BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE

Volume 10 (2024)
Pages 10–29

BANGMOD INTERNATIONAL JOURNAL OF
MATHEMATICAL & COMPUTATIONAL SCIENCE ISSN: 2408-154X

https://bangmodjmcs.com/

RELAXED DOUBLE INERTIA AND VISCOSITY ALGORITHMS FOR THE MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS

Solomon Gebregiorgis[1], **Poom Kumam**[2,*]

[1] *Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand*
E-mails: solomonggty@gmail.com
[2] *Center of Excellence in Theoretical and Computational Science (TaCS-CoE), SCL 802 Fixed Point Laboratory, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand*
E-mails: poom.kum@kmutt.ac.th

**Corresponding author.*

Received: 2 April 2024 / Accepted: 3 May 2024

Abstract In this paper, we investigate a multiple-sets split feasibility problem with multiple output sets in infinite-dimensional Hilbert spaces. To address this problem, we propose relaxed inertial self-adaptive algorithms that do not use the least squares method and prove strong convergence results for the sequences generated by these algorithms. Finally, we validate the performances of the proposed algorithms by using a numerical example.

MSC: 47H09

Keywords: Multiple-sets split feasibility problems with multiple output sets; generalized Fermat-Torricelli problem; relaxed; double inertia; viscosity

Published online: 29 June 2024

Please cite this article as: S. Gebregiorgis et al., Relaxed double inertia and viscosity algorithms for the multiple-sets split feasibility problem with multiple output sets, Bangmod J-MCS., Vol. 10 (2024) 10–29.

TaCS Center of Excellence Publications

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https://doi.org/10.58715/bangmodjmcs.2024.10.2

1. Introduction

The split feasibility problem (SFP) which was introduced by Censor and Elfving [1] is stated as follows. Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $T : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is to find a point

$$x^* \in C \text{ such that } Tx^* \in Q. \tag{1.1}$$

The SFP provides a unified model for many inverse problems which have many real life applications such as medical image reconstraction and signal processing (see [2, 3]). Due to its immense potential several generalizations of the SFP have been studied by many authors, see, for instance, the multiple-sets SFP (MSSFP) [4–6], the SFP with multiple output sets (SFPMOS) [7–12], the split variational inequality problem (SVIP) [11, 13, 14], the multiple-sets split variational inequality problem (MSSVIP) [15], the split variational inequality problem with multiple output sets (SVIPMOS) [16], and the multiple-sets split feasibility problem with multiple output sets (MSSFPMOS) [17, 18].

In 2021, Kim et al. [18], introduced the following MSSFPMOS in general Hilbert spaces: Let C_i, $i = 1, 2, \dots, s$, and Q_{jk}, $j = 1, 2, \dots, p$, $k = 1, \cdots, r_j$ be nonempty, closed and convex subsets of real Hilbert spaces H and H_j, respectively, and $T_j : H \to H_j$ be bounded linear operators. The MSSFPMOS is to find an element x^* such that

$$x^* \in \Omega := \Big(C := \cap_{i=1}^{s} C_i\Big) \cap \Big(\cap_{j=1}^{p} T_j^{-1}\Big(\cap_{k=1}^{r_j} Q_{jk}\Big)\Big) \neq \emptyset. \tag{1.2}$$

In 2023, Reich and Tuyen [19, 20] developed self adaptive algorithms for solving the split feasibility problem with multiple output sets which is a special case of the generalized Fermat-Torricelli problem and proved strong and weak convergence theorems. The important advantage of these algorithms is that they do not use the least square approximation unlike most algorithms.

Motivated by the above works specially that of Riech and Tuyen [19] and Kim et al. [18], we propose inertial self-adaptive relaxed CQ-algorithms for solving the MSSFPMOS (1.2) and prove their corresponding strong convergence.

The rest of this paper is organized as follows. We begin by stating some basic definitions and lemmas in Section 2. We give convergence analysis of our proposed algorithms in Section 3. We provide a numerical experiment in Section 4 to validate our proposed algorithms.

2. Preliminaries

This section states some notations, definitions, and lemmas which are required in the proofs of our theorems.

The weak ω-limit set of $\{t_n\}$ is given by

$$\omega_\omega(t_n) = \big\{t \in H : \exists \{t_{n_k}\} \subseteq \{t_n\} \text{ such that } t_{n_k} \rightharpoonup t\big\}.$$

Let C be a nonempty closed convex subset of a real Hilbert space H.

A mapping $S : C \to C$ is called nonexpansive on C if

$$\|Su - Sv\| \le \|u - v\|, \ \forall u, v \in C.$$

It is well known that for every element $a_1 \in H$, there exists a unique nearest point in C, denoted by $P_C(a_1)$ such that

$$\|a_1 - P_C(a_1)\| = \min\{\|a_1 - a_2\| : a_2 \in C\}.$$

The operator $P_C : H \to C$ is called a metric projection of H onto C. It has got important characterization shown below:

$$\langle a_1 - P_C a_1, a_2 - P_C a_1\rangle \leq 0, \tag{2.1}$$

for all $a_1 \in H$ and $a_2 \in C$. We can deduce from (2.1) that the operator P_C is a nonexpansive mapping.

Lemma 2.1. (see [21]) *For all $a_1, a_2 \in H$, the following inequalities hold.*

(1) $\|P_C(a_1) - P_C(a_2)\|^2 \leq \langle P_C(a_1) - P_C(a_2),\ a - b\rangle$;
(2) $\langle a_1 - a_2, (I - P_C)a_1 - (I - P_C)a_2\rangle \geq \|(I - P_C)a_1 - (I - P_C)a_2\|^2$.

Definition 2.2. (see [21]) A $g : H \to (-\infty, +\infty]$ be a given function. Then g is σ-strongly convex if

$$g(\delta a_1 + (1-\delta)a_2) + \frac{\sigma}{2}\delta(1-\delta)\|a_1 - a_2\|^2 \leq \delta g(a_1) + (1-\delta)g(a_2),$$

for all $\delta \in (0, 1)$ and for all $a_1, a_2 \in H$ where $\sigma > 0$.

Lemma 2.3. (see [21]) *Let $g : H \to (-\infty, +\infty]$ be a ϱ-strongly convex function. Then*

$$g(b) \geq g(a) + \langle \xi,\ b - a\rangle + \frac{\varrho}{2}\|b - a\|^2,\ \xi \in \partial g(a),$$

for all $a, b \in H$.

Lemma 2.4. (see [22, 23]) *Let $\{s_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers, such that*

$$s_{n+1} \leq (1 - \sigma_n)s_n + \varepsilon_n + \gamma_n,\ n \geq 1,$$

where $\{\sigma_n\} \subset (0, 1)$ and $\{\varepsilon_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then the following results hold:

(1) *If $\varepsilon_n \leq \sigma_n M$ for some $M \geq 0$, then $\{s_n\}$ is a bounded sequence*;
(2) *If $\sum_{n=1}^{\infty}\sigma_n = \infty$ and $\limsup_{n\to\infty} \frac{\varepsilon_n}{\sigma_n} \leq 0$, then $s_n \to 0$ as $n \to \infty$.*

Lemma 2.5. (see [24]) *Let $\{s_n\}$ be a non-negative real sequence, such that*

$$s_{n+1} \leq (1 - \sigma_n)s_n + \sigma_n\mu_n,\ n \geq 1,$$
$$s_{n+1} \leq s_n - \phi_n + \varphi_n,\ n \geq 1,$$

where $\{\sigma_n\} \subset (0, 1)$, $\{\phi_n\} \subset [0, \infty)$, and $\{\mu_n\}$, $\{\varphi_n\} \subset (-\infty, \infty)$. In addition, suppose the following conditions hold.

(1) $\sum_{n=1}^{\infty}\sigma_n = \infty$;
(2) $\lim_{n\to\infty}\varphi_n = 0$;
(3) $\lim_{k\to\infty}\phi_{n_k} = 0$ *implies* $\limsup_{k\to\infty}\mu_{n_k} \leq 0$ *for every subsequence $\{n_k\}$ of $\{n\}$.*

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. [25] *Let $\{s_k\} \subset (0, \infty)$, $\{t_k\} \subset (-\infty, \infty)$, and $\{u_k\} \subset (0, 1)$ satisfying $\sum_{n=1}^{\infty}u_k = \infty$, and $s_{k+1} \leq (1 - u_k)s_k + u_k t_k$, $\forall k \geq 1$. If $\limsup_{i\to\infty} t_{k_i} \leq 0$ and for every subsequence $\{s_{k_i}\}$ of $\{s_k\}$, $\liminf_{i\to\infty}(s_{k_i+1} - s_{k_i}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.*

3. Main Results

In this section, we state our algorithms and analyze their convergence. For simplicity let $J_1 := \{1, 2, \dots, s\}$, $J_2 := \{1, 2, \dots, p\}$, and $J_3 := \{1, 2, \dots, r_j\}$.

We take the following assumptions to undergo the analysis.

(C1) The nonempty level sets C and Q in the MSSFPMOS (1.2) are defined as follows

$$C_i = \{x \in H : c_i(x) \leq 0\} \text{ and } Q_{jk} = \{y \in H_j : q_{jk}(y) \leq 0\}, \tag{3.1}$$

where $c_i : H \to (-\infty, +\infty]$ for all $i \in J_1$ and $q_{jk} : H_j \to (-\infty, +\infty]$ for all $j \in J_2, k \in J_3$ are ϖ_i-strongly and ω_j-strongly convex subdifferentiable function, respectively. Then c_i and q_{jk} are also lower semicontinuous (See, [21] Theorem 9.1)

(C2) Let c_i and q_{jk} defined in (3.1), respectively. Assume that at least one subgradient $\xi_i \in \partial c_i(x)$ and $\mu_{jk} \in \partial q_{jk}(y)$ can be computed for any $x \in H$ and $y \in H_j$. Moreover, both $\partial c_i (i \in J_1)$ and $\partial q_{jk} (j \in J_2, k \in J_3)$ are bounded operators (bounded on bounded sets). The sets C_i^n $(i \in J_1)$ and Q_{jk}^n $(j \in J_2, k \in J_3)$ are constructed as follows:

$$C_i^n = \Big\{x \in H : c_i(x_n) + \langle \xi_i^n,\ x - x_n\rangle + \frac{\varpi_i}{2}\|x - x_n\|^2 \leq 0\Big\}, \tag{3.2}$$

where $\xi_i^n \in \partial c_i(x_n)$ and

$$Q_{jk}^n = \Big\{y \in H_j : q_{jk}(T_j x_n) + \langle \eta_{jk}^n,\ y - T_j x_n\rangle + \frac{\omega_j}{2}\|y - T_j x_n\|^2 \leq 0\Big\}, \tag{3.3}$$

where $\eta_{jk}^n \in \partial q_{jk}(T_j x_n)$.

Now, we introduce our proposed algorithms for solving the MSSFPMOS (1.2).

Algorithm 1 A strongly convergent method with double inertial steps for solving the MSSFPMOS (1.2)

Step 0. Choose the sequences $\{\sigma_n\} \subset [0, 1)$, $\{\rho_n\} \subset (0, 2)$, and $\{\theta_n\} \subset [0, 1)$. Take the weights α_i^n $(i \in J_1) > 0$ and the constant parameters β_{jk} $(j \in J_2, k \in J_3) > 0$ such that

$$\sum_{i=1}^{s} \alpha_i^n = 1 \text{ and } \inf_{i \in I_n} \alpha_i^n > \alpha > 0, \text{ where } I_n = \{i \in J_1 : \alpha_i^n > 0\}. \tag{3.4}$$

Select initial points $t_0, t_1 \in C$. Assume that t_n has been constructed.

Step 1. For t_{n-1} and t_n , choose $\theta \in (0, 1)$ such that $0 \leq \theta_n \leq \hat{\theta}_n$ where

$$\tilde{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{||t_n - t_{n-1}||}\right\} & \text{if } t_n \neq t_{n-1}, \\ \theta & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\epsilon_n \subset (0, \infty)$ such that $\lim_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0$.

Step 2. Compute

$$w_n = t_n + \theta_n(t_n - t_{n-1}). \tag{3.6}$$

Step 3. Compute

$$v_n = (1 - \sigma_n)w_n. \tag{3.7}$$

Step 4. Compute

$$d_{jk}^n = \begin{cases} \dfrac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n}{\left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n\right\|} & \text{if } T_j v_n \notin Q_{jk}^n, \\ 0 & \text{if } T_j v_n \in Q_{jk}^n, \end{cases} \tag{3.8}$$

for all $j \in J_2$ and $k \in J_3$.

Step 5. If $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n = 0$, then stop. If not, compute t_{n+1} via

$$t_{n+1} = \sum_{i=1}^{s} \alpha_i^n P_{C_i^n} \left(v_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right), \tag{3.9}$$

where C_i^n and Q_{jk}^n are defined as in (3.2) and (3.3), respectively and

$$\tau_n := \frac{\rho_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} f_{jk}(T_j v_n)}{\left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2}, \tag{3.10}$$

where $f_{jk}(v) = \left\| \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j} \right)(v) \right\|$ for all $v \in \mathcal{H}_j$ and for all $j \in J_2, k \in J_3$.

Step 6. Set $n := n + 1$ and return to **Step 1.**

Proposition 3.1. *In Algorithm 1, if $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n = 0$, then v_n is a solution to the MSSFPMOS (1.2).*

Proof. For each n, let $\Delta_n = \{(j,k) : d_{jk}^n \neq 0\}$.
Following the same line of proof as in Proposition 6 of [26], we get $\left\| \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j} \right) T_j v_n \right\| = 0$ for all $(j,k) \in \Delta_n$ which in turn implies that $T_j v_n \in Q_{jk}^n$ for all $(j,k) \in \Delta_n$. If $(j,k) \notin \Delta_n$, then $d_{jk}^n = 0$ which in turn implies that $T_j v_n \in Q_{jk}^n$. Consequently, $T_j v_n \in Q_{jk}^n$ for all $j \in J_2$ and $k \in J_3$. Now, since $v_n \in C_i^n$ and $T_j v_n \in Q_{jk}^n$ for all $j \in J_2, k \in J_3$, we conclude that v_n is a solution to the MSSFPMOS (1.2). ∎

Next, we analyze the convergence of Algorithm 1.

Theorem 3.2. *Let $\Omega \neq \emptyset$. Then the sequence $\{t_n\}$ generated by Algorithm 1 converges strongly to a point $t^* = P_\Omega 0$ under the following conditions.*

(A1) $\{\sigma_n\} \subset [0,1)$ *such that* $\lim_{k\to\infty} \sigma_n = 0$ *and* $\sum_{k=0}^{\infty} \sigma_n = \infty$.

(A2) $\{\rho_n\} \subset (0,\infty)$ *such that* $0 < \underline{\rho} \leq \inf_n \rho_n \leq \sup_n \rho_n \leq \overline{\rho} < 2$.

Proof. Let $t^* \in \Omega$, then we have

$$\begin{aligned} &\left\| v_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n - t^* \right\|^2 \\ = &\left\| (v_n - t^*) - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \left\langle \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n, v_n - t^* \right\rangle \\
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle T_j^* d_{jk}^n, v_n - t^* \right\rangle \\
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle d_{jk}^n, T_j v_n - T_j t^* \right\rangle \\
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \\
&\quad -2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle \frac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n}{\left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n\right\|}, T_j v_n - T_j t^* \right\rangle \\
&= \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 \\
&\quad -2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\langle \frac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n - \left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j t^*}{\left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n\right\|}, T_j v_n - T_j t^* \right\rangle \\
&\leq \|v_n - t^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \right\|^2 - 2\tau_n \sum_{(j,k)\in\Delta_n} \beta_{jk} \left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j v_n\right\|.
\end{aligned}
\tag{3.11}
$$

Substituting (3.10) into (3.11) and simplifying, we get

$$
\left\| v_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n - t^* \right\|^2 \leq \|v_n - t^*\|^2 - \rho_n(2-\rho_n) g_{jk}(v_n) \tag{3.12}
$$

$$
\leq \|v_n - t^*\|^2, \tag{3.13}
$$

where $g_{jk}(v_n) = \left(\frac{\sum_{(j,k)\in\Delta_n} \beta_{jk} f_{jk}(T_j v_n)}{\left\|\sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n\right\|} \right)^2$.

By Lemma 2.1 (2), (3.9), and (3.13), we also obtain

$$\begin{aligned}\|t_{n+1}-t^*\| &= \left\|\sum_{i=1}^{s}\alpha_i^n P_{C_i^n}\left(v_n-\tau_n\sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^n\right)-t^*\right\| \\ &= \left\|\sum_{i=1}^{s}\alpha_i^n P_{C_i^n}\left(v_n-\tau_n\sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^n\right)-\sum_{i=1}^{s}\alpha_i^n P_{C_i^n}t^*\right\| \\ &\leq \left\|v_n-\tau_n\sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^n-t^*\right\| \quad (3.14) \\ &\leq \|v_n-t^*\|. \quad (3.15)\end{aligned}$$

Using (3.6), we have

$$\begin{aligned}\|w_n-t^*\| &= \|t_n+\theta_n(t_n-t_{n-1})-t^*\| \\ &\leq \|t_n-t^*\|+\theta_n\|t_n-t_{n-1}\| \\ &= \|t_n-t^*\|+\sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right]. \quad (3.16)\end{aligned}$$

Using (3.7) and (3.16), we also obtain the following estimation

$$\begin{aligned}\|v_n-t^*\| &= \|(1-\sigma_n)(w_n-t^*)-\sigma_n t^*\| \\ &\leq (1-\sigma_n)\|w_n-t^*\|+\sigma_n\|t^*\| \\ &\leq (1-\sigma_n)\left[\|t_n-t^*\|+\theta_n\|t_n-t_{n-1}\|\right]+\sigma_n\|t^*\| \\ &= (1-\sigma_n)\|t_n-t^*\|+(1-\sigma_n)\theta_n\|t_n-t_{n-1}\|+\sigma_n\|t^*\| \\ &\leq (1-\sigma_n)\|t_n-t^*\|+\sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|+\|t^*\|\right]. \quad (3.17)\end{aligned}$$

Therefore, we have from (3.15) and (3.17) that

$$\|t_{n+1}-t^*\| \leq (1-\sigma_n)\|t_n-t^*\|+\sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|+\|t^*\|\right]. \quad (3.18)$$

Since $\lim_{n\to\infty}\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\| = 0$, the sequence $\left\{\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right\}$ is bounded. Now, setting

$$M_{max} = \max\left\{\sup_{n\in\mathbb{N}}\left\{\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right\},\ \|t^*\|\right\}. \quad (3.19)$$

We can see that all the conditions of Lemma 2.4 are satisfied and hence the sequence $\{\|t_n-t^*\|\}$ is bounded. This in turn implies that the sequence $\{t_n\}$ is bounded. Consequently, $\{w_n\}$, $\{v_n\}$, and $\{T_j v_n\}$ (for each $j\in J_2$) are also bounded.
Using (3.6), we have

$$\begin{aligned}\|w_n-t^*\|^2 &= \|t_n+\theta_n(t_n-t_{n-1})-t^*\|^2 \\ &= \|(t_n-t^*)+\theta_n(t_n-t_{n-1})\|^2 \\ &\leq \|t_n-t^*\|^2+\theta_n^2\|(t_n-t_{n-1})\|^2+2\theta_n\langle t_n-t^*, t_n-t_{n-1}\rangle \\ &\leq \|t_n-t^*\|^2+\sigma_n\left[\frac{\theta_n^2}{\sigma_n^2}\|t_n-t_{n-1}\|^2+2\frac{\theta_n}{\sigma_n}\|t_n-t^*\|\|t_n-t_{n-1}\|\right].\end{aligned}$$

Using (3.7) and (3.20), we have

$$\begin{aligned}
& \|v_n - t^*\|^2 = \|(1-\sigma_n)w_n - t^*\|^2 \\
= \quad & \|(1-\sigma_n)(w_n - t^*) - \sigma_n t^*\|^2 \\
\leq \quad & (1-\sigma_n)^2\|w_n - t^*\|^2 + 2\sigma_n\langle v_n - t^*, -t^*\rangle \\
\leq \quad & (1-\sigma_n)\|t_n - t^*\|^2 + \sigma_n\bigg[\frac{\theta_n^2}{\sigma_n^2}\|t_n - t_{n-1}\|^2 + 2\frac{\theta_n}{\sigma_n}\|t_n - t^*\|\|t_n - t_{n-1}\| \\
& +2\|v_n - t_{n+1}\|\|t^*\|\bigg] + 2\sigma_n\langle t_{n+1} - t^*, -t^*\rangle. \qquad (3.20)
\end{aligned}$$

Using (3.13), (3.14), and (3.20), we also obtain

$$\begin{aligned}
\|t_{n+1} - t^*\|^2 \quad \leq \quad & \left\|v_n - \tau_n \sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^n - t^*\right\|^2 \\
\leq \quad & \|v_n - t^*\|^2 - \rho_n(1-\rho_n)g_{jk}(v_n) \\
\leq \quad & (1-\sigma_n)\|t_n - t^*\|^2 + \sigma_n\Bigg[\left(\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\|\right)^2 \\
& +2\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\|\|t_n - t^*\| + 2\|v_n - t_{n+1}\|\|t^*\| + 2\langle -t^*, t_{n+1} - t^*\rangle\Bigg] \\
& -\rho_n(2-\rho_n)g_{jk}(v_n). \qquad (3.21)
\end{aligned}$$

Now, letting

$$\begin{aligned}
s_n \quad &= \quad \|t_n - t^*\|^2, \\
\phi_n \quad &= \quad \left(\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\|\right)^2 + 2\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\|\|t_n - t^*\| \\
& \quad\quad +2\|v_n - t_{n+1}\|\|t^*\| + 2\langle -t^*, t_{n+1} - t^*\rangle, \\
\varphi_n \quad &= \quad \sigma_n\phi_n, \text{ and} \\
\mu_n \quad &= \quad \rho_n(2-\rho_n)g_{jk}(v_n).
\end{aligned}$$

Now, (3.21) reduces to

$$s_{n+1} \leq (1-\sigma_n)s_n + \sigma_n\phi_n \text{ and } s_{n+1} \leq (1-\sigma_n)s_n - \mu_n + \varphi_n. \qquad (3.22)$$

Assume, $\lim_{l\to\infty}\mu_{n_l} = 0$. It follows that

$$\lim_{l\to\infty}\frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_j v_{n_l})}{\left\|\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^* d_{jk}^{n_l})\right\|} = 0, \qquad (3.23)$$

for all $(j,k)\in\Delta_{n_l}$. By using $\|d_{jk}^{n_l}\| = 1$ for all $j\in J_2$ and $k\in J_3$, we get

$$0 \leq \frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_j v_{n_l})}{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}\|T_j^*\|} \leq \frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_j v_{n_l})}{\left\|\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^* d_{jk}^{n_l})\right\|}.$$

It follows from (3.24) that, $\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_j v_{n_l}) = 0$, or equivalently

$$\lim_{l\to\infty}\|(I^{\mathcal{H}_j} - P^{\mathcal{H}_j}_{Q_{jk}^{n_l}})T_j v_{n_l}\| = 0,$$

for all $(j,k) \in \Delta_{n_l}$.
From the definition of Δ_{n_l} and $d_{jk}^{n_l}$, we have $T_j v_{n_l} \in Q_{jk}^{n_l}$ when $(j,k) \notin \Delta_{n_l}$ and hence $\|(I^{\mathcal{H}_j} - P^{\mathcal{H}_j}_{Q_{jk}^{n_l}})T_j v_{n_l}\| = 0$. As a result, we get

$$\lim_{l\to\infty} \|(I^{\mathcal{H}_j} - P^{\mathcal{H}_j}_{Q_{jk}^{n_l}})T_j v_{n_l}\| = 0, \tag{3.24}$$

for all $j \in J_2$ and $k \in J_3$.
Using (3.7), we get

$$\begin{aligned} \|v_n - t_n\| &= \big\|(1-\sigma_n)w_n - t_n\big\| \\ &= \big\|(1-\sigma_n)\theta_n(t_n - t_{n-1}) - \sigma_n t_n\big\| \\ &\le (1-\sigma_n)\theta_n\|t_n - t_{n-1}\| + \sigma_n\|t_n\| \\ &= \sigma_n\Big[(1-\sigma_n)\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| + \|t_n\|\Big]. \end{aligned} \tag{3.25}$$

By Lemma 2.1 (2) and (3.9), we obtain

$$\begin{aligned} \|t_{n_l+1} - v_{n_l}\| &= \Big\|\sum_{i=1}^{s}\alpha_i^{n_l}P_{C_i^{n_l}}\Big(v_{n_l} - \tau_{n_l}\sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^{n_l}\Big) - \sum_{i=1}^{s}\alpha_i^{n_l}P_{C_i^{n_l}}v_{n_l}\Big\| \\ &\le \tau_{n_l}\Bigg\|\sum_{j=1}^{p}\sum_{k=1}^{r_j}\beta_{jk}T_j^* d_{jk}^{n_l}\Bigg\| \\ &= \rho_{n_l} g_{jk}(v_{n_l}) \\ &\le \bar{\rho} g_{jk}(v_{n_l}) \to 0 \text{ as } l \to \infty, \end{aligned} \tag{3.26}$$

which implies that

$$\lim_{l\to\infty} \|t_{n_l+1} - v_{n_l}\| = 0. \tag{3.27}$$

Moreover, since

$$\|t_{n_l+1} - t_{n_l}\| \le \|t_{n_l+1} - v_{n_l}\| + \|v_{n_l} - t_{n_l}\|,$$

we obtain that

$$\lim_{l\to\infty} \|t_{n_l+1} - t_{n_l}\| = 0. \tag{3.28}$$

Next, we need to show that $\omega_w(t_n) \subset \Omega$. Since $\{t_n\}$ is bounded, $\omega_w(t_n) \neq \emptyset$. Let $\bar{t} \in \omega_w(t_n)$. It follows that there exists a subsequence $\{t_{n_l}\}$ of $\{t_n\}$ such that $t_{n_l} \rightharpoonup \bar{t}$. Since $\|v_n - t_n\| \to 0$, we have $v_{n_l} \rightharpoonup \bar{t}$. Now, due to the linearity and boundedness of T_j, we have $T_j v_{n_l} \rightharpoonup T_j\bar{t}$.

We claim that $\bar{t} \in \Omega$. To show this it is suffices to show that $\bar{t} \in C_i^n$ for all $i \in J_1$ and $T_j(\bar{t}) \in Q_{jk}^n$ for all $j \in J_2, k \in J_3$.
From the assumption (C2), we can see that ∂q_{jk} is bounded on bounded set for each $j \in J_2, k \in J_3$. It follows that we can find a constant $\eta > 0$ such that $\|\eta_{jk}^{n_l}\| \le \eta$, where $\eta_{jk}^{n_l} \in \partial q_{jk}(T_j v_{n_l})$ for each $j \in J_2, k \in J_3$.

Now, using (3.3), (3.24), and the fact that $P_{Q_{jk}}^{n_l}\Big(T_j v_{n_l}\Big) \in Q_{jk}^{n_l}$, we get

$$\begin{aligned} q_{jk}\Big(T_j v_{n_l}\Big) &\leq \Big\langle \eta_{jk}^{n_l},\ T_j v_{n_l} - P_{Q_{jk}^{n_l}}\Big(T_j v_{n_l}\Big)\Big\rangle - \frac{\omega_j}{2}\Big\|T_j v_{n_l} - P_{Q_{jk}^{n_l}}\Big(T_j v_{n_l}\Big)\Big\|^2 \\ &\leq \Big\langle \eta_{jk}^{n_l},\ T_j v_{n_l} - P_{Q_{jk}^{n_l}}\Big(T_j v_{n_l}\Big)\Big\rangle \\ &\leq \Big\|\eta_{jk}^{n_l}\Big\|\Big\|\Big(I - P_{Q_{jk}^{n_l}}\Big)T_j v_{n_l}\Big\| \\ &\leq \eta\Big\|\Big(I - P_{Q_{jk}^{n_l}}\Big)T_j v_{n_l}\Big\| \to 0. \end{aligned} \tag{3.29}$$

Noting q_{jk} is weakly lower semi-continuous, it follows that

$$q_{jk}(T_j\bar{t}) \leq \liminf_{l\to\infty} q_{jk}\Big(T_j v_{n_l}\Big) \leq \lim_{l\to\infty} \eta\Big\|\Big(I - P_{Q_{jk}^{n_l}}\Big)T_j v_{n_l}\Big\| = 0,$$

for all $j \in J_2, k \in J_3$. It turns out that, $T_j\bar{t} \in Q_{jk}$ for all $j \in J_2, k \in J_3$.

Again, from the assumption (C2), we can see that ∂c_i is bounded on bounded set for each $i \in J_1$. It follows that there is a constant $\xi > 0$ such that $\|\xi_i^{n_l}\| \leq \xi$, where $\xi_i^{n_l} \in \partial c_i(v_{n_l})$. That is the sequence $\{\xi_i^{n_l}\}$ is bounded.
By using (3.2) and (3.27), we have as $l \to \infty$ that

$$\begin{aligned} c_i(v_{n_l}) &\leq \Big\langle \xi_i^{n_l},\ v_{n_l} - t_{n_l+1}\Big\rangle - \frac{\varpi_i}{2}\Big\|v_{n_l} - t_{n_l+1}\Big\|^2 \\ &\leq \Big\|\xi_i^{n_l}\Big\|\Big\|v_{n_l} - t_{n_l+1}\Big\| \\ &\leq \xi\Big\|v_{n_l} - t_{n_l+1}\Big\| \to 0. \end{aligned} \tag{3.30}$$

Noting c_i is weakly lower semi-continuous, it follows that

$$c_i(\bar{t}) \leq \liminf_{l\to\infty} c_i(v_{n_l}) \leq \lim_{l\to\infty} \xi\Big\|v_{n_l} - t_{n_l+1}\Big\| = 0,$$

for all $i \in J_1$. Thus, $\bar{t} \in C_i^n$, i.e., $\omega_\omega(t_n) \subseteq \Omega$ for each $i \in J_1$.

For $t^* = P_\Omega 0$ and $t_{n_l} \rightharpoonup \bar{t} \in \Omega$, it follows from (2.1) that $\langle -t^*, \bar{t} - t^*\rangle \leq 0$. So,

$$\limsup_{n\to\infty}\langle -t^*, t_n - t^*\rangle = \limsup_{l\to\infty}\langle -t^*, t_{n_l} - t^*\rangle = \langle -t^*, \bar{t} - t^*\rangle \leq 0. \tag{3.31}$$

It follows from (3.28) and (3.31), that

$$\limsup_{n\to\infty}\langle -t^*, t_{n+1} - t^*\rangle = \limsup_{n\to\infty}(\langle -t^*, t_{n+1} - t_n\rangle + \langle -t^*, t_n - t^*\rangle) \leq 0, \tag{3.32}$$

and hence

$$\begin{aligned} \limsup_{l\to\infty}\phi_{n_l} = \limsup_{l\to\infty}\Bigg[\bigg(\frac{\theta_{n_l}}{\sigma_{n_l}}\|t_{n_l} - t_{n_l-1}\|\bigg)^2 + 2\frac{\theta_{n_l}}{\sigma_{n_l}}\|t_{n_l} - t_{n_l-1}\|\|t_{n_l} - t^*\| \\ + 2\|v_n - t_{n+1}\|\|t^*\| + \langle -t^*, t_{n_l+1} - t^*\rangle\Bigg] \leq 0. \end{aligned}$$

Now, applying Lemma 2.5 to (3.21), we have that $\lim\limits_{n\to\infty}\|t_n - t^*\| = 0$. This completes the proof of Theorem 3.2.

∎

Our second proposed algorithm is given below.

Algorithm 2 A strongly convergent viscosity type method for solving the MSSFPMOS (1.2)

Step 0. Choose the sequences $\{\sigma_n\} \subset [0,1)$, $\{\rho_n\} \subset (0,2)$, and $\{\theta_n\} \subset [0,1)$. Take the weights α_i^n $(i \in J_1) > 0$ and the constant parameters β_{jk} $(j \in J_2, k \in J_3) > 0$ such that

$$\sum_{i=1}^{s} \alpha_i^n = 1 \text{ and } \inf_{i \in I_n} \alpha_i^n > \alpha > 0, \text{ where } I_n = \{i \in J_1 : \alpha_i^n > 0\}. \tag{3.33}$$

Select initial points $t_0, t_1 \in C$. Assume that t_n has been constructed.

Step 1. For t_{n-1} and t_n , choose $\theta \in (0,1)$ such that $0 \leq \theta_n \leq \hat{\theta}_n$ where

$$\tilde{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{||t_n - t_{n-1}||}\right\} & \text{if } t_n \neq t_{n-1}, \\ \theta & \text{otherwise,} \end{cases} \tag{3.34}$$

where $\epsilon_n \subset (0,\infty)$ such that $\lim_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0$.

Step 2. Given the iterates $t_{n-1}, t_n \in H$, then compute

$$w_n = t_n + \theta_n(t_n - t_{n-1}). \tag{3.35}$$

Step 3. Compute

$$d_{jk}^n = \begin{cases} \dfrac{\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j w_n}{\left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right) T_j w_n\right\|} & \text{if } T_j w_n \notin Q_{jk}^n, \\ 0 & \text{if } T_j w_n \in Q_{jk}^n, \end{cases} \tag{3.36}$$

for all $j \in J_2$ and $k \in J_3$.

Step 4. If $\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* v_{jk}^n = 0$, then stop. If not, compute z_n via

$$z_n = \sum_{i=1}^{s} \alpha_i^n P_{C_i^n}\left(w_n - \tau_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\right), \tag{3.37}$$

where C_i^n and Q_{jk}^n are defined as in (3.2) and (3.3), respectively and

$$\tau_n := \frac{\rho_n \sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} f_{jk}(T_j w_n)}{\left\|\sum_{j=1}^{p} \sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\right\|}, \tag{3.38}$$

where $f_{jk}(w) = \left\|\left(I^{\mathcal{H}_j} - P_{Q_{jk}^n}^{\mathcal{H}_j}\right)(w)\right\|$ for all $w \in \mathcal{H}_j$, and for all $j \in J_2, k \in J_3$.

Step 5.

$$t_{n+1} = \sigma_n v(t_n) + (1 - \sigma_n) z_n, \tag{3.39}$$

where $v : \mathcal{H} \to C$ is a λ-contraction mapping where $\lambda \in [0,1)$.

Step 6. Set $n := n + 1$ and return to **Step 1.**

Theorem 3.3. *Assume that $\Omega \neq \emptyset$. Then the sequence $\{t_n\}$ generated by Algorithm 2 converges strongly to a point $t^* = P_\Omega v(t^*)$ under the assumption (A1) and (A2).*

Proof. Following the same steps as in getting inequalities (3.11) to (3.15), we get

$$\begin{aligned} \|z_n - t^*\| &= \Big\| \sum_{i=1}^{s} \alpha_i^n P_{C_i^n}\Big(w_n - \tau_n \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\Big) - t^* \Big\| \\ &\leq \|w_n - t^*\|. \end{aligned} \tag{3.40}$$

By using (3.39) and (3.40), we have

$$\begin{aligned} \|t_{n+1} - t^*\| &= \|\sigma_n(v(t_n) - t^*) + (1-\sigma_n)(z_n - t^*)\| \\ &= \sigma_n\|v(t_n) - t^*\| + (1-\sigma_n)\|z_n - t^*\| \\ &\leq \sigma_n\|v(t_n) - v(t^*)\| + \sigma_n\|v(t^*) - t^*\| + (1-\alpha_n)\|z_n - t^*\| \\ &\leq \lambda\sigma_n\|t_n - t^*\| + \sigma_n\|v(t^*) - t^*\| + (1-\sigma_n)\|z_n - t^*\| \\ &\leq \sigma_n\|v(t_n) - v(t^*)\| + \sigma_n\|v(t^*) - t^*\| + (1-\alpha_n)\|z_n - t^*\| \\ &\leq \lambda\sigma_n\|t_n - t^*\| + \sigma_n\|v(t^*) - t^*\| + (1-\sigma_n)\|w_n - t^*\|. \end{aligned} \tag{3.41}$$

Using (3.16) and (3.41), we have

$$\|t_{n+1}-t^*\| \leq [1-(1-\lambda)\sigma_n]\|t_n-t^*\|+\sigma_n\left[\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|+\|v(t^*)-t^*\|\right]. \tag{3.42}$$

By the condition of σ_n, we have $\lim_{n\to\infty} \frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| = 0$. Hence, we can find a constant $M \geq 0$ such that

$$\frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| \leq M.$$

Now, (3.42) becomes

$$\begin{aligned} \|t_{n+1} - t^*\| &\leq [1-(1-\eta)\sigma_n]\|t_n - t^*\| + \sigma_n\big[M + \|v(t) - t^*\|\big] \\ &= [1-(1-\eta)\sigma_n]\|t_n - t^*\| + \sigma_n(1-\eta)\left[\frac{M + \|v(t) - t^*\|}{1-\eta}\right]. \end{aligned}$$

Proceeding inductively, we arrive at

$$\|t_{n+1} - t^*\| \leq \max\left\{\|t_1 - t^*\|, \frac{M + \|v(t^*) - t^*\|}{1-\eta}\right\},$$

for all $n \geq 1$ which proves that $\{t_n\}$ is bounded.
Again, using (3.39), we have

$$\begin{aligned} \|t_{n+1} - t^*\|^2 &= \|\sigma_n(v(t_n) - t^*) + (1-\sigma_n)(z_n - t^*)\|^2 \\ &= \|\sigma_n(v(t_n) - v(t^*) + v(t^*) - t^*) + (1-\sigma_n)(z_n - t^*)\|^2 \\ &= \|[\sigma_n(v(t_n) - v(t^*)) + (1-\sigma_n)(z_n - t^*)] + \sigma_n(v(t^*) - t^*)\|^2 \\ &\leq ||\sigma_n(v(t_n) - v(t^*)) + (1-\sigma_n)(z_n - t^*)||^2 + \\ &\quad + 2\sigma_n\langle v(t^*) - t^*, t_{n+1} - t^*\rangle \\ &\leq \sigma_n||v(t_n) - v(t^*)||^2 + (1-\sigma_n)||z_n - t^*||^2 \\ &\quad + 2\sigma_n\langle v(t^*) - t^*, t_{n+1} - t^*\rangle \\ &\leq \lambda\sigma_n||t_n - t^*||^2 + (1-\sigma_n)||z_n - t^*||^2 \\ &\quad + 2\sigma_n\langle v(t^*) - t^*, t_{n+1} - t^*\rangle. \end{aligned}$$

(3.43)

Using the inequality(3.15) (after replacing t_{n+1} for z_n and w_n for v_n), (3.20), and (3.43), we get

$$\begin{aligned}\|t_{n+1}-t^*\|^2 \quad &\leq \quad [1-(1-\lambda)\sigma_n]\|t_n-t^*\|^2\\ &\quad + \sigma_n\left[\left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)^2+2\left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)\|t_n-t^*\|\right]\\ &\quad -\rho_n(1-\rho_n)g_{jk}(v_n)+2\sigma_n\langle v(t^*)-t^*,t_{n+1}-t^*\rangle.\end{aligned} \tag{3.44}$$

It follows that

$$\rho_n(1-\rho_n)g_{jk}(v_n) \quad \leq \quad \|t_n-t^*\|^2-\|t_{n+1}-t^*\|^2+\sigma_n N, \tag{3.45}$$

where

$N \ = \ \sup_{n\geq 1}\left\{\left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)^2+2\|t_n-t^*\|\left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)+2\langle v(t^*)-t^*,t_{n+1}-t^*\rangle\right\}.$

Let $q_n := \|t_n-t^*\|^2$ and

$\gamma_n := \left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)^2+2\|t_n-t^*\|\left(\frac{\theta_n}{\sigma_n}\|t_n-t_{n-1}\|\right)+2\langle v(t^*)-t^*,t_{n+1}-t^*\rangle.$

Then (3.44) becomes

$$q_{n+1} \ \leq \ \Big[1-(1-\lambda)\sigma_n\Big]q_n+\sigma_n\gamma_n. \tag{3.46}$$

Our next task is to show the strong convergence of the sequence $\{t_n\}$ to t^*.
With out loss of generality, we can assume that q_n has a subsequence $\{q_{n_l}\}$ such that

$$\liminf_{l\to\infty}(q_{n_l+1}-q_{n_l}) \ \geq \ 0. \tag{3.47}$$

Passing limit supremum on both sides of (3.45) and using conditions (A1) and (A2), we get

$$\begin{aligned}\limsup_{l\to\infty} g_{jk}(v_{n_l}) \quad &\leq \quad \frac{1}{\underline{\rho}(2-\overline{\rho})}\left(\limsup_{l\to\infty}(q_{n_l}-q_{n_l+1})+N\limsup_{i\to\infty}\sigma_{n_l}\right)\\ &= \quad -\liminf_{l\to\infty}(q_{n_l+1}-q_{n_l})+N\limsup_{i\to\infty}\sigma_{n_l}\\ &\leq \quad 0.\end{aligned}$$

It follows that

$$\lim_{l\to\infty}\frac{\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}f_{jk}(T_jw_{n_l})}{\left\|\sum_{(j,k)\in\Delta_{n_l}}\beta_{jk}T_j^*d_{jk}^{n_l})\right\|} = 0, \tag{3.48}$$

By an argument similar to the one used in the proof of Theorem 3.2, we obtain

$$\lim_{l\to\infty}\|(I^{\mathcal{H}_j}-P^{\mathcal{H}_j}_{Q^{n_l}_{jk}})T_jw_{n_l}\| = 0, \tag{3.49}$$

for all $j\in J_2$ and $k\in J_3$.

Next, we show that $\omega_w(t_n)\subset\Omega$. Since $\{t_n\}$ is bounded, $\omega_w(t_n)\neq\emptyset$. Let $\bar{t}\in\omega_w(t_n)$, then we may assume that there exists a subsequence $\{t_{n_l}\}$ of $\{t_n\}$ such that $t_{n_l}\rightharpoonup\bar{t}$.

Furthermore, $\|z_n - t_n\| \to 0$, and hence $z_{n_l} \rightharpoonup \bar{t}$, and since T_j is linear and bounded, $T_j z_{n_l} \rightharpoonup T_j \bar{t}$.

By the Banach contraction principle, there exists a unique point $t^* = P_\Omega v(t^*)$. It then follows from (2.1) that

$$\langle v(t^*) - t^*, z - t^* \rangle \leq 0, \tag{3.50}$$

for all $z \in \Omega$. Next, we prove that $\limsup_{l\to\infty} \gamma_{n_l} \leq 0$. Indeed, let $t_{n_{l_m}}$ such that

$$\limsup_{l\to\infty} \langle v(t^*) - t^*, t_{n_l} - t^* \rangle = \lim_{m\to\infty} \langle v(t^*) - t^*, t_{n_{l_m}} - t^* \rangle. \tag{3.51}$$

Again, by an argument similar to the one used in the proof of Theorem 3.2, we can prove that $\bar{t}$ is a solution of the MSSFPMOS (1.2). Thus, by using (3.50) and (3.51), we get

$$\langle v(t^*) - t^*, \bar{t} - t^* \rangle \leq 0. \tag{3.52}$$

In order to prove that $\limsup_{l\to\infty} \gamma_{n_l} \leq 0$, we need to show that $\lim_{l\to\infty} \|t_{n_l+1} - t_{n_l}\| = 0$. Now, taking in to account the boundedness of $\{z_n\}$ and using (3.39), we obtain

$$\begin{aligned} \|t_{n+1} - z_n\| &= \sigma_n \|v(t_n) - z_n\| \\ &\leq \sigma_n (\|v(t_n)\| + \|z_n\|) \\ &\leq \sigma_n (\|v(t_n) - t^* + t^*\| + \|z_n\|) \\ &\leq \sigma_n (\|v(t_n) - t^*\| + \|t^*\| + \|z_n\|) \\ &\leq \sigma_n (N_1 + N_2 + \|t^*\|) \to 0, \end{aligned} \tag{3.53}$$

where $N_1 = \sup_n\{\|v(t_n) - t^*\|\}$ and $N_2 = \sup_n\{\|y_n\|\} < \infty$.
Using the definitions of z_n and τ_n, we obtain

$$\begin{aligned} \|z_n - t_n\| &= \Big\| \sum_{i=1}^{s} \alpha_i^n P_{C_i^n}\Big(w_n - \tau_n \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\Big) - t_n \Big\| \\ &= \Big\| \sum_{i=1}^{s} \alpha_i^n P_{C_i^n}\Big(w_n - \tau_n \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n\Big) - \sum_{i=1}^{s} \alpha_i^n P_{C_i^n} t_n \Big\| \\ &\leq \Big\| w_n - \tau_n \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n - t_n \Big\| \\ &\leq \|w_n - t_n\| + \tau_n \Big\| \sum_{j=1}^{p}\sum_{k=1}^{r_j} \beta_{jk} T_j^* d_{jk}^n \Big\| \\ &\leq \frac{\theta_n}{\sigma_n}\|t_n - t_{n-1}\| + \rho_n \frac{\sum_{(j,k)\in\Delta_n} \beta_{jk} f_{jk}(T_j w_n)}{\Big\|\sum_{(j,k)\in\Delta_n} \beta_{jk} T_j^* d_{jk}^n)\Big\|}. \end{aligned} \tag{3.54}$$

Using (3.34), (3.48), and (3.54), it follows that

$$\lim_{l\to\infty} \|z_n - t_n\| = 0. \tag{3.55}$$

Now, combining (3.53) and (3.55), we get

$$\|t_{n_l+1} - t_{n_l}\| \leq \|t_{n_l+1} - z_{n_l}\| + \|z_{n_l} - t_{n_l}\| \to 0 \text{ as } l \to \infty,$$

that is

$$\lim_{l\to\infty} \|t_{n_l+1} - t_{n_l}\| = 0. \tag{3.56}$$

It follows from (3.52) and (3.56), that

$$\begin{aligned} & \limsup_{n\to\infty} \langle v(t^*) - t^*, t_{n+1} - t^* \rangle \\ = & \limsup_{n\to\infty} \Big(\langle v(t^*) - t^*, t_{n+1} - t_n + \langle v(t^*) - t^*, t_n - t^* \rangle \Big) \\ \leq & \ 0, \end{aligned} \tag{3.57}$$

and hence

$$\begin{aligned} & \limsup_{l\to\infty} \gamma_{n_l} \\ = & \limsup_{l\to\infty} \left[\left(\frac{\theta_{n_l}}{\sigma_{n_l}} \|t_{n_l} - t_{n_l-1}\| \right)^2 + 2\frac{\theta_{n_l}}{\sigma_{n_l}} \|t_{n_l} - t_{n_l-1}\| \|t_{n_l} - t^*\| + \langle v(t^*) - t^*, t_{n_l+1} - t^* \rangle \right] \\ \leq & \ 0. \end{aligned}$$

Hence, all the assumptions of Lemma 2.6 are satisfied. Therefore, we conclude that $t_n \to t^*$. This completes the proof of Theorem 3.3.

∎

4. Numerical Experiment

In this section, we show the validity of a special case of Algorithm 1 and Algorithm 2 (when $k = 1$).

Example 4.1. Let $H = \mathbb{R}^S$, $H_1 = \mathbb{R}^R$, $H_2 = \mathbb{R}^N$, $H_3 = \mathbb{R}^M$, $H_4 = \mathbb{R}^L$.

Let $C_1 = \{x \in \mathbb{R}^S : \|x - \mathbf{o}_1\|^2 \leq \mathbf{r}_1^2\}$, $C_2 = \{x \in \mathbb{R}^S : \|x - \mathbf{o}_2\|^2 \leq \mathbf{r}_2^2\}$, $C_3 = \{x \in \mathbb{R}^S : \|x - \mathbf{o}_3\|^2 \leq \mathbf{r}_3^2\}$, and $C_4 = \{x \in \mathbb{R}^S : \|x - \mathbf{o}_4\|^2 \leq \mathbf{r}_4^2\}$ where $\mathbf{o}_1$, $\mathbf{o}_2$, $\mathbf{o}_3$, $\mathbf{o}_4 \in \mathbb{R}^S$ and $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4 \in \mathbb{R}$. Clearly C_1, C_2, C_3, and C_4 are nonempty closed and convex subsets of H.

Let $Q_1 = \{T_1x \in \mathbb{R}^R : \|T_1x - \mathbf{c}_1\|^2 \leq \varrho_1^2\}$, $Q_2 = \{T_2x \in \mathbb{R}^N : \|T_2x - \mathbf{c}_2\|^2 \leq \varrho_2^2\}$, $Q_3 = \{T_3x \in \mathbb{R}^M : \|T_3x - \mathbf{c}_3\|^2 \leq \varrho_3^2\}$, and $Q_4 = \{T_4x \in \mathbb{R}^L : \|T_4x - \mathbf{c}_4\|^2 \leq \varrho_4^2\}$ where $\mathbf{c}_1 \in \mathbb{R}^R$, $\mathbf{c}_2 \in \mathbb{R}^N$, $\mathbf{c}_3 \in \mathbb{R}^M$, $\mathbf{c}_4 \in \mathbb{R}^L$ and $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in \mathbb{R}$.

Let $T_1 : \mathbb{R}^S \to \mathbb{R}^R$, $T_2 : \mathbb{R}^S \to \mathbb{R}^N$, $T_3 : \mathbb{R}^S \to \mathbb{R}^M$, $T_4 : \mathbb{R}^S \to \mathbb{R}^L$ where their entries are randomly generated in the closed interval $[-5, 5]$.

Now, we construct the balls C_i^n $(i = 1, 2, 3, 4)$ and Q_j^n $(j = 1, 2, 3, 4)$ given in (3.2) and (3.3) of the sets C_i and Q_j, respectively, as follows.

For any $x \in \mathbb{R}^S$, we have $c_i(x) = \|x - \mathbf{o}_i\|^2 - \mathbf{r}_i^2$ for $i = 1, 2, 3, 4$ and $q_j(T_jx) = \|T_jx - \mathbf{c}_j\|^2 - \varrho_j^2$ for $j = 1, 2, 3, 4$. In what follows the subgradients ξ_i^n and η_j^n of respectively $c_i(y_n)$ and $q_j(T_jy_n)$ can be calculated respectively at the points y_n and T_jy_n by $\xi_i^n(y_n) = 2(y_n - \mathbf{o}_i)$ and $\eta_j^n(T_jy_n) = 2T_j^*(T_jy_n - \mathbf{c}_j)$. The metric projections onto the balls C_i^n $(i = 1, 2, 3, 4)$ and Q_j^n $(j = 1, 2, 3, 4)$, can be easily calculated.

We randomly generate the coordinates of $\mathbf{o}_i$ and $\mathbf{c}_j$ in $[-1, 1]$ and, $\mathbf{r}_i$ and ϱ_j in $[S, 2S]$, $[R, 2R]$, $[N, 2N]$, $[M, 2M]$, and $[L, 2L]$, respectively. We take the initial points as $t_0 = 100(1, 1, \ldots, 1)^T \in \mathbb{R}^S$ and $t_1 = -10(1, 1, \ldots, 1)^T \in \mathbb{R}^S$.

The parameters are chosen in such away that for $i = 1, 2, 3, 4$, we take $\alpha_i^n = \frac{i}{10}$ and $\varpi_i = 0.5$. For $j = 1, 2, 3, 4$, we take $\beta_j = \frac{j}{10}$ and $\omega_j = 1.5$, $\rho_n = \frac{n}{4n+1}$, $\sigma_n = \frac{1}{n+1}$,

$\theta = 0.3$, and $\epsilon_n = \frac{1}{(n+1)^3}$. We use $Error_n = \|t_{n+1} - t_n\|^2 < 10^{-8}$ as a stopping criterion in this example. The algorithms are coded in MATLAB 2023b on a personal computer (13th Gen Intel(R) Core(TM) i7-1355U 1.70 GHz, and a 16.0 GB RAM). All results are reported in Table 1, Table 2, Figure 1, and Figure 2.

TABLE 1. Numerical results of Algorithm 1 (when $k = 1$) for different choices of S, R, N, M, L

Dimensions	Iter. (n)	$CPU(s)$	$Error_n$
$S = 3, R = 6, N = 9, M = 12, L = 15$	136	0.002492	9.9209e-09
$S = 15, R = 30, N = 45, M = 60, L = 75$	346	0.008785	9.9804e-09
$S = 30, R = 60, N = 90, M = 120, L = 150$	558	0.022037	9.9594e-09
$S = 100, R = 200, N = 300, M = 400, L = 500$	1517	0.809077	9.9800e-09

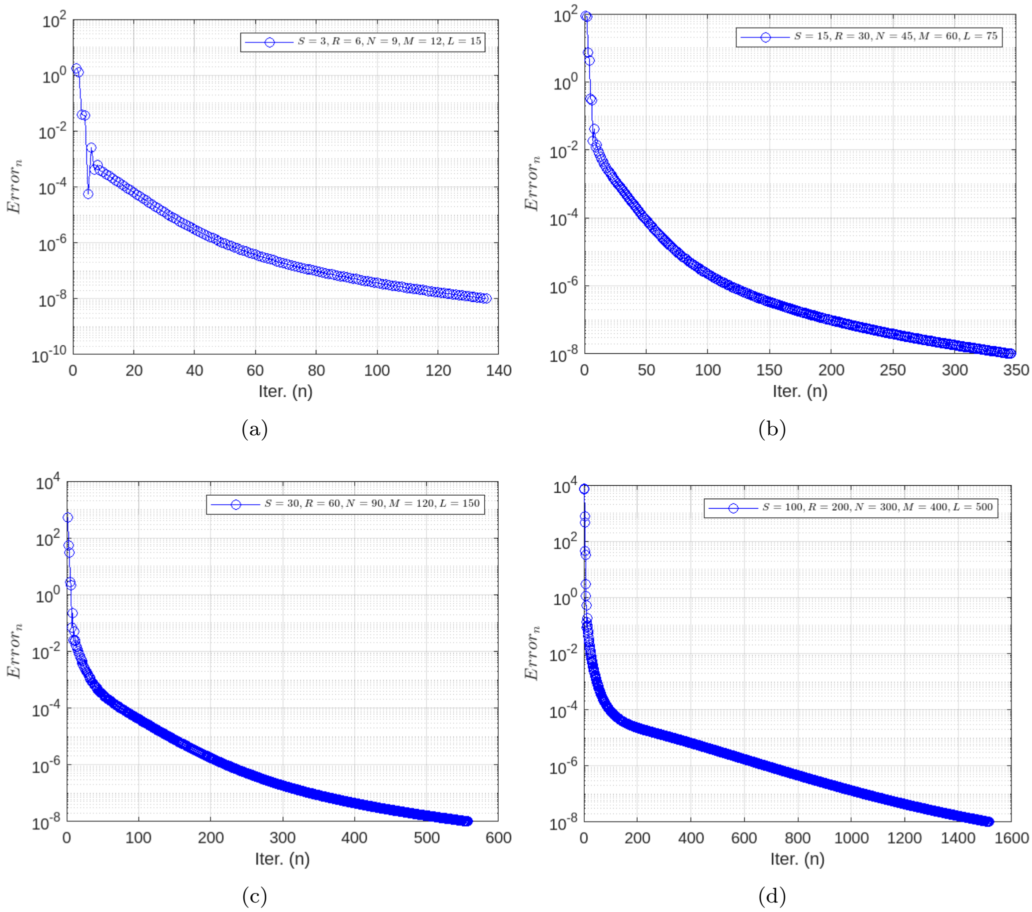

FIGURE 1. Iter. (n) vs $Error_n$, experimental results of Algorithm 1 (when $k = 1$) for different choices of S, R, N, M, L

Table 2. Numerical results of Algorithm 2 (when $k = 1$) for different choices of S, R, N, M, L

Dimensions	Iter. (n)	$CPU(s)$	$Error_n$
$S = 3, R = 6, N = 9, M = 12, L = 15$	35	0.000988	9.6381e-09
$S = 15, R = 30, N = 45, M = 60, L = 75$	75	0.002604	9.8955e-09
$S = 30, R = 60, N = 90, M = 120, L = 150$	145	0.005994	9.9132e-09
$S = 100, R = 200, N = 300, M = 400, L = 500$	257	0.157988	9.8768e-09

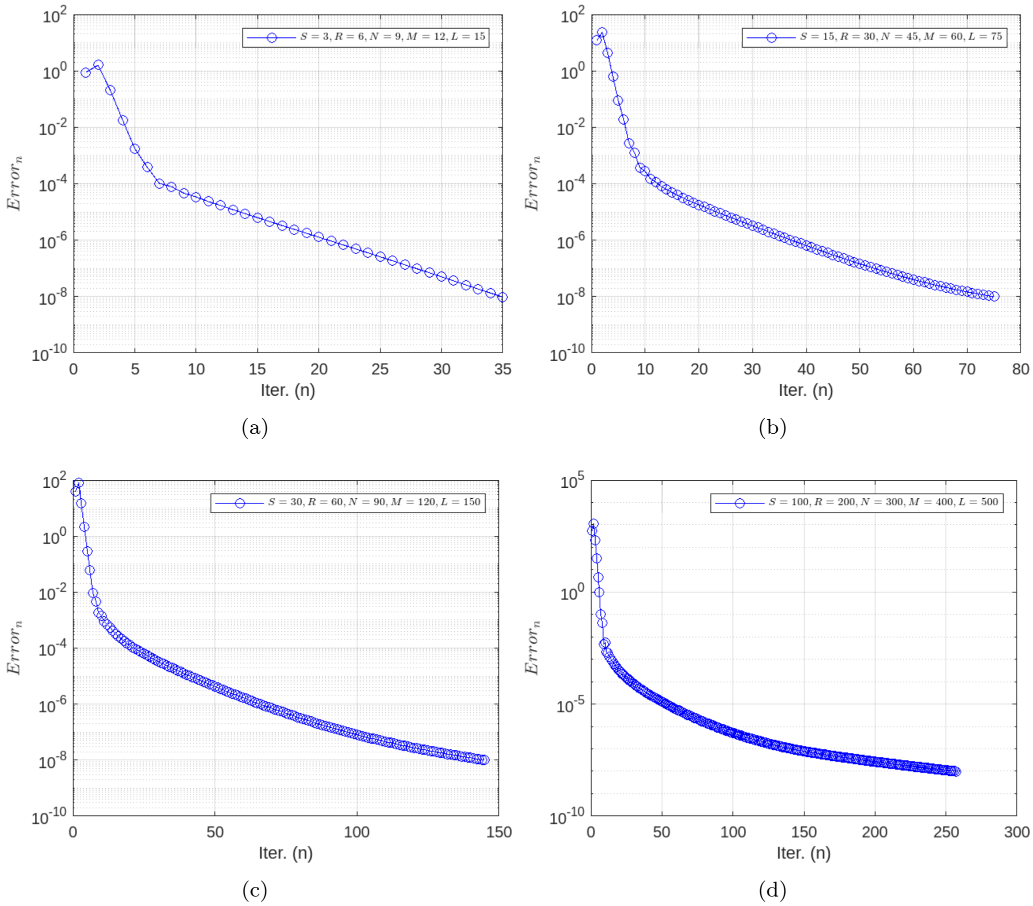

Figure 2. Iter. (n) vs $Error_n$, experimental results of Algorithm 2 (when $k = 1$) for different choices of S, R, N, M, L

5. Conclusion

In this paper, we study a multiple-sets split feasibility problem with multiple output sets in infinite-dimensional Hilbert spaces. We propose relaxed inertial self-adaptive algorithms and prove strong convergence results for the sequences generated by these

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algorithms. These algorithms generalize the algorithms developed by Kim et al. [18] and Reich and Tuyen [19]. The important advantage of our proposed algorithms is that they do not use the least square approximation unlike most algorithms. Finally, we validate the performance of the proposed algorithms by using a numerical example and the numerical results show that our proposed algorithms perform well.

Acknowledgements

The authors would like to thank the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT and the Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi for all the necessary supports.

Authors contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

Funding

This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT. The first author was supported by the Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi with Grant No. 51/2565.

Competing interests

The authors declare no competing interests.

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https://doi.org/10.58715/bangmodjmcs.2024.10.2
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