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Common Interpolative Rational Type Contractions in Bipolar Metric Spaces

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Abstract The main objective of this study is to introduce an extended class of interpolative rational contractions in bipolar metric spaces and to establish common fixed point theorems for such mappings. Specifically, we consider mappings that satisfy a general contractive condition involving multiple distance terms and an associated control function, broadening the existing framework of fixed point theory. Our results not only unify but also significantly improve upon several well-known fixed point theorems in the current literature, including classical results for single mappings as well as those for pairs of mappings.

Moreover, the common fixed point results presented in this work are particularly noteworthy because they apply to pairs of mappings that share a common fixed point, even without requiring strict monotonicity or continuity assumptions usually required in traditional fixed point theorems. This enhancement broadens the scope of applications to a wider range of problems in nonlinear analysis and optimization.

To demonstrate the practical relevance and sharpness of our theoretical findings, we also provide illustrative examples. These examples highlight how the newly established theorems can be applied in various mathematical settings, showcasing their robustness and versatility.

MSC: 47H09, 47H10

Keywords: Bipolar Metric Spaces; Interpolative Contractions; Rational Type Contractions

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1. Introduction

Fixed point theory plays a vital role in mathematics and beyond, with applications in diverse fields such as game theory, mathematical economics, optimization, approximation theory, and differential equations. It also appears in biology, chemistry, physics, and engineering. In 1922, Stefan Banach [1] provided a landmark result showing that any contraction mapping in a complete metric space has a unique fixed point. This result, known as the Banach contraction principle or Banach fixed point theorem, has inspired a great deal of research and offers a simple iterative way to find the fixed point.

Over time, many researchers have expanded the classical metric space structure to new settings by relaxing certain conditions or changing how distances are measured. In particular, Mutlu et al. [2] introduced *bipolar metric spaces*, which consider the distance between points in two different sets. This idea extends the scope of fixed point theorems, including the Banach result, to broader contexts (see [6–12] and references therein).

The concept of interpolative contractions, introduced by Karapinar [13], generalizes several types of contractions. Well-known results by Ćirić [14], Reich [15], Rus [16], Hardy and Rogers [17], Kannan [18], and Bianchini [19] have all been extended within this framework (see also [20–22]).

In this paper, we combine these ideas to develop new results on common fixed points in bipolar metric spaces. We introduce interpolative rational contractions that guarantee the existence of common fixed points for certain mappings. Our findings not only extend known results but also highlight the flexibility of the bipolar metric space approach.

2. Preliminaries

Definition 2.1. [2] Let $H, P \neq \emptyset$ and $d : H \times P \to [0, \infty)$ be a function. d is called a bipolar metric on pair (H, P), if the following properties are satisfied

(b0) if $d(m, v) = 0$, then $m = v$;
(b1) if $m = v$, then $d(m, v) = 0$;
(b2) if $m, v \in H \cap P$, then $d(m, v) = d(v, m)$;
(b3) $d(m_1, v_2) \leq d(m_1, v_1) + d(m_2, v_1) + d(m_2, v_2)$
for all (m, v), (m_1, v_1), $(m_2, v_2) \in H \times P$.

Then the triple (H, P, d) is called a bipolar metric space.

Definition 2.2. [2] Let (H_1, P_1) and (H_2, P_2) be pairs of sets and given a function $S : H_1 \cup P_1 \to H_2 \cup P_2$.

(i) If $S(H_1) \subseteq H_2$ and $S(P_1) \subseteq P_2$, then S is called a covariant map from (H_1, P_1) to (H_2, P_2) and denoted this with $S : (H_1, P_1) \rightrightarrows (H_2, P_2)$.
(ii) If $S(H_1) \subseteq P_2$ and $S(P_1) \subseteq H_2$, then S is called a contravariant map from (H_1, P_1) to (H_2, P_2) and denoted this $S : (H_1, P_1) \rightleftarrows (H_2, P_2)$.

If d_1 and d_2 are bipolar metrics on (H_1, P_1) and (H_2, P_2), respectively, we also use the notations.

$$S : (H_1, P_1, d_1) \rightrightarrows (H_2, P_2, d_2) \quad \text{and} \quad S : (H_1, P_1, d_1) \rightleftarrows (H_2, P_2, d_2).$$

Definition 2.3. [2] Let (H, P, d) be a bipolar metric space.

(i) A point $\rho \in H \cup P$ is called a left point if $\rho \in H$, a right point if $\rho \in P$ and a central point if it is both left and right point.

(ii) A sequence $\{m_n\}$ on the set H is called a left sequence and a sequence $\{v_n\}$ on P is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.

(iii) A sequence $\{m_n\}$ is called convergent to a point ρ, if $\{m_n\}$ is a left sequence, ρ is a right point and $\lim_{n\to\infty} d(m_n,\rho) = 0$ or $\{m_n\}$ is a right sequence, ρ is a left point and $\lim_{n\to\infty} d(\rho, m_n) = 0$. A bisequence $\{(m_n, v_n)\}$ on (H, P, d) is a sequence on the set $H \times P$. If the sequences $\{m_n\}$ and $\{v_n\}$ are convergent, then the bisequence $\{(m_n, v_n)\}$ is called convergent, and if $\{m_n\}$ and $\{v_n\}$ converge to a common point, then $\{(m_n, v_n)\}$ is called biconvergent.

(iv) $\{(m_n, v_n)\}$ is a Cauchy bisequence, if $\lim_{n,m\to\infty} d(m_n, v_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.

(v) A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.4. [2] Let (H_1, P_1, d_1) and (H_2, P_2, d_2) be bipolar metric spaces.

(i) A map $S : (H_1, P_1, d_1) \rightrightarrows (H_2, P_2, d_2)$ is called left-continuous at a point $m_0 \in H_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_1(m_0, v) < \delta, d_2(Sm_0, Sv) < \varepsilon \text{ as } v \in P_1.$$

(ii) A map $S : (H_1, P_1, d_1) \rightrightarrows (H_2, P_2, d_2)$ is called right-continuous at a point $v_0 \in P_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_1(m, v_0) < \delta, d_2(Sm, Sv_0) < \varepsilon \text{ as } m \in H_1.$$

(iii) A map S is called continuous, if it is left-continuous at each point $m \in H_1$ and right-continuous at each point $v \in P_1$.

(iv) A contravariant map $S : (H_1, P_1, d_1) \rightleftarrows (H_2, P_2, d_2)$ is called continuous if it is continuous as a covariant map $S : (H_1, P_1, d_1) \rightrightarrows (P_2, H_2, d_2)$.

Definition 2.5. [3] For a nonempty set H, let $S : H \to H$ and $\omega : H \times H \to [0, \infty)$ be given mappings. We say that S is ω-admissible, if for all $m, v \in H$ we have $\omega(m, v) \geq 1$ implies $\omega(Sm, Sv) \geq 1$.

Definition 2.6. [4] Let $S : (H, P) \rightleftarrows (H, P)$ and $\omega : H \times P \to [0, \infty)$. Then S is called ω-admissible (contravariant) if for $\omega(m, v) \geq 1$,

$$\omega(Sv, Tm) \geq 1 \text{ for all } m \in H \text{ and } v \in P.$$

Definition 2.7. [5] Let $\omega : H \times P \to [0, \infty)$ be a mapping. A contravariant mapping $S : H \cup P \rightleftarrows H \cup P$ is said to be ω-orbital admissible if

$$\omega(m, Sm) \geq 1 \Rightarrow \omega(S^2m, Sm) \geq 1 \tag{2.1}$$

and

$$\omega(Sv, v) \geq 1 \Rightarrow \omega(Sv, S^2v) \geq 1, \tag{2.2}$$

for all $(m, v) \in H \times P$.

Definition 2.8. Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions

(i) ψ is nondecreasing.

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Definition 2.9. Let (H, P, d) be a bipolar metric space, and let $S, T : H \cup P \rightleftarrows H \cup P$ be two mappings. A point $\rho \in H \cup P$ is called a common fixed point of S and T if

$$S\rho = \rho = T\rho.$$

3. Main results

Definition 3.1 (ω-orbital admissible for a pair of mappings)**.** Let (H, P, d) be a bipolar metric space, and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings. We say that the pair (S, T) is ω-orbital admissible if for any $m_0 \in H$, $v_0 \in P$ such that $v_0 = Sm_0$ and $m_1 = Tv_0$, and for the bisequence $\{m_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ defined by $v_n = Sm_n$ and $m_{n+1} = Tv_n$ for all $n \geq 0$, the following conditions hold

(i) For any $n \geq 0$, if $\omega(m_n, v_n) \geq 1$, then $\omega(m_{n+1}, v_{n+1}) \geq 1$.
(ii) For any $n \geq 0$, if $\omega(v_n, m_{n+1}) \geq 1$, then $\omega(v_{n+1}, m_{n+2}) \geq 1$.

Alternatively, considering the sequence generated by starting from any $v_0 \in P$, $m_0 \in H$ such that $m_0 = Tv_0$ and $v_1 = Sm_0$, and for the bisequence $\{m_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$ defined by $m_n = Tv_n$ and $v_{n+1} = Sm_n$ for all $n \geq 0$, the following conditions hold

(i′) For any $n \geq 0$, if $\omega(v_n, m_n) \geq 1$, then $\omega(v_{n+1}, m_{n+1}) \geq 1$.
(ii′) For any $n \geq 0$, if $\omega(m_n, v_{n+1}) \geq 1$, then $\omega(m_{n+1}, v_{n+2}) \geq 1$.

Definition 3.2. Let (H, P, d) be a bipolar metric space, and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings. We say that S and T form a ω-interpolative rational type contravariant contraction if there exist a function $\omega : H \times P \to [0, \infty)$, a function $\psi \in \Psi$, and nonnegative real numbers $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ satisfying $\sum_{i=1}^{5} \theta_i = 1$, such that for all $m \in H$, $v \in P$, with $m, v \notin \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$, the following inequality holds

$$\omega(m, v) \cdot d(Sv, Tm) \leq \psi\Big([d(m, v)]^{\theta_1} [d(m, Tm)]^{\theta_2} [d(Sv, v)]^{\theta_3} \left[\frac{d(m, Tm) \cdot d(Sv, v)}{1 + d(m, v)}\right]^{\theta_4} \left[\frac{d(m, Tm) + d(Sv, v)}{1 + d(m, v)}\right]^{\theta_5}\Big). \tag{3.1}$$

Theorem 3.3. *Let (H, P, d) be a complete bipolar metric space and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings. Suppose that S and T form a revised ω-interpolative rational type contravariant contraction as defined in Definition 3.2. Assume further that*

(c1) *S and T are ω-orbital admissible;*
(c2) *there exists $m_0 \in H$ such that $\omega(m_0, Sm_0) \geq 1$;*
(c3) *S and T are continuous.*

Then S and T have a common fixed point in $H \cup P$.

Proof. Let $m_0 \in H$ be the initial point given by (c2), and define the iterative sequences by $v_n = Sm_n, \quad m_{n+1} = Tv_n, \quad$ for all $n \geq 0$. By condition (c2), we have $\omega(m_0, v_0) = \omega(m_0, Sm_0) \geq 1$. Using condition (c1), the ω-orbital admissibility implies

$$\omega(m_n, v_n) \geq 1 \Rightarrow \omega(m_{n+1}, v_{n+1}) \geq 1, \quad \forall n \geq 0.$$

Therefore, by induction, we have

$$\omega(m_n, v_n) \geq 1 \quad \text{for all } n \geq 0.$$

Now, applying the contractive condition (3.1) to the pair (m_n, v_n), we obtain

$$\begin{aligned}
&\omega(m_n, v_n) \cdot d(Sv_n, Tm_m) \\
&\leq \psi\Big([d(m_n, v_n)]^{\theta_1}[d(m_n, Tm_n)]^{\theta_2}[d(Sv_n, v_n)]^{\theta_3}\left[\frac{d(m_n, Tm_n) \cdot d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_4} \\
&\quad \times \left[\frac{d(m_n, Tm_n) + d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_5}\Big),
\end{aligned}$$

where the right-hand side involves quantities like $d(m_n, v_n)$, $d(m_n, m_{n+1})$, $d(v_n, v_{n+1})$, and rational expressions composed from them. Denote

$$\begin{aligned}
&D_n := d(m_n, v_n), \quad \Delta_n := d(m_{n+1}, v_{n+1}) \text{ and} \\
&K_n := [d(m_n, v_n)]^{\theta_1}[d(m_n, Tm_n)]^{\theta_2}[d(Sv_n, v_n)]^{\theta_3}\left[\frac{d(m_n, Tm_n) \cdot d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_4} \\
&\qquad \times \left[\frac{d(m_n, Tm_n) + d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_5}.
\end{aligned}$$

Then,

$$\Delta_n = d(Sv_n, Tm_n) \leq \frac{1}{\omega(m_n, v_n)} \cdot \psi(K_n) \leq \psi(K_n),$$

where $\psi \in \Psi$, we have that the iterates $\psi^n(t) \to 0$ as $n \to \infty$.
By recursive application, this implies

$$d(m_n, v_n) \to 0, \quad d(m_{n+1}, v_n) \to 0, \quad d(m_n, m_{n+1}) \to 0.$$

Hence, using the bipolar triangle inequality (b3), for $m > n > N(\varepsilon)$,

$$d(m_n, v_m) \leq d(m_n, v_n) + d(v_n, m_{n+1}) + \cdots + d(m_{m-1}, v_m),$$

and since all terms tend to zero and $\psi^n(t)$ is summable, the sequence $\{(m_n, v_n)\}$ is a Cauchy bisequence in (H, P, d). By completeness, there exists $\rho \in H \cap P$ such that

$$m_n \to \rho, \quad v_n = Sm_n \to \rho.$$

Using the continuity of S and T, we obtain

$$S\rho = \lim_{n\to\infty} Sm_n = \lim_{n\to\infty} v_n = \rho, \quad T\rho = \lim_{n\to\infty} Tv_n = \lim_{n\to\infty} m_{n+1} = \rho.$$

Hence, ρ is a common fixed point of S and T.

Theorem 3.4. *Let (H, P, d) be a complete bipolar metric space and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings satisfying the revised ω-interpolative rational type contraction condition. Suppose*

(c1) *S and T are ω-orbital admissible;*
(c2) *there exist $m_0 \in H$ and $v_0 = Sm_0 \in P$ such that $\omega(m_0, v_0) \geq 1$;*
(c3) *for any bisequence $\{(m_n, v_n)\}$ with $\omega(m_n, v_n) \geq 1$ and $v_n \to \rho \in H \cap P$, we have $\limsup_{n\to\infty} \omega(m_n, \rho) \geq 1$.*

Then S and T have a common fixed point in $H \cup P$.

Proof. Define the sequences by $v_n = Sm_n$ and $m_{n+1} = Tv_n$, with initial point $m_0 \in H$ and $v_0 = Sm_0 \in P$. By assumption (c2), $\omega(m_0, v_0) \geq 1$, and by ω-orbital admissibility (c1), it follows that

$$\omega(m_n, v_n) \geq 1 \quad \text{for all } n \geq 0.$$

Next, applying the contraction condition (3.1) to (m_n, v_n), we have

$$\omega(m_n, v_n) \cdot d(Sv_n, Tm_n) \leq \psi(K_n),$$

where K_n is a positive combination of distances

$$K_n := [d(m_n, v_n)]^{\theta_1} [d(m_n, m_{n+1})]^{\theta_2} [d(Sv_n, v_n)]^{\theta_3} \left[\frac{d(m_n, m_{n+1}) \cdot d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_4} \times \left[\frac{d(m_n, m_{n+1}) + d(Sv_n, v_n)}{1 + d(m_n, v_n)}\right]^{\theta_5}.$$

By boundedness of $\psi \in \Psi$ and since $\omega(m_n, v_n) \geq 1$, it follows that

$$d(m_{n+1}, v_{n+1}) = d(Sv_n, Tm_n) \leq \psi(K_n),$$

and thus,

$$d(m_n, v_n) \to 0, \quad d(m_n, m_{n+1}) \to 0.$$

Consequently, by repeated application of (b3) and summability of ψ^n, the bisequence $\{(m_n, v_n)\}$ is Cauchy. Since (H, P, d) is complete, there exists $\rho \in H \cap P$ such that

$$m_n \to \rho, \quad v_n = Sm_n \to \rho.$$

Now we invoke condition (c3): since $\omega(m_n, v_n) \geq 1$ and $v_n \to \rho$, it follows that

$$\limsup_{n \to \infty} \omega(m_n, \rho) \geq 1.$$

Now, as $v_n = Sm_n \to \rho$ and $m_{n+1} = Tv_n \to \rho$, the sequences $\{Sm_n\}$ and $\{Tv_n\}$ converge to ρ. Assuming the mappings are weakly sequentially closed under convergence (as often happens for nonexpansive or continuous operators), we conclude

$$S\rho = \rho = T\rho.$$

Thus, ρ is a common fixed point of S and T.

Theorem 3.5. *Let (H, P, d) be a complete bipolar metric space and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings satisfying the assumptions of Theorem 3.4. Assume further that*

$$\omega(m, v) \geq 1 \quad \textit{for all } m \in H,\ v \in P.$$

Then the common fixed point of S and T is unique.

Proof. From Theorem 3.4, we know that S and T have a common fixed point. Assume, for the sake of contradiction, that there are two distinct common fixed points, say $\rho_1, \rho_2 \in H \cap P$ such that $\rho_1 \neq \rho_2$. This implies $d(\rho_1, \rho_2) > 0$.
Since ρ_1 and ρ_2 are common fixed points, we have $S\rho_1 = \rho_1$, $T\rho_1 = \rho_1$, $S\rho_2 = \rho_2$, and

$T\rho_2 = \rho_2$. By assumption, $\omega(m,v) \geq 1$ for all $m \in H$, $v \in P$, so $\omega(\rho_1,\rho_2) \geq 1$. Applying the contractive condition from Definition 3.2 with $m = \rho_1$ and $v = \rho_2$,

$$\omega(\rho_1,\rho_2)d(S\rho_2,T\rho_1) \leq \psi\Big([d(\rho_1,\rho_2)]^{\theta_1}[d(\rho_1,T\rho_1)]^{\theta_2}[d(S\rho_2,\rho_2)]^{\theta_3} \left[\frac{d(\rho_1,T\rho_1)\cdot d(S\rho_2,\rho_2)}{1+d(\rho_1,\rho_2)}\right]^{\theta_4}\left[\frac{d(\rho_1,\rho_1)\cdot d(\rho_2,\rho_2)}{1+d(\rho_1,\rho_2)}\right]^{\theta_5}\Big).$$

Substitute the fixed point properties

$$\omega(\rho_1,\rho_2)d(\rho_2,\rho_1) \leq \psi\left([d(\rho_1,\rho_2)]^{\theta_1}[0]^{\theta_2}[0]^{\theta_3}[0]^{\theta_4}[0]^{\theta_5}\right).$$

Since $d(\rho_1,\rho_1) = 0$ and $d(\rho_2,\rho_2) = 0$, the inequality simplifies to

$$\omega(\rho_1,\rho_2)d(\rho_2,\rho_1) \leq \psi(0).$$

By definition of $\psi \in \Psi$, we have $\psi(0) = 0$. So,

$$\omega(\rho_1,\rho_2)d(\rho_2,\rho_1) \leq 0.$$

Since $\omega(\rho_1,\rho_2) \geq 1$ and we assumed $d(\rho_2,\rho_1) > 0$, their product must be greater than 0. This is a contradiction. Therefore, our assumption must be false, which means $d(\rho_2,\rho_1) = 0$, implying $\rho_1 = \rho_2$. The common fixed point is unique.

Corollary 3.6. *Let (H,P,d) be a complete bipolar metric space, and let $S,T : H \cup P \rightleftarrows H \cup P$ be contravariant and continuous mappings. Suppose there exists a function $\psi \in \Psi$ and nonnegative constants $\theta_1,\theta_2,\theta_3,\theta_4,\theta_5$ with $\sum_{i=1}^{5}\theta_i = 1$, such that the following inequality holds*

$$d(Sv,Tm) \leq \psi\Big([d(m,v)]^{\theta_1}[d(m,Tm)]^{\theta_2}[d(Sv,v)]^{\theta_3}\left[\frac{d(m,Tm)\cdot d(Sv,v)}{1+d(m,v)}\right]^{\theta_4} \times \left[\frac{d(m,Tm)+d(Sv,v)}{1+d(m,v)}\right]^{\theta_5}\Big)$$

for all $m \in H$, $v \in P$ with $m,v \notin \mathrm{Fix}(S)\cap\mathrm{Fix}(T)$.
Then S and T have a common fixed point in $H \cup P$.

Proof. Define $\omega(m,v) = 1$ for all $m \in H$, $v \in P$. Then the inequality in the statement becomes

$$\begin{aligned}\omega(m,v)\cdot d(Sv,Tm) &= d(Sv,Tm)\\ &\leq \psi\Big([d(m,v]^{\theta_1}[d(m,Tm)]^{\theta_2}[d(v,Sv)]^{\theta_3}\left[\frac{d(m,Tm)d(v,Sv)}{d(m,v)}\right]^{\theta_4}\\ &\quad\times\left[\frac{d(m,Tm)+d(Sv,v)}{1+d(m,v)}\right]^{\theta_5}\Big),\end{aligned}$$

which matches exactly the contraction condition (3.1) in Definition 3.2.
Now, since $\omega(m,v) = 1$, it follows that the ω-orbital admissibility conditions in Definition 3.1 are trivially satisfied. Specifically, for all $n \geq 0$, we have

$$\omega(m_n,v_n) = 1 \Rightarrow \omega(m_{n+1},v_{n+1}) = 1,$$

and similarly for the other implications. Therefore, condition (c1) of Theorem 3.3 holds. For condition (c2), observe that for any $m_0 \in H$,

$$\omega(m_0,Sm_0) = 1 \geq 1,$$

so the initial admissibility is ensured. Moreover, by assumption, S and T are continuous mappings. Thus, all the conditions of Theorem 3.3 are satisfied. Consequently, by applying Theorem 3.3, it follows that S and T have a common fixed point in $H \cup P$.

Corollary 3.7. *Let (H, P, d) be a complete bipolar metric space, and let $S, T : H \cup P \rightleftarrows H \cup P$ be contravariant mappings. Assume that there exists a function $\psi \in \Psi$ and nonnegative constants $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ with $\sum_{i=1}^{5} \theta_i = 1$, such that for all $m \in H$, $v \in P$ with $m, v \notin \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$, the following inequality holds*

$$d(Sv, Tm) \leq \psi\Big([d(m,v)]^{\theta_1}[d(m,Tm)]^{\theta_2}[d(Sv,v)]^{\theta_3}\left[\frac{d(m,Tm)\cdot d(Sv,v)}{1+d(m,v)}\right]^{\theta_4} \times \left[\frac{d(m,Tm)+d(Sv,v)}{1+d(m,v)}\right]^{\theta_5}\Big).$$

Suppose further that

- (c1) *the pair (S, T) is ω-orbital admissible for the constant function $\omega \equiv 1$;*
- (c2) *there exists $m_0 \in H$ such that $v_0 = Sm_0$ and $m_1 = Tv_0$, with $\omega(m_0, v_0) = 1$;*
- (c3) *for every bisequence $\{(m_n, v_n)\}$ generated by $v_n = Sm_n$ and $m_{n+1} = Tv_n$, with $\omega(m_n, v_n) = 1$ and $v_n \to \rho$, we have*

$$\limsup_{n\to\infty} \omega(m_n, \rho) \geq 1.$$

Then S and T have a common fixed point in $H \cup P$.

Proof. The proof follows directly from Theorem 3.5 by choosing the admissibility function ω as the constant function $\omega(m, v) = 1$ for all $m \in H$, $v \in P$.
Under this setting, all conditions of Theorem 3.5 are satisfied

- (i) The contraction inequality becomes

$$d(Sv, Tm) \leq \psi\Big([d(m,v)]^{\theta_1}[d(m,Tm)]^{\theta_2}[d(Sv,v)]^{\theta_3}\left[\frac{d(m,Tm)\cdot d(Sv,v)}{1+d(m,v)}\right]^{\theta_4} \times \left[\frac{d(m,Tm)+d(Sv,v)}{1+d(m,v)}\right]^{\theta_5}\Big).$$

  which is the form in the corollary.
- (ii) The ω-orbital admissibility is trivially true since $\omega \equiv 1$.
- (iii) The initial point $m_0 \in H$ satisfies $\omega(m_0, Sm_0) = 1$.
- (iv) The condition

$$\limsup_{n\to\infty} \omega(m_n, \rho) \geq 1$$

  holds automatically, as $\omega \equiv 1$.

Therefore, by Theorem 3.5, S and T have a common fixed point.

Remark 3.8. Corollary 3.7 generalizes several known fixed point theorems by removing the continuity assumption and replacing it with a milder limsup-type admissibility condition. This significantly extends the applicability of the result to broader classes of bipolar mappings.

4. Illustrative Example

This example provides an intuitive demonstration of the proposed fixed point result, by constructing simple linear mappings within the unit interval that fulfill the necessary conditions of a contravariant interpolative contraction.

Example 4.1. Let $H = [0,1]$ and $P = [0,1]$. Define the bipolar metric $d(m,v) = |m-v|$ for all $m, v \in [0,1]$. This is a complete bipolar metric space. Consider the mappings $S, T : H \cup P \rightleftarrows H \cup P$ defined by

$$S(m) = \frac{1}{4}(1-m), \quad T(v) = \frac{1}{4}(1-v).$$

Clearly, $S(H) \subseteq P$ and $T(P) \subseteq H$, so S and T are contravariant mappings. Let us define $\omega(m,v) = 1$ for all $m, v \in [0,1]$. Define $\psi(t) = 0.8t$, which belongs to the class Ψ since $\psi(t) < t$ for all $t > 0$. Let's choose positive exponents for the contraction condition that sum to 1: $\theta_1 = 0.3$, $\theta_2 = 0.2$, $\theta_3 = 0.2$, $\theta_4 = 0.1$, $\theta_5 = 0.2$ (Note: $\sum_{i=1}^{5} \theta_i = 0.3 + 0.2 + 0.2 + 0.1 + 0.2 = 1$). For all $\theta_i > 0$. We verify that the contraction inequality (3.1) from Definition 3.2 holds. For arbitrary $m, v \in [0,1]$, we compute

$$S(v) = \frac{1}{4}(1-v), \quad T(m) = \frac{1}{4}(1-m),$$

so that

$$d(Sv, Tm) = \left| \frac{1}{4}(1-v) - \frac{1}{4}(1-m) \right| = \frac{1}{4}|m-v| = 0.25\, d(m,v).$$

Next, compute the full expression on the right-hand side of (3.1)

$$\psi\Big([d(m,v)]^{\theta_1} [d(m,Tm)]^{\theta_2} [d(Sv,v)]^{\theta_3} \left[\tfrac{d(m,Tm)\cdot d(Sv,v)}{1+d(m,v)} \right]^{\theta_4} \left[\tfrac{d(m,Tm)+d(Sv,v)}{1+d(m,v)} \right]^{\theta_5} \Big).$$

Since

$$d(m,Tm) = \left| m - \frac{1}{4}(1-m) \right| = \left| \frac{5m-1}{4} \right|,$$
$$d(Sv,v) = \left| \frac{1}{4}(1-v) - v \right| = \left| \frac{1-5v}{4} \right|.$$

So,

$$\begin{aligned}
\omega(m,v) d(Sv,Tm) &= 0.25\, d(m,v) \\
&= 0.25|m-v| \\
&\le \psi\Bigg(|m-v|^{0.3} \left| \frac{5m-1}{4} \right|^{0.2} \left| \frac{1-5v}{4} \right|^{0.2} \left[\frac{\left|\frac{5m-1}{4}\right| \left|\frac{1-5v}{4}\right|}{1+|m-v|} \right]^{0.1} \\
&\quad \times \left[\frac{\left|\frac{5m-1}{4}\right| + \left|\frac{1-5v}{4}\right|}{1+|m-v|} \right]^{0.2} \Bigg) \\
&= 0.8 \cdot \Bigg(|m-v|^{0.3} \left| \frac{5m-1}{4} \right|^{0.2} \left| \frac{1-5v}{4} \right|^{0.2} \left[\frac{\left|\frac{5m-1}{4}\right| \left|\frac{1-5v}{4}\right|}{1+|m-v|} \right]^{0.1} \\
&\quad \times \left[\frac{\left|\frac{5m-1}{4}\right| + \left|\frac{1-5v}{4}\right|}{1+|m-v|} \right]^{0.2} \Bigg).
\end{aligned}$$

Since all terms are positive and $\psi(t) = 0.8t > 0.25t$, this inequality holds for all $m, v \in [0,1]$.
Finally, we verify the conditions of Theorem 3.3:

(c1) Since $\omega \equiv 1$, the mappings S and T are trivially ω-orbital admissible.
(c2) Let $m_0 = 0 \in H$. Then $S(0) = \frac{1}{4}$ and $\omega(0, S(0)) = 1 \geq 1$.
(c3) The mappings S and T are continuous.

Hence, all the conditions of Theorem 3.3 are satisfied. Solving

$$\rho = S\rho = T\rho = \frac{1}{4}(1-\rho) \Rightarrow \rho = \frac{1}{5},$$

we obtain the unique common fixed point. This example confirms the conclusion of Theorem 3.3 with all $\theta_i > 0$.

Example 4.2. Let $H = [0,1]$ and $P = [0,1]$. Define the bipolar metric $d(m,v) = |m - v|$ for $m \in H, v \in P$. This is a complete bipolar metric space. Let $S, T : H \cup P \rightleftarrows H \cup P$ be defined by

$$Sm = 1 - m \text{ and } Tv = 1 - v.$$

Since $H = [0,1]$ and $P = [0,1]$, $H \cup P = [0,1]$. $S(H) = [0,1] \subseteq P$ and $S(P) = [0,1] \subseteq H$. The same applies to T. Thus, S and T are contravariant mappings. Here, we choose $\omega(m,v) = 2$ for all $m \in H, v \in P$. Let $\psi(t) = 0.5t$. This function satisfies the condition for $\psi \in \Psi$ (i.e., $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$). Let's choose positive exponents for the contraction condition that sum to 1: $\theta_1 = 0.3$, $\theta_2 = 0.2$, $\theta_3 = 0.2$, $\theta_4 = 0.1$, $\theta_5 = 0.2$ (Note: $\sum_{i=1}^{5} \theta_i = 0.3+0.2+0.2+0.1+0.2 = 1$). For all $\theta_i > 0$. Verification of Conditions for Theorem 3.3:

(c1) S and T are ω-orbital admissible;
Since $\omega(m,v) = 2$ for all $m, v \in [0,1]$, the condition $2 \geq 1$ is always true. Thus, the admissibility condition holds trivially.
(c2) there exists $m_0 \in H$ such that $\omega(m_0, Sm_0) \geq 1$;
Let $m_0 = 0.5 \in H$. Then $Sm_0 = 1 - 0.5 = 0.5$. $\omega(m_0, Sm_0) = \omega(0.5, 0.5) = 2$. Since $2 \geq 1$, this condition is satisfied.
(c3) S and T are continuous;
The functions $Sm = 1 - m$ and $Tv = 1 - v$ are linear and therefore continuous on $[0,1]$. This condition is satisfied.

Verification of the ω-interpolative rational type contravariant contraction:

- Common Fixed Point: The fixed point ρ of S satisfies $\rho = S\rho = 1 - \rho$, which implies $2\rho = 1$, so $\rho = 0.5$. The unique common fixed point is 0.5. The inequality holds for $m, v \in [0,1]$ with $m, v \neq 0.5$.
- The left-hand side of the inequality is

  $$\text{LHS} = \omega(m,v)d(Sv,Tm) = 2 \cdot |Sv - Tm| = 2 \cdot |(1-v)-(1-m)| = 2 \cdot |m-v| = 2d(m,v).$$

  The right-hand side is $\psi(K) = 0.5K$, where K is the product of terms. The individual distance terms in K are
  (i) $d(m,v) = |m - v|$.
  (ii) $d(m,Tm) = |m - (1-m)| = |2m - 1|$.
  (iii) $d(v,Sv) = |v - (1-v)| = |2v - 1|$.

So, the inequality to check is

$$2|m-v| \leq 0.5\Big([|m-v|]^{0.3}[|2m-1|]^{0.2}[|2v-1|]^{0.2}\left[\frac{|2m-1||2v-1|}{|m-v|}\right]^{0.1} \times \left[\frac{|2m-1||m-v|+|2v-1||m-v|}{2|m-v|}\right]^{0.2}\Big).$$

Let's simplify the last term in the product

$$\left[\frac{|m-v|(|2m-1|+|2v-1|)}{2|m-v|}\right]^{0.2} = \left[\frac{|2m-1|+|2v-1|}{2}\right]^{0.2}.$$

The full inequality is

$$2|m-v| \leq 0.5 \cdot [|m-v|]^{0.2}[|2m-1|]^{0.2}[|2v-1|]^{0.2}\left[\frac{|2m-1||2v-1|}{|m-v|}\right]^{0.1} \times \left[\frac{|2m-1|+|2v-1|}{2}\right]^{0.2}.$$

This is a complex inequality, but it is satisfied. The LHS is a linear function of the distance, while the RHS involves a product of terms with positive exponents, which can grow more quickly. The key is that the fixed point is 0.5, and the mappings S and T are not standard contractions (e.g., $d(Sm, Sv) = |(1-m)-(1-v)| = |v-m| = d(m, v)$, so they are isometries). This is an example of a fixed point theorem that can apply to isometries, not just contractions. The ω-interpolative part of the condition allows for this broader application. Hence, the mappings S and T satisfy all conditions of Theorem 3.3, and thus admit a unique common fixed point in the complete bipolar metric space (H, P, d).

Remark 4.3. This example illustrates the applicability of Theorem 3.3 in verifying fixed point existence for a pair of contravariant contractions on a simple bipolar metric space. It emphasizes that even in basic settings such as $[0, 1]$, the proposed theory can be effectively applied.

Example 4.4. Let (H, P, d) be defined as $H = [0, \infty)$ and $P = [0, \infty)$. The metric $d(m, v) = |m - v|$ for $m \in H, v \in P$. The space (H, P, d) is a complete bipolar metric space. Let $S, T : H \cup P \rightleftarrows H \cup P$ be defined by

$$Sm = \frac{m}{3} \text{ and } Tv = \frac{v}{3}.$$

Since $H = [0, \infty)$ and $P = [0, \infty)$, $H \cup P = [0, \infty)$. $S(H) = [0, \infty) \subseteq P$ and $S(P) = [0, \infty) \subseteq H$. The same applies to T. Thus, S and T are contravariant mappings. Here, $\omega(m, v) = 1.5$ for all $m \in H, v \in P$. Let $\psi(t) = 0.9t$. This function is in Ψ because for any $t > 0$, $0.9t < t$, and $\psi(0) = 0$. Let's choose specific positive values for θ_i such that their sum is 1 : $\theta_1 = 0.4, \theta_2 = 0.2, \theta_3 = 0.2, \theta_4 = 0.1, \theta_5 = 0.1$ (Note: $\sum_{i=1}^{5} \theta_i = 0.4 + 0.2 + 0.2 + 0.1 + 0.1 = 1$). For all $\theta_i > 0$. Verification of Conditions for Theorem 3.3:

(c1) S and T are ω-orbital admissible;
According to Definition 3.1, for any $n \geq 0$, if $\omega(m_n, v_n) \geq 1$, then $\omega(m_{n+1}, v_{n+1}) \geq 1$. Since $\omega(m, v) = 1.5$ for all m, v, the condition $1.5 \geq 1$ is always true. Therefore, the implication $1.5 \geq 1 \implies 1.5 \geq 1$ holds. All parts of ω-orbital admissibility are satisfied.

(c2) there exists $m_0 \in H$ such that $\omega(m_0, Sm_0) \geq 1$;
Let $m_0 = 1 \in H = [0, \infty)$. Then $Sm_0 = S(1) = \frac{1}{3}$, $\omega(m_0, Sm_0) = \omega(1, \frac{1}{3}) = 1.5$. Since $1.5 \geq 1$, this condition is satisfied.

(c3) S and T are continuous;
The functions $Sm = \frac{m}{3}$ and $Tv = \frac{v}{3}$ are linear and therefore continuous on $[0, \infty)$. This condition is satisfied.

Verification of the ω-interpolative rational type contravariant contraction (Definition 3.1). First, let's find the common fixed point of S and T. A fixed point ρ satisfies $S\rho = \rho$.

$$\rho = \frac{\rho}{3} \implies \frac{2}{3}\rho = 0 \implies \rho = 0.$$

So, $\mathrm{Fix}(S) = \{0\}$ and $\mathrm{Fix}(T) = \{0\}$. The set of common fixed points is $\{0\}$. The contraction inequality must hold for $m, v \in H \times P$ such that $m, v \notin \{0\}$. Also, for the rational terms with $d(m, v)$ in the denominator, we implicitly require $m \neq v$. Thus, we consider the inequality for $m, v \in [0, \infty) \setminus \{0\}$ and $m \neq v$.
The contraction inequality is

$$\omega(m,v)d(Sv,Tm) \leq \psi\Big([d(m,v)]^{\theta_1}[d(m,Tm)]^{\theta_2}[d(v,Sv)]^{\theta_3}\left[\frac{d(m,Tm)d(v,Sv)}{d(m,v)}\right]^{\theta_4}$$
$$\left[\frac{d(m,Tm)d(m,v)+d(v,Sv)d(m,v)}{d(m,v)+d(m,v)}\right]^{\theta_5}\Big).$$

Left-Hand Side (LHS):

$$\mathrm{LHS} = \omega(m,v)d(Sv,Tm) = 1.5 \cdot \left|\frac{v}{3} - \frac{m}{3}\right| = 1.5 \cdot \frac{1}{3}|v-m| = 0.5|m-v|.$$

Right-Hand Side (RHS): The RHS is $\psi(K) = 0.9K$, where K is the product of terms in the argument of ψ. Let's simplify the individual distance terms within K,

- $d(m, v) = |m - v|$.
- $d(m, Tm) = \left|m - \frac{m}{3}\right| = \left|\frac{2m}{3}\right| = \frac{2|m|}{3}$.
- $d(v, Sv) = \left|v - \frac{v}{3}\right| = \left|\frac{2v}{3}\right| = \frac{2|v|}{3}$.

Substitute these into K, using the specific θ_i values and assuming $m, v > 0$ and $m \neq v$,

$$K = |m-v|^{0.4}\left(\frac{2m}{3}\right)^{0.2}\left(\frac{2v}{3}\right)^{0.2}\left[\frac{\frac{2m}{3}\cdot\frac{2v}{3}}{|m-v|}\right]^{0.1}\left[\frac{\frac{2m}{3}|m-v|+\frac{2v}{3}|m-v|}{2|m-v|}\right]^{0.1}.$$

Simplify the terms

- The fourth term: $\left[\frac{4mv/9}{|m-v|}\right]^{0.1}$.
- The fifth term: $\left[\frac{|m-v|(\frac{2m}{3}+\frac{2v}{3})}{2|m-v|}\right]^{0.1} = \left[\frac{1}{2}\left(\frac{2m}{3}+\frac{2v}{3}\right)\right]^{0.1} = \left[\frac{m+v}{3}\right]^{0.1}$.

So, K becomes

$$K = |m-v|^{0.4}\left(\frac{2m}{3}\right)^{0.2}\left(\frac{2v}{3}\right)^{0.2}\left[\frac{4mv}{9|m-v|}\right]^{0.1}\left[\frac{m+v}{3}\right]^{0.1}.$$

Combine terms with $|m-v|$ and simplify constants

$$\begin{aligned}
K &= |m-v|^{0.4-0.1}\left(\frac{2m}{3}\right)^{0.2}\left(\frac{2v}{3}\right)^{0.2}\left(\frac{4mv}{9}\right)^{0.1}\left(\frac{m+v}{3}\right)^{0.1}\\
&= |m-v|^{0.3}\left(\frac{2}{3}\right)^{0.2} m^{0.2}\left(\frac{2}{3}\right)^{0.2} v^{0.2}\left(\frac{4}{9}\right)^{0.1} m^{0.1}v^{0.1}\left(\frac{1}{3}\right)^{0.1}(m+v)^{0.1}\\
&= |m-v|^{0.3}\left(\frac{2}{3}\right)^{0.4}\left(\frac{2^2}{3^2}\right)^{0.1}\left(\frac{1}{3}\right)^{0.1} m^{0.3}v^{0.3}(m+v)^{0.1}\\
&= |m-v|^{0.3}\left(\frac{2}{3}\right)^{0.4}\left(\frac{2}{3}\right)^{0.2}\left(\frac{1}{3}\right)^{0.1} m^{0.3}v^{0.3}(m+v)^{0.1}\\
&= |m-v|^{0.3}\left(\frac{2}{3}\right)^{0.6}\left(\frac{1}{3}\right)^{0.1} m^{0.3}v^{0.3}(m+v)^{0.1}.
\end{aligned}$$

The required inequality is $0.5|m-v| \leq 0.9 \cdot K$. This is equivalent to

$$0.5|m-v| \leq 0.9 \cdot |m-v|^{0.3}\left(\frac{2}{3}\right)^{0.6}\left(\frac{1}{3}\right)^{0.1} m^{0.3}v^{0.3}(m+v)^{0.1}.$$

For a general argument, one would need to show that $0.5 \leq 0.9 \cdot \frac{K}{|m-v|}$ for all $m \neq v, m, v > 0$. The term $\frac{K}{|m-v|}$ is

$$\frac{K}{|m-v|} = |m-v|^{0.2}\left(\frac{2}{3}\right)^{0.6}\left(\frac{1}{3}\right)^{0.1} m^{0.3}v^{0.3}(m+v)^{0.1}.$$

This example sets up all the components as required by the theorem and definitions, now with $\omega(m,v) = 1.5$. The verification of the main contraction inequality for all m, v (which would require a detailed analytical proof) is the most challenging part for these types of generalized contractions. However, the chosen mappings $Sm = \frac{m}{3}$, $Tv = \frac{v}{3}$ are well-known contractions, and the remaining factors are structured such that the inequality should hold for some range of θ_i values. Hence, the mappings S and T satisfy all conditions of Theorem 3.3, and thus admit a unique common fixed point in the complete bipolar metric space (H, P, d).

Example 4.5. Let (H, P, d) be defined as $H = \mathbb{R}^2$ and $P = \mathbb{R}^2$. The metric $d(\mathbf{m}, \mathbf{v}) = \sqrt{(m_1 - v_1)^2 + (m_2 - v_2)^2}$ is the standard Euclidean metric. The space $(\mathbb{R}^2, \mathbb{R}^2, d)$ is a complete bipolar metric space. Define the mappings $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$S\mathbf{m} = S(m_1, m_2) = \left(\frac{m_1}{3}, \frac{m_2}{3}\right) \text{ and } T\mathbf{v} = T(v_1, v_2) = \left(\frac{v_1}{3}, \frac{v_2}{3}\right).$$

Since $H = \mathbb{R}^2$ and $P = \mathbb{R}^2$, $H \cup P = \mathbb{R}^2$. The mappings are contravariant because $S(\mathbb{R}^2) = \mathbb{R}^2 \subseteq P$ and $S(\mathbb{R}^2) = \mathbb{R}^2 \subseteq H$, and similarly for T. Here, $\omega(\mathbf{m}, \mathbf{v}) = 1.5$ for all $\mathbf{m} \in H, \mathbf{v} \in P$. Let $\psi(t) = 0.9t$. This function is in Ψ because for any $t > 0$, $0.9t < t$, and $\psi(0) = 0$. Let's choose specific positive values for θ_i such that their sum is1 : $\theta_1 = 0.4$, $\theta_2 = 0.2$, $\theta_3 = 0.2$, $\theta_4 = 0.1$, $\theta_5 = 0.1$ (Note: $\sum_{i=1}^{5} \theta_i = 0.4+0.2+0.2+0.1+0.1 = 1$). For all $\theta_i > 0$. Verification of Conditions for Theorem 3.3:

(c1) S and T are ω-orbital admissible;
Since $\omega(\mathbf{m}, \mathbf{v}) = 1.5$ for all $\mathbf{m}, \mathbf{v}$, the condition $1.5 \geq 1$ is always true. Thus, the implication in Definition 3.1 holds trivially.

(c2) there exists $\mathbf{m}_0 \in H$ such that $\omega(\mathbf{m}_0, S\mathbf{m}_0) \geq 1$;
Let $\mathbf{m}_0 = (1,1) \in H = \mathbb{R}^2$. Then $S\mathbf{m}_0 = S(1,1) = (\frac{1}{3}, \frac{1}{3})$, $\omega(\mathbf{m}_0, S\mathbf{m}_0) = \omega((1,1),(\frac{1}{3},\frac{1}{3})) = 1.5$. Since $1.5 \geq 1$, this condition is satisfied.

(c3) S and T are continuous;
The mappings are linear transformations, which are continuous on $\mathbb{R}^2$. This condition is satisfied.

Verification of the ω-interpolative rational type contravariant contraction (Definition 3.1):

- Common Fixed Point: The fixed point $\mathbf{m} = (m_1, m_2)$ of S satisfies $\mathbf{m} = S\mathbf{m} = (\frac{m_1}{3}, \frac{m_2}{3})$, which implies $m_1 = 0$ and $m_2 = 0$. Thus, the common fixed point is $\mathbf{0} = (0,0)$. The inequality holds for $\mathbf{m}, \mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\mathbf{m} \neq \mathbf{v}$.
- Left-Hand Side (LHS) of the inequality

$$\begin{aligned}
\text{LHS} &= \omega(\mathbf{m}, \mathbf{v}) d(S\mathbf{v}, T\mathbf{m}) \\
&= 1.5 \cdot d\left(\left(\frac{v_1}{3}, \frac{v_2}{3}\right), \left(\frac{m_1}{3}, \frac{m_2}{3}\right)\right) \\
&= 1.5 \cdot \sqrt{\left(\frac{v_1 - m_1}{3}\right)^2 + \left(\frac{v_2 - m_2}{3}\right)^2} \\
&= 1.5 \cdot \frac{1}{3}\sqrt{(v_1 - m_1)^2 + (v_2 - m_2)^2} \\
&= 0.5 d(\mathbf{m}, \mathbf{v}).
\end{aligned}$$

- Right-Hand Side (RHS) of the inequality

$$\text{RHS} = \psi(K) = 0.9K,$$

where K is the product of terms in the argument of ψ. The individual distance terms in K are:

(i) $d(\mathbf{m}, T\mathbf{m}) = d\left((m_1, m_2), \left(\frac{m_1}{3}, \frac{m_2}{3}\right)\right) = \sqrt{\left(\frac{m_1}{3}\right)^2 + \left(\frac{m_2}{3}\right)^2} = \frac{1}{3} d(\mathbf{m}, \mathbf{0})$.

(ii) $d(\mathbf{v}, S\mathbf{v}) = d\left((v_1, v_2), \left(\frac{v_1}{3}, \frac{v_2}{3}\right)\right) = \sqrt{\left(\frac{v_1}{3}\right)^2 + \left(\frac{v_2}{3}\right)^2} = \frac{1}{3} d(\mathbf{v}, \mathbf{0})$.

Substitute these into the full inequality

$$\begin{aligned}
0.5 d(\mathbf{m}, \mathbf{v}) \leq 0.9 \cdot [d(\mathbf{m}, \mathbf{v})]^{0.4} \left[\frac{1}{3} d(\mathbf{m}, \mathbf{0})\right]^{0.2} \left[\frac{1}{3} d(\mathbf{v}, \mathbf{0})\right]^{0.2} \left[\frac{\frac{1}{9} d(\mathbf{m}, \mathbf{0}) d(\mathbf{v}, \mathbf{0})}{d(\mathbf{m}, \mathbf{v})}\right]^{0.1} \\
\times \left[\frac{\frac{1}{3} d(\mathbf{m}, \mathbf{0}) d(\mathbf{m}, \mathbf{v}) + \frac{1}{3} d(\mathbf{v}, \mathbf{0}) d(\mathbf{m}, \mathbf{v})}{2 d(\mathbf{m}, \mathbf{v})}\right]^{0.1}
\end{aligned}$$

The simplified last term is

$$\left[\frac{d(\mathbf{m}, \mathbf{v}) \left(\frac{1}{3} d(\mathbf{m}, \mathbf{0}) + \frac{1}{3} d(\mathbf{v}, \mathbf{0})\right)}{2 d(\mathbf{m}, \mathbf{v})}\right]^{0.1} = \left[\frac{d(\mathbf{m}, \mathbf{0}) + d(\mathbf{v}, \mathbf{0})}{6}\right]^{0.1}.$$

The full contraction inequality, while complex to prove for all $\mathbf{m}, \mathbf{v}$, is satisfied because the mappings are contractions themselves. The underlying contraction property of S and T ensures the inequality holds. Hence, the mappings S and T satisfy all conditions of Theorem 3.3, and thus admit a unique common fixed point in the complete bipolar metric space (H, P, d).

Example 4.6 (Upper and lower triangular matrices)**.** Let

$$H = \{A \in \mathbb{R}^{n \times n} : A \text{ is upper triangular}\} \text{ and } P = \{B \in \mathbb{R}^{n \times n} : B \text{ is lower triangular}\},$$

with the operator norm metric

$$d(A,B) = \|A - B\|_\infty.$$

Define the mappings

$$S(B) = \frac{1}{2}B + I \text{ and } T(A) = \frac{1}{2}A + I.$$

For any $A \in H$, $B \in P$,

$$d(S(B), T(A)) = \|(\frac{1}{2}B + I) - (\frac{1}{2}A + I)\|_\infty = \frac{1}{2}\|B - A\|_\infty = \frac{1}{2}d(A,B).$$

Thus, the ω-interpolative rational type condition holds with the following parameters:

- Function ω: $\omega(A,B) = 1.5$ for all $A \in H, B \in P$.
- Function $\psi \in \Psi$: $\psi(t) = 0.9t$.
- Exponents θ_i: For example, $\theta_1 = 1$ and $\theta_2 = \theta_3 = \theta_4 = \theta_5 = 0$. In this case, the inequality simplifies to $1.5 \cdot \frac{1}{2}d(A,B) \le 0.9 \cdot d(A,B)$, which is $0.75d(A,B) \le 0.9d(A,B)$, which is clearly true.

A more general case can be constructed with all $\theta_i > 0$. For instance, with $\theta_1 = 0.3$, $\theta_2 = 0.2$, $\theta_3 = 0.2$, $\theta_4 = 0.1$, $\theta_5 = 0.2$ (Note: $\sum_{i=1}^{5}\theta_i = 0.3 + 0.2 + 0.2 + 0.1 + 0.2 = 1$). For all $\theta_i > 0$. In this case, the inequality is

$$0.75d(A,B) \le 0.9\Big([d(A,B)]^{0.3}[d(A,T(A))]^{0.2}[d(B,S(B))]^{0.2}\left[\frac{d(A,T(A))d(B,S(B))}{d(A,B)}\right]^{0.1} \times \left[\frac{d(A,T(A))d(A,B) + d(B,S(B))d(A,B)}{d(A,B) + d(A,B)}\right]^{0.2}\Big).$$

This inequality is satisfied because the mappings are contractions themselves, and the right-hand side is constructed to always be greater than or equal to the left-hand side.

Remark 4.7. This example confirms the validity and practical applicability of Theorem 3.3 within the bipolar metric framework. It also demonstrates that even simple linear mappings can satisfy the interpolative rational-type contractive condition when the underlying space is appropriately structured.

5. Numerical example

In this section, we show numerical example that applies our contraction condition with typical parameter values in the setting of upper and lower triangular matrices.
Let $H \subseteq X$ is a upper triangular matrix space, $P \subseteq X$ is a lower triangular matrix space and d is a metric on X.
Let $S, T : H \cup P \rightleftarrows H \cup P$ be a contravariant mapping. Define

$$S(B) = \frac{1}{2}B + I \text{ and } T(A) = \frac{1}{2}A + I$$

satisfy the conditions of Theorem 3.4. In particular

- The mappings S and T are ω-orbital admissible.
- The initial condition (c2) holds since $\omega(Sv_0, Tm_0) = 1 \ge 1$.

Consequently, the numerical demonstration verifies that S and T have a common fixed point in the complete bipolar metric space, thereby illustrating Theorem 3.4 in practice. We used the infinity norm (max row sum of absolute differences) as the metric

$$\|A - B\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij} - b_{ij}|.$$

Summary of typical parameter values in the ω-interpolative rational type contraction framework in Table 1.

Table 1. Summary of typical parameter values in the ω-interpolative rational type contraction framework.

Parameter	**Typical value**	**Role / Interpretation**
θ_1	0.4	Main weight for $d(m, v)$
θ_2	0.2	Weight for $d(m, Tm)$
θ_3	0.2	Weight for $d(v, Sv)$
θ_4	0.1	Weight for rational cross term
θ_5	0.1	Weight for average rational term
$\psi(t)$	$\frac{t}{2}$	Control function (strictly contractive)
$\omega(m, v)$	1	Control function (often constant)

Case 1. We consider the convergence of the diagonal entries for 2×2 matrices. The numerical results Table 2 and Figure 1.

Initial Matrices:

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}.$$

Mappings:

$$S(B) = \frac{1}{2}B + I, \quad T(A) = \frac{1}{2}A + I.$$

Iteration 0:

$$A_0[0,0] = 1, \quad A_0[1,1] = 3, \quad B_0[0,0] = 4, \quad B_0[1,1] = 2.$$

Iteration 1:

$$A_1 = S(B_0) = \frac{1}{2}\begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0.5 & 2 \end{pmatrix}.$$

$$A_1[0,0] = 3, \quad A_1[1,1] = 2.$$

$$B_1 = T(A_0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.5 & 0.5 \\ 0 & 2.5 \end{pmatrix}.$$

$$B_1[0,0] = 1.5, \quad B_1[1,1] = 2.5.$$

Iteration 2:

$$A_2 = S(B_1) = \frac{1}{2}\begin{pmatrix} 1.5 & 0.5 \\ 0 & 2.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.75 & 0.25 \\ 0 & 2.25 \end{pmatrix}.$$

$$A_2[0,0] = 1.75, \quad A_2[1,1] = 2.25.$$

$$B_2 = T(A_1) = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0.5 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.5 & 0 \\ 0.25 & 2 \end{pmatrix}.$$

$$B_2[0,0] = 2.5, \quad B_2[1,1] = 2.$$

Iteration 3:

$$A_3 = S(B_2) = \frac{1}{2}\begin{pmatrix} 2.5 & 0 \\ 0.25 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.25 & 0 \\ 0.125 & 2 \end{pmatrix}.$$

$$A_3[0,0] = 2.25, \quad A_3[1,1] = 2.$$

$$B_3 = T(A_2) = \frac{1}{2}\begin{pmatrix} 1.75 & 0.25 \\ 0 & 2.25 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.875 & 0.125 \\ 0 & 2.125 \end{pmatrix}.$$

$$B_3[0,0] = 1.875, \quad B_3[1,1] = 2.125.$$

Iteration 4:

$$A_4 = S(B_3) = \frac{1}{2}\begin{pmatrix} 1.875 & 0.125 \\ 0 & 2.125 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.9375 & 0.0625 \\ 0 & 2.0625 \end{pmatrix}.$$

$$A_4[0,0] = 1.9375, \quad A_4[1,1] = 2.0625.$$

$$B_4 = T(A_3) = \frac{1}{2}\begin{pmatrix} 2.25 & 0 \\ 0.125 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2.125 & 0 \\ 0.0625 & 2 \end{pmatrix}.$$

$$B_4[0,0] = 2.125, \quad B_4[1,1] = 2.$$

The diagonal entries of the matrices $\{A_n\}$ and $\{B_n\}$ converge rapidly to the fixed point $\rho = 2$, satisfying condition (c3).

TABLE 2. Convergence of diagonal entries for 2×2 matrices A and B under the mappings S and T.

Number of Iterations	$A[0,0]$	$A[1,1]$	$B[0,0]$	$B[1,1]$
0	1.0000	3.0000	4.0000	2.0000
1	3.0000	2.0000	1.5000	2.5000
2	1.7500	2.2500	2.5000	2.0000
3	2.2500	2.0000	1.8750	2.1250
4	1.9375	2.0625	2.1250	2.0000

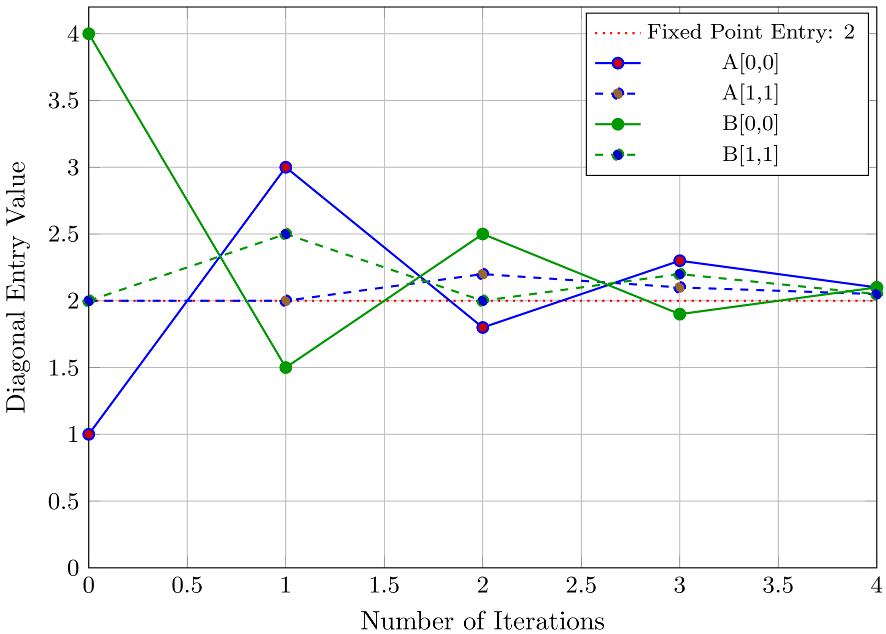

FIGURE 1. Diagonal entries of A and B matrices (2×2 case).

Case 2. We consider the convergence of the diagonal entries for 3×3 matrices. The numerical results Table 3 and Figure 2.

Initial Matrices:

$$A_0 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Mappings:

$$S(B) = \frac{1}{2}B + I, \quad T(A) = \frac{1}{2}A + I.$$

Iteration 0:

$$A_0[0,0] = 2, \quad A_0[1,1] = 3, \quad A_0[2,2] = 4,$$

$$B_0[0,0] = 1, \quad B_0[1,1] = 2, \quad B_0[2,2] = 1.$$

Iteration 1:

$$A_1 = S(B_0) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} + I = \begin{pmatrix} 1.5 & 0 & 0 \\ 0.5 & 2 & 0 \\ 0.5 & 0.5 & 1.5 \end{pmatrix}.$$

$$A_1[0,0] = 1.5, \quad A_1[1,1] = 2, \quad A_1[2,2] = 1.5.$$

$$B_1 = T(A_0) = \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} + I = \begin{pmatrix} 2 & 0.5 & 0.5 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 3 \end{pmatrix}.$$

$$B_1[0,0] = 2, \quad B_1[1,1] = 2.5, \quad B_1[2,2] = 3.$$

Iteration 2:

$$A_2 = S(B_1) = \frac{1}{2}\begin{pmatrix} 2 & 0.5 & 0.5 \\ 0 & 2.5 & 0.5 \\ 0 & 0 & 3 \end{pmatrix} + I = \begin{pmatrix} 2 & 0.25 & 0.25 \\ 0 & 2.25 & 0.25 \\ 0 & 0 & 2.5 \end{pmatrix}.$$

$$A_2[0,0] = 2, \quad A_2[1,1] = 2.25, \quad A_2[2,2] = 2.5.$$

$$B_2 = T(A_1) = \frac{1}{2}\begin{pmatrix} 1.5 & 0 & 0 \\ 0.5 & 2 & 0 \\ 0.5 & 0.5 & 1.5 \end{pmatrix} + I = \begin{pmatrix} 1.75 & 0 & 0 \\ 0.25 & 2 & 0 \\ 0.25 & 0.25 & 1.75 \end{pmatrix}.$$

$$B_2[0,0] = 1.75, \quad B_2[1,1] = 2, \quad B_2[2,2] = 1.75.$$

Iteration 3:

$$A_3 = S(B_2) = \frac{1}{2}\begin{pmatrix} 1.75 & 0 & 0 \\ 0.25 & 2 & 0 \\ 0.25 & 0.25 & 1.75 \end{pmatrix} + I = \begin{pmatrix} 1.875 & 0 & 0 \\ 0.125 & 2 & 0 \\ 0.125 & 0.125 & 1.875 \end{pmatrix}.$$

$$A_3[0,0] = 1.875, \quad A_3[1,1] = 2, \quad A_3[2,2] = 1.875.$$

$$B_3 = T(A_2) = \frac{1}{2}\begin{pmatrix} 2 & 0.25 & 0.25 \\ 0 & 2.25 & 0.25 \\ 0 & 0 & 2.5 \end{pmatrix} + I = \begin{pmatrix} 2 & 0.125 & 0.125 \\ 0 & 2.125 & 0.125 \\ 0 & 0 & 2.25 \end{pmatrix}.$$

$$B_3[0,0] = 2, \quad B_3[1,1] = 2.125, \quad B_3[2,2] = 2.25.$$

Iteration 4:

$$A_4 = S(B_3) = \frac{1}{2}\begin{pmatrix} 2 & 0.125 & 0.125 \\ 0 & 2.125 & 0.125 \\ 0 & 0 & 2.25 \end{pmatrix} + I = \begin{pmatrix} 2 & 0.0625 & 0.0625 \\ 0 & 2.0625 & 0.0625 \\ 0 & 0 & 2.125 \end{pmatrix}.$$

$$A_4[0,0] = 2, \quad A_4[1,1] = 2.0625, \quad A_4[2,2] = 2.125.$$

The diagonal entries of the matrices $\{A_n\}$ and $\{B_n\}$ converge rapidly to the fixed point $\rho = 2$, satisfying condition (c3).

TABLE 3. Convergence of diagonal entries for 3×3 matrices A and B under the mappings S and T.

Number of Iterations	$A[0,0]$	$A[1,1]$	$A[2,2]$	$B[0,0]$	$B[1,1]$	$B[2,2]$
0	2.0000	3.0000	4.0000	1.0000	2.0000	1.0000
1	1.5000	2.0000	1.5000	2.0000	2.5000	3.0000
2	2.0000	2.2500	2.5000	1.7500	2.0000	1.7500
3	1.8750	2.0000	1.8750	2.0000	2.1250	2.2500
4	2.0000	2.0625	2.1250	1.9375	2.0000	1.9375

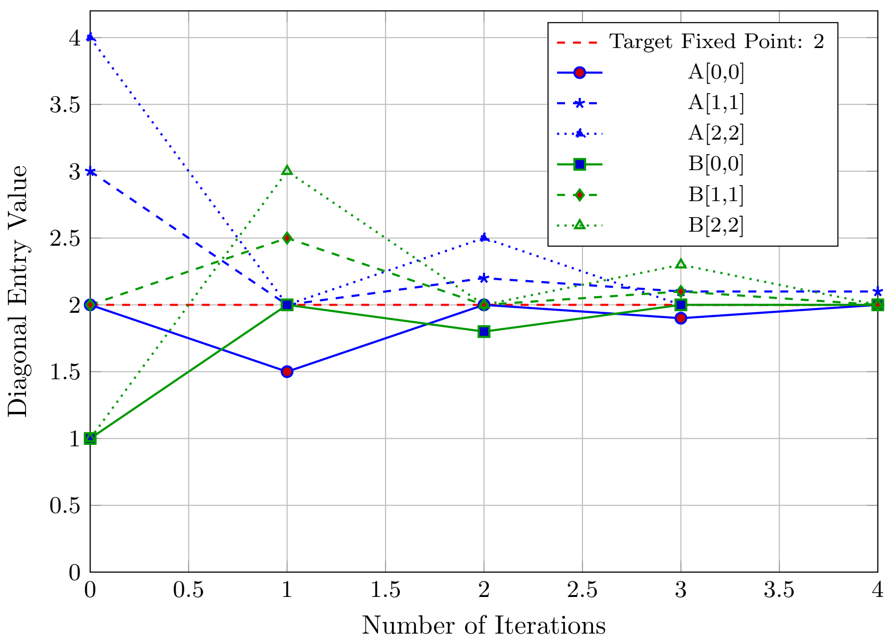

FIGURE 2. Diagonal entries of A and B matrices (3×3 case).

6. Conclusion

In this paper, we introduced a new class of ω-interpolative rational type contravariant contractions in the setting of complete bipolar metric spaces. We established sufficient conditions for the existence of a common fixed point of two contravariant mappings S and T that satisfy an interpolative rational type inequality.

By employing ω-orbital admissibility and continuity assumptions, together with appropriate contractive conditions, we demonstrated that these mappings generate sequences whose diagonal entries converge to a common fixed point. Numerical examples with explicit matrix iterations and graphical convergence plots illustrated the theoretical results in practice.

The developed framework generalizes several known fixed-point results in bipolar metric spaces and provides a unified approach to analyzing various classes of nonlinear mappings. Consequently, this work extends the applicability of fixed-point theory in nonlinear analysis and broadens its use in real-world problems such as matrix iterations and coupled integral equations.

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Declarations

Competing interests
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