, by (3.11) it follows that $\{d(x_n, x^*)\}$ converges for every $x^* \in F(T_\lambda)$, we have

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{k \rightarrow \infty} d(x_{n_k}, x)$$

which completes the proof.

Intuitively, a fixed point iteration procedure is numerically stable if “small” changes in the initial data or in the data that are involved in the computation process will produce a “small” influence on the computed value of the fixed point. Following this idea, Harder and Hicks introduced the following concept of stability [31].



Definition 3.8. Let (X, d) be a metric space and $T : X \rightarrow X$ a mapping, $x_0 \in X$ and let us assume that iteration procedure

$$x_{n+1} = f(T, x_n), n = 0, 1, 2, \dots, \quad (3.17)$$

where $f(T, x_n)$ implicitly includes all parameters defining the fixed-point iteration. Suppose that $\{x_n\}_{n=0}^\infty$ converges to a fixed point x^* of T .

Let $\{y_n\}_{n=0}^\infty$ be an arbitrary sequence in X and set

$$\varepsilon_n = d(y_{n+1}, f(T, y_n)), \text{ for } n = 0, 1, 2, \dots \quad (3.18)$$

We shall say that the fixed point iteration procedure (3.17) is T -stable or stable with respect to T if and only if

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} y_n = x^*.$$

Theorem 3.9. Let (X, d) be a complete $\text{CAT}_p(0)$ space, with $p \geq 2$. Let K be a nonempty closed, bounded and convex subset of X and let $T_\lambda : K \rightarrow K$ be a Enriched Kannan mapping. Let $\{x_n\}$ be the sequence of CR iteration defined by (2.5) with sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}_{n=1}^\infty$ in $[0, 1]$ such that $\sum_{n=0}^\infty \alpha_n = \infty$. Then the sequence defined by (2.5) is stable with respect to T_λ .

Proof. Suppose $\{y_n\} \subset K$ be an arbitrary sequence,

$$\varepsilon_n = d(y_{n+1}, (1 - \alpha_n)b_n \oplus \alpha_n T_\lambda b_n),$$

where $b_n = (1 - \beta_n)T_\lambda y_n \oplus \beta_n T_\lambda c_n$, $c_n = (1 - \gamma_n)y_n \oplus \gamma_n T_\lambda y_n$. Note that similarly from, (3.3), (3.4), (3.5) we have,

$$d(T_\lambda y_n, x^*) \leq \delta d(y_n, x^*), \quad d(T_\lambda y_n, x^*) \leq \delta d(y_n, x^*), \quad d(T_\lambda y_n, x^*) \leq \delta d(y_n, x^*), \quad (3.19)$$

where $\delta \in [0, 1)$. Now from (3.19) and Lemma 2.3, we get

$$\begin{aligned} d(y_{n+1}, x^*) &\leq d(y_{n+1}, (1 - \alpha_n)b_n \oplus \alpha_n T_\lambda b_n) \\ &\quad + d((1 - \alpha_n)(b_n, x^*) \oplus \alpha_n T_\lambda b_n, x^*) \\ &\leq \varepsilon_n + (1 - \alpha_n)d(b_n, x^*) + \alpha_n d(T_\lambda b_n, x^*) \\ &\leq \varepsilon_n + (1 - \alpha_n)d(b_n, x^*) + \delta \alpha_n d(b_n, x^*) \\ &\leq \varepsilon_n + (1 - \alpha_n(1 - \delta))d(b_n, x^*). \end{aligned} \quad (3.20)$$

Now, we have the following estimates:

$$\begin{aligned} d(b_n, x^*) &= d((1 - \beta_n)T_\lambda y_n \oplus \beta_n T_\lambda c_n, x^*) \\ &\leq (1 - \beta_n)d(T_\lambda y_n, x^*) + \beta_n d(T_\lambda c_n, x^*) \\ &= (1 - \beta_n)\delta d(y_n, x^*) + \beta_n \delta d(b_n, x^*). \end{aligned} \quad (3.21)$$

Also,

$$\begin{aligned} d(c_n, x^*) &= d((1 - \gamma_n)y_n \oplus \gamma_n T_\lambda y_n, x^*) \\ &\leq (1 - \gamma_n)d(y_n, x^*) + \gamma_n d(T_\lambda y_n, x^*) \\ &\leq (1 - \gamma_n)d(y_n, x^*) + \gamma_n \delta d(y_n, x^*) \\ &\leq (1 - \gamma_n(1 - \delta))d(y_n, x^*). \end{aligned} \quad (3.22)$$



Replacing (3.22) into (3.21) we get,

$$d(b_n, x^*) \leq (1 - \beta_n)\delta d(y_n, x^*) + \beta_n\delta(1 - \gamma(1 - \delta))d(y_n, x^*). \quad (3.23)$$

Using the estimates $(1 - \beta_n)\delta \leq (1 - \beta_n)$ and $\beta_n\delta(1 - \gamma(1 - \delta)) \leq \beta_n\delta$, inequality (3.8) yields

$$d(b_n, x^*) \leq (1 - \beta_n(1 - \delta))d(y_n, x^*). \quad (3.24)$$

Therefore, from (3.24) and (3.20) it follows that

$$\begin{aligned} d(y_{n+1}, x^*) &\leq \varepsilon_n + (1 - \alpha_n(1 - \delta))(1 - \beta_n(1 - \delta))d(y_n, x^*) \\ &\leq \varepsilon_n + (1 - \alpha_n(1 - \delta))d(y_n, x^*). \end{aligned} \quad (3.25)$$

Now suppose $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Since $\delta < 1$ and $\alpha_n > 0$, it results by Lemma 2.10, that $\lim_{n \rightarrow \infty} y_n = x^*$.

Conversely, assume $\lim_{n \rightarrow \infty} d(y_{n+1}, x^*) = 0$. Then using again (3.19), Lemma 2.3 and the triangle inequality, we have

$$\begin{aligned} \varepsilon_n &= d(y_{n+1}, (1 - \alpha_n)b_n \oplus \alpha_n T_\lambda b_n) \\ &\leq d(y_{n+1}, x^*) + d((1 - \alpha_n)b_n \oplus \alpha_n T_\lambda b_n, x^*) \\ &\leq d(y_{n+1}, x^*) + (1 - \alpha_n)d(b_n, x^*) + \alpha_n d(T_\lambda b_n, x^*) \\ &\leq d(y_{n+1}, x^*) + (1 - \alpha_n)d(b_n, x^*) + \alpha_n d(b_n, x^*). \end{aligned} \quad (3.26)$$

Using estimates (3.24) and (3.26) yields

$$\varepsilon_n \leq d(y_{n+1}, x^*) + (1 - \alpha_n(1 - \delta))d(y_n, x^*).$$

Again from Lemma 2.10 $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Therefore $\{x_n\}$ defined by (2.5) is stable with respect to T_λ .

The above Theorem 3.9 contributes significantly to the analysis of the stability of CR iteration with respect to enriched Kannan assignment. The study demonstrates that the iterative process is robust to small perturbations in the initial data or in the computational steps. This robustness is crucial for practical applications, as it guarantees the reliability of the algorithm despite possible errors.

4. APPLICATION

The split feasibility problem (SFP), introduced by Censor and Elfving in 1994 [32], is formulated as:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \quad (4.1)$$

where C and Q are closed convex subsets of the Hilbert spaces H_1 and H_2 , respectively, and $A : H_1 \rightarrow H_2$ is a bounded linear operator. Recently, Feng et al. [33] proposed a novel three-step iterative algorithm for solving the split feasibility problem in Hilbert spaces. Under appropriate assumptions, the sequence generated by this new iterative algorithm converges strongly to a solution of the SFP.

Assuming that the SFP (4.1) is consistent (i.e., it admits at least one solution), and denoting by S the solution set of (4.1), it follows from [33] that $x^* \in C$ is a solution of (4.1) if and only if it solves the fixed point problem:

$$x = P_C(I - \gamma A^*(I - P_Q)A)x,$$



where P_C and P_Q represent the nearest point projections onto C and Q , respectively, $\gamma > 0$, and A^* denotes the adjoint operator of A . Consequently, the solution set of the SFP (4.1) coincides with the fixed points of T , i.e., $F(T) = C \cap A^{-1}Q \neq \emptyset$. For further details, we refer to [34, 35].

Furthermore, Byrne [36] proved that if k is the spectral radius of A^*A and $\gamma \in (0, \frac{2}{k})$, then the operator

$$T = P_C(I - \gamma A^*(I - P_Q)A)$$

is averaged and nonexpansive, and the so-called CQ algorithm converges weakly to a solution of the SFP.

Extensive research has been conducted in this area due to its applications in modeling real-world problems, including inverse problems in signal processing, radiotherapy, data compression, among others. Moreover, numerous algorithms have been developed by various authors to solve the SFP and related optimization problems. Notably, Berinde and Pacurar [37] presented two Krasnoselskij projection-type algorithms for solving split feasibility problems and variational inequality problems in the class of enriched mappings. In this work, we propose an alternative approach by considering enriched mappings instead of nonexpansive mappings, which are inherently continuous. We examine an SFP with a unique solution, as we will demonstrate in the following theorem, provided that the proposed algorithm (4.2) converges strongly. Recall that a complete normed space is a $\text{CAT}(0)$ space if and only if it is a Hilbert space. Additionally, note that a $\text{CAT}_p(0)$ space with $p = 2$ coincides with the classical $\text{CAT}(0)$ space.

Theorem 4.1. *Assume that the SFP problem (4.1) is consistent, $\gamma \in (0, \frac{2}{k})$, and $T = P_C(I - \gamma A^*(I - P_Q)A)$ is an enriched mapping. Then, the sequence defined by*

$$\begin{cases} x_{n+1} &= (1 - \alpha_n)y_n \oplus \alpha_n(P_C(I - \gamma A^*(I - P_Q)A))y_n, \\ y_n &= (1 - \beta_n)(P_C(I - \gamma A^*(I - P_Q)A))x_n \oplus \beta_n(P_C(I - \gamma A^*(I - P_Q)A))z_n, \\ z_n &= (1 - \gamma_n)x_n \oplus \gamma_n(P_C(I - \gamma A^*(I - P_Q)A))x_n, \end{cases} \quad (4.2)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[0, 1]$, converges strongly to the unique solution x^* of the SFP problem (4.1) for any initial point $x_0 \in C$.

Proof. We apply Theorem 3.5 for $K = C$ and $T_\lambda = P_C(I - \gamma A^*(I - P_Q)A)$. ■

5. APPLICATION OF CR ITERATION IN IMAGE RECONSTRUCTION

The following papers [38–40] demonstrate that image restoration and signal recovery are interconnected and fundamental fields in information processing, with a significant impact on improving the quality and usefulness of visual and non-visual data in a wide range of practical applications. In this example, we implement a numerical reconstruction scheme based on the CR iteration algorithm in the context of the Split Feasibility Problem (SFP), using the theoretical foundations developed in this paper. Specifically, we simulate a tomographic image reconstruction problem where the forward operator A corresponds to the Radon transform and the constraint sets C and Q are convex subsets of Euclidean spaces, representing feasible solutions in the image and projection domains, respectively. We consider a standard Shepp–Logan phantom $x^* \in \mathbb{R}^n$ of resolution 128×128 and generate its sinogram data $Q = Ax^*$ by applying the Radon transform over 180 uniformly distributed angles in the interval $[0^\circ, 180^\circ)$. The goal is to recover an approximation



of x^* from the sinogram data using the CR iteration, which is well defined under the assumptions introduced in Section 4.

The reconstruction is carried out by iteratively applying the operator

$$T(x) = P_C(x - \gamma A^*(Ax - Q)),$$

where A^* denotes an approximation of the adjoint operator of A via unfiltered back-projection and P_C is the projection onto the box constraint $C = [0, 1]^n$. The iteration parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are chosen as constant sequences in $(0, 1)$, satisfying the requirements of Theorem 4.1.

The numerical results confirm that the CR iteration scheme successfully reconstructs the structural content of the phantom image. The final reconstruction displays meaningful contrast and spatial coherence, even without imposing additional regularization. Furthermore, the histogram of the reconstructed image exhibits a nontrivial distribution of intensity values, indicating that the method avoids convergence to trivial or constant solutions. These outcomes validate the practical applicability of the convergence results obtained in this paper for iterative schemes in split feasibility problems involving composite nonexpansive operators in abstract metric settings.

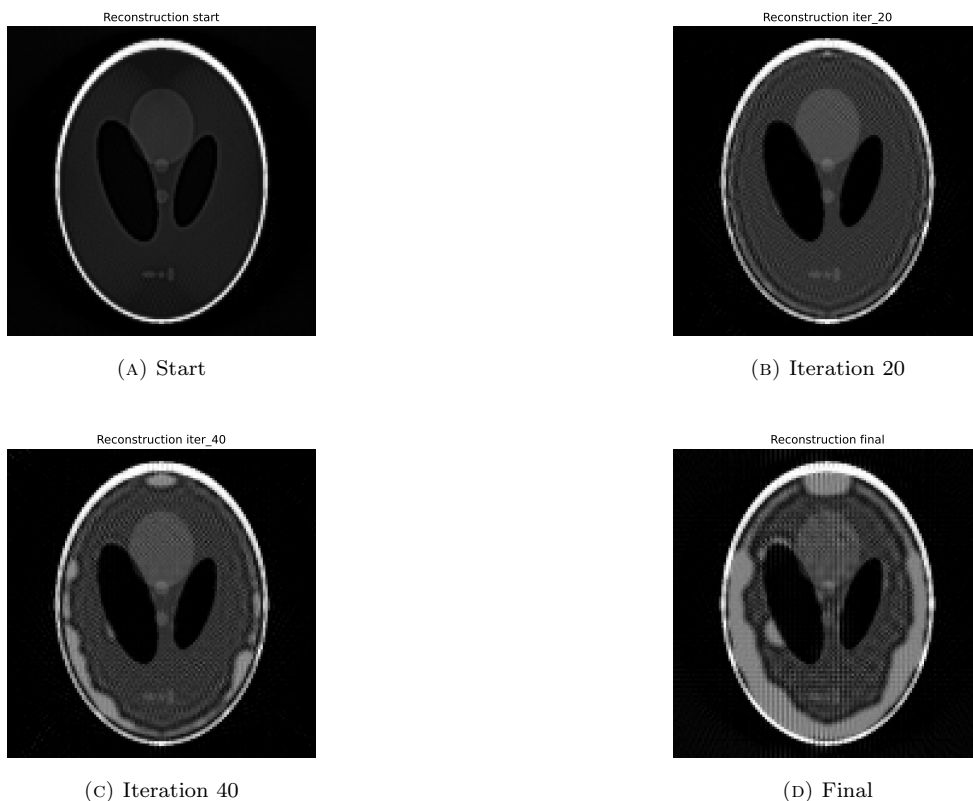


FIGURE 1. Evolution of the reconstructed image using the CR iteration algorithm at selected iterations: initial state, intermediate stages and final result.



TABLE 1. Simulation parameters for the CR iteration reconstruction experiment.

Parameter	Value / Description
Phantom size	128×128 (Shepp–Logan phantom)
Projection angles	180 uniformly spaced in $[0^\circ, 180^\circ)$
Forward operator A	Radon transform
Adjoint operator A^*	Unfiltered backprojection (normalized)
Constraint set C	$[0, 1]^n$ (pixel intensities clipped)
Initial guess x_0	Zero image: $x_0 = \mathbf{0}$
CR iterations n	60
Relaxation parameter γ_n	Constant: 0.4
Weights α_n, β_n	Constant: $\alpha_n = 0.7, \beta_n = 0.5$
Implementation	Python 3.11 with <code>scikit-image</code> , <code>matplotlib</code>
Output format	Reconstruction saved as EPS (300 dpi)

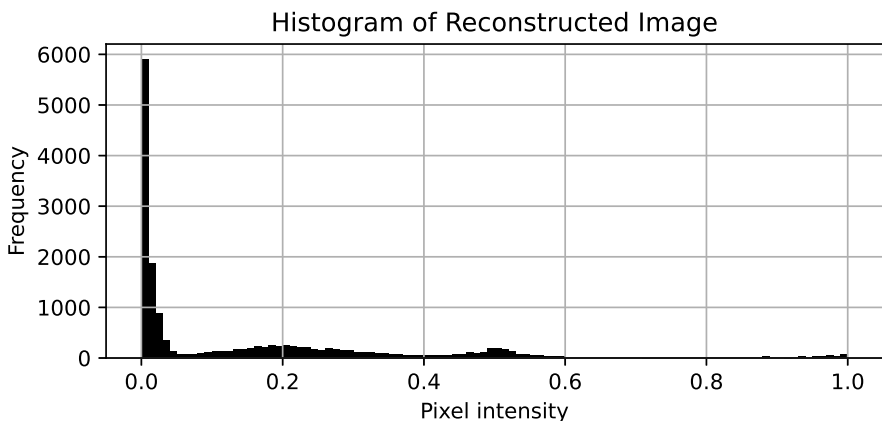


FIGURE 2. Histogram of pixel intensities in the image reconstructed via CR iteration. The distribution confirms that the algorithm avoids convergence to a trivial constant solution and recovers a meaningful range of intensity values.

6. NUMERICAL EXAMPLES

Example 6.1. Let $[0, 1] \subset \mathbb{R}$, where $(\mathbb{R}, d)$ is endowed with the usual metric d induced by the absolute value $|\cdot|$. We claim that $([0, 1], d)$ is a $\text{CAT}_p(0)$ space. In fact, let $x_1, x_2, x_3 \in \Delta \subset [0, 1]$ (Δ is a degenerate triangle). Let us define the comparison triangle $\bar{\Delta}$ in ℓ_p with vertices $\bar{x}_i = (x_i, 0, \dots)$, $i = 1, 2, 3$. Then

$$d(x_i, x_j) = |x_i - x_j|$$

and

$$\|\bar{x}_i - \bar{x}_j\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |x_i^k - x_j^k|^p \right)^{1/p} = |x_i - x_j|.$$



Let us consider the Kannan selfmapping on $[0, 1]$:

$$Tx = \begin{cases} \frac{x}{4}, & x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & x \in [\frac{1}{2}, 1]. \end{cases}$$

The sequence $\{x_n\}$ is generated via the CR iteration (2.5), with z_n , y_n , and x_n defined by (2.2), starting from $x_0 = \frac{1}{2}$. Three configurations of $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, all in $[0, 1]$ and satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, are considered: (i) **Condition A**: $\alpha_n = \beta_n = \gamma_n = 0.5$ (constant case); (ii) **Condition B**: $\alpha_n = \frac{n}{n+1}$, $\beta_n = 0.6$, $\gamma_n = \frac{1}{n+2}$ (increasing $\{\alpha_n\}$, constant $\{\beta_n\}$, decreasing $\{\gamma_n\}$); (iii) **Condition C**: $\alpha_n = 0.7$, $\beta_n = \frac{1}{n+1}$, $\gamma_n = 0.3 + \frac{0.2}{n+1}$ (constant $\{\alpha_n\}$, decreasing $\{\beta_n\}$, $\{\gamma_n\}$ converging to 0.3). In all cases, $x = 0$ is the unique fixed point of T , attained at a comparable rate, with slightly faster convergence under Condition C. See Figure 3.

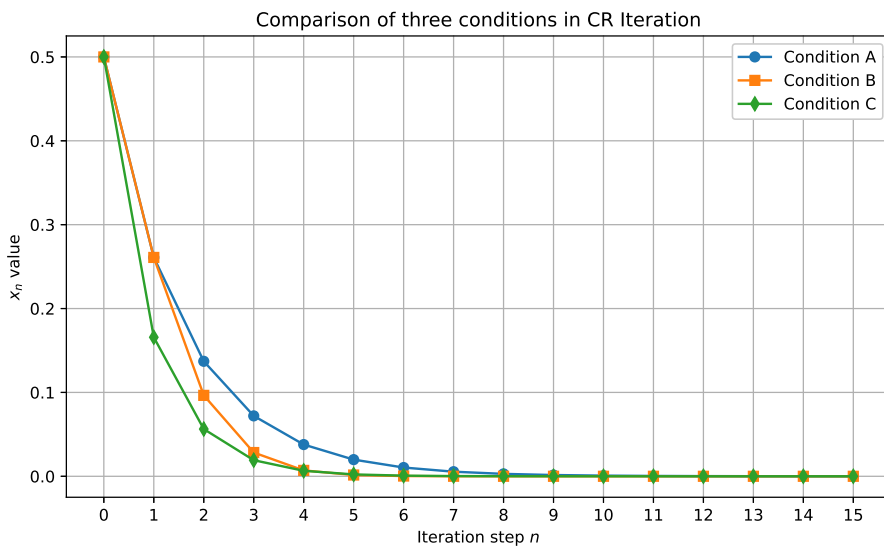


FIGURE 3. Comparison of the convergence behavior of the CR iteration scheme under Conditions A, B, and C for the example 6.1.

Example 6.2. Let $[-1, 1] \subset \mathbb{R}$. Similarly to Example 6.1 we claim that $([-1, 1], d)$ is a $\text{CAT}_p(0)$ space, also the usual metric d induced by the absolute value $|\cdot|$. Let us consider the Kannan selfmapping on $[-1, 1]$:

$$Tx = \begin{cases} \frac{x}{2}, & x \in (-1, 1) \\ 0 & \text{if } x = -1 \text{ or } x = 1. \end{cases}$$

Considering the same sequences of Example 6.1, as well as condition one and condition two. For $x_0 = -\frac{1}{2}$ we have the convergence that is illustrated in the Figure 4. It is clear that the fixed point for T is $x = 0$. We see that under condition two it is slightly faster than **condition C**.



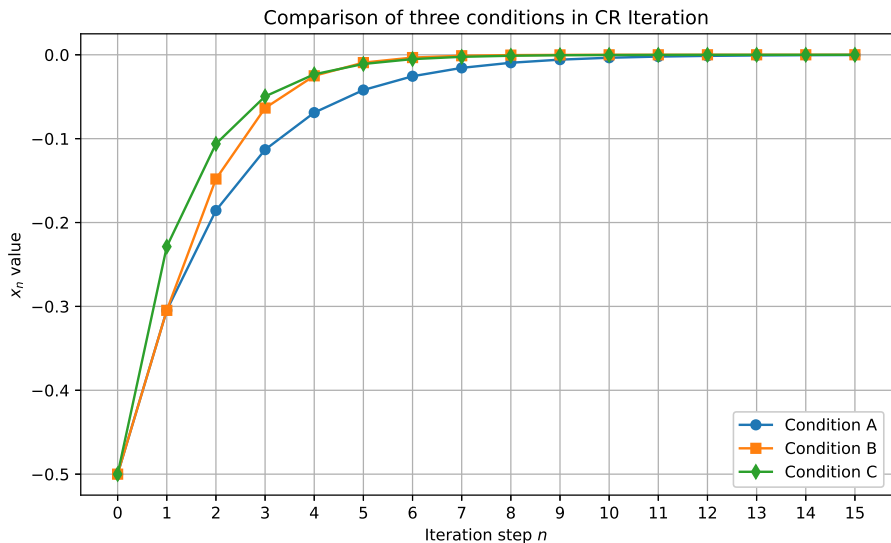


FIGURE 4. Comparison of the convergence behavior of the CR iteration scheme under Conditions A, B, and C for the example 6.2.

Remark 6.3. Our results immediately apply to an $CAT(0)$ space with $p = 2$, because $CAT_2(0)$ space is the classical $CAT(0)$ space.

Remark 6.4. We established some conditions under which a sequence k -converges to a fixed point of an enriched mapping satisfying the demiclosedness-type property.

Remark 6.5. The Lemma 3.2 generalizes Kirk and Panyanak’s Proposition 3.7 of [16] from $CAT(0)$ to $CAT_p(0)$ for enriched mappings.

In essence, this research provides a comprehensive theoretical framework for approximating fixed points of enriched Kannan mappings in the generalized non-linear setting of $CAT_p(0)$ spaces, along with demonstrating the stability and practical utility of the proposed iterative algorithms. The findings extend existing knowledge from $CAT(0)$ to $CAT_p(0)$ spaces for enriched mappings and have substantial implications for solving a broad range of problems in nonlinear analysis, optimization and applied sciences.

COMPETING INTERESTS

The author declares that they have no competing interests. All mathematical content and analysis were solely developed by the author.

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