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Convergence Theorem of Inertial Projection and Contraction Methods for Pseudomonotone Variational Inequalities Problem

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Abstract This paper presents a novel convergence theorem for an inertial projection and contraction-based method aimed at solving pseudomonotone variational inequality problems (VIP) in real Hilbert spaces. The proposed algorithm combines inertial dynamics and contraction projections to enhance the convergence speed and stability of solutions, even when the operator involved is not strongly monotone but pseudomonotone. We establish the strong convergence of the method under mild conditions, demonstrating its effectiveness in non-strongly monotone settings. The theoretical analysis is complemented by numerical experiments, which show that the proposed method significantly outperforms classical projection and extragradient methods in terms of both computational efficiency and iteration count. This work contributes to the broader field of nonlinear optimization by providing a robust and efficient solution approach for VIP with pseudomonotone operators, and it opens pathways for further research on adaptive and large-scale extensions of the method.

MSC: 47H09, 47H10

Keywords: Variational Inequality Problem; Projection and Contraction Method; Pseudomonotone Mapping

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1. Introduction

Variational inequality problem (VIP) have become a central tool in both pure and applied sciences, offering a unified framework for a wide variety of problems. It is well known that many problems in society and science can be formulated using variational inequality models, which play an important role in nonlinear optimization theory and its applications. Recently, VIP have attracted significant attention from researchers interested not only in theoretical developments but also in numerical approaches to solving such problems.

The variational inequality problem can be stated as follows:

$$\text{Find } z^* \in C \text{ such that } \langle Mz^*, z - z^* \rangle \geq 0, \quad \forall z \in C, \tag{1.1}$$

where C is a nonempty, closed, and convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and $M : H \to H$ is a nonlinear mapping. The solution set is denoted by $VI(C, M)$.

It is a well-established fact that a point z^* constitutes a solution to the VIP in equation (1.1) if and only if it resolves the associated fixed-point equation:

$$z^* = P_C(z^* - \tau M z^*), \quad \forall tau > 0, \tag{1.2}$$

where P_C is the projection operator from H onto C.

One of the projection methods for solving the problem described in (1.1) is the extragradient method, which was introduced by Korpelevich [1] and independently by Antipin [2]. The extragradient method is formulated as follows:

$$\begin{cases} s_n = P_C(w_n - \tau_n M w_n), \\ z_{n+1} = P_C(s_n - \tau_n M s_n), \end{cases} \tag{1.3}$$

where $\tau_n \in (0, 1/L)$ and $M : C \to H$ be monotone and L-Lipschitz continuous operator. Recently, the extragradient method has produced conclusive results under the assumptions of monotonicity and Lipschitz continuity of the mappings (see, e.g., [4–6]).

In 2021, Tian and Xu [7] introduced the following inertial projection and contraction method to circumvent this obstacle.

$$\begin{cases} w_0, w_1 \in H, \\ s_n = w_n + \zeta_n (w_n - w_{n-1}), \\ t_n = P_C(s_n - \tau_n M s_n), \\ d(s_n, t_n) = (s_n - t_n) - \tau_n (M s_n - M t_n), \\ v_n = s_n - \lambda \eta_n d(s_n, t_n), \\ s_{n+1} = (1 - \alpha_n - \beta_n) s_n + \beta_n v_n, \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\zeta_n\}$ are control sequences in $(0, 1)$, $\lambda \in (0, 2)$ and

$$\eta_n = \begin{cases} \dfrac{\varphi(s_n, t_n)}{\|d(s_n, t_n)\|^2}, & \text{if } d(s_n, t_n) \neq 0, \\ 0, & \text{if } d(s_n, t_n) \neq 0, \end{cases}.$$

where $\varphi(s_n, t_n) = \langle s_n - t_n, d(s_n, t_n) \rangle$. The step size τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, \dots\}$ such that

$$\tau\|Ms_n - Mt_n\| \leq \kappa\|s_n - t_n\|, \quad \gamma > 0, l \in (0,1), \kappa \in (0,1).$$

Equation (1.4) assumes that M is uniformly continuous on C and pseudomonotone. In order to resolve variational inequalities and associated optimization problems, a range of numerical techniques featuring inertial exponentiation steps have been proposed; refer to [8–15] as well as the cited works.

Jolaoso et al. [16] propose a modified inertial projection and contraction algorithm to solve the pseudomonotone VIP, as follows:
Given $\tau_1 > 0$, $\mu \in (0,1)$ and $\lambda \in \left(1, \frac{2}{\sigma}\right)$, where $\sigma \in (1,2)$. Choose $\{\zeta_n\} \subset [0,1)$.
1. Let $w_0, w_1 \in H$ be arbitrary. Given the iterates w_{n-1} and w_n $(n \geq 1)$. Compute

$$s_n = w_n + \zeta_n(w_n - w_{n-1}).$$

2. Compute

$$t_n = P_C(s_n - \tau_n M s_n).$$

If $s_n = t_n$ or $Mt_n = 0$, then stop and t_n is a solution of VIP. Otherwise, go to:
3. Compute

$$s_{n+1} = s_n - \lambda\eta_n d_n,$$

where η_n and d_n are defined as follows:

$$\eta_n := (1-\mu)\frac{\|s_n - t_n\|^2}{\|d_n\|^2}, \quad d_n := s_n - t_n - \tau_n(Ms_n - Mt_n), \tag{3.1}$$

and update stepsize by

$$\tau_{n+1} = \min\left\{\frac{\mu\|s_n - t_n\|}{\|Ms_n - Mt_n\|}, \tau_n\right\}. \tag{3.2}$$

Set $n := n+1$ and return to **Step 1**.

Conversely, the inertial method has garnered significant interest and attention from researchers. Recently, this technique has been frequently employed to accelerate the convergence rates of algorithms for various optimization problems (see, for example, [17–20]).

Motivated by the aforementioned works, we propose two modified inertial projection and contraction methods with self-adaptive step size rules to solve the pseudomonotone variational inequality problem in real Hilbert spaces. These adaptive step size rules are designed to enhance efficiency and flexibility in computations, eliminating the need for a line search procedure, which can be time-consuming and costly. Additionally, we establish weak and strong convergence theorems for the proposed methods without requiring prior knowledge of the Lipschitz constant of the mapping or assuming the weak sequential continuity of the mapping.

The remainder of the paper is structured as follows: Section 2, we present some preliminary results necessary for our work. In Section 3, we prove weak and strong convergence theorems for the proposed methods. Finally, in Section 4, we provide numerical experiments that include comparisons with other algorithms and applications of the proposed algorithms in the image deblurring problem.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The weak convergence of $\{w_n\}$ to z^* is denoted by $w_n \rightharpoonup z^*$ as $n \to \infty$, while the strong convergence of $\{w_n\}$ to z^* is written as $w_n \to z^*$ as $n \to \infty$. For all $s, z \in H$, we have

$$\|s + z\|^2 \leq \|s\|^2 + 2\langle z, s + z \rangle.$$

Definition 2.1. [21] Let $M : H \to H$ be an operator. Then:

(1) M is called L-Lipschitz continuous with constant $L > 0$ if

$$\|Ms - Mz\| \leq L\|s - z\| \quad \forall s, z \in H.$$

If $L = 1$, then M is called nonexpansive, if $L \in (0, 1)$, M is called a contraction.

(2) M is called monotone, if

$$\langle Ms - Mz, s - z \rangle \geq 0 \quad \forall s, z \in H.$$

(3) M is called pseudomonotone in the sense of Karamardian [22] if

$$\langle Ms, z - s \rangle \geq 0 \Rightarrow \langle Mz, z - s \rangle \geq 0 \quad \forall s, z \in H. \tag{2.1}$$

(4) M is called *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Mr - Mt, r - t \rangle \geq \alpha\|r - t\|^2 \quad \forall r, t \in H.$$

(5) M is called *α-strongly pseudomonotone* if there exists a constant $\alpha > 0$ such that

$$\langle Mz, s - z \rangle \geq 0 \Rightarrow \langle Ms, s - z \rangle \geq \alpha\|z - s\|^2 \quad \forall s, z \in H.$$

(6) The operator M is called sequentially weakly continuous if for each sequence $\{w_n\}$ we have: $w_n \rightharpoonup z^*$ weakly implies $Mw_n \rightharpoonup Mz^*$ weakly.

We note that (2.1) represents only one of several definitions of pseudomonotonicity available in the literature. For every point $z \in H$, there exists a unique nearest point in C, denoted by $P_C z$, such that

$$\|z - P_C z\| \leq \|z - s\| \quad \text{for all } s \in C.$$

The mapping P_C is referred to as the metric projection of H onto C. It is well known that P_C is nonexpansive. For further properties of the metric projection, the reader is referred to Section 3 in [23].

Lemma 2.2. [23] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $z \in H$ and $r \in C$. Then $r = P_C z \iff \langle z - r, r - s \rangle \geq 0 \quad \forall s \in C$. Moreover,*

$$\|P_C z - P_C s\|^2 \leq \langle P_C z - P_C s, z - s \rangle \quad \forall s, z \in C.$$

Lemma 2.3. [24] *Consider the problem $\mathrm{Sol}(C, M)$ with C being a nonempty, closed, convex subset of a real Hilbert space H, and $M : C \to H$ being pseudomonotone and continuous. Then $z^* \in \mathrm{Sol}(C, M)$ if and only if*

$$\langle Mz, z - z^* \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.4. [25] *Let $M : C \to H$ be a mapping. For $z \in H$ and $\alpha \geq \beta > 0$, the following inequalities hold:*

$$\frac{\|z - P_C(z - \alpha Mz)\|}{\alpha} \leq \frac{\|z - P_C(z - \beta Mz)\|}{\beta} \leq \frac{\|z - P_C(z - \alpha Mz)\|}{\alpha}.$$

Lemma 2.5. [12] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences in $[0,+\infty)$ such that*

$$a_{n+1} \leq a_n + c_n(a_n - a_{n-1}) + b_n \quad \forall n \geq 1, \quad \sum_{n=1}^{\infty} b_n < +\infty,$$

and there exists a real number c with $0 \leq c_n \leq c < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(i) $\sum_{n=1}^{\infty}[a_n - a_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;
(ii) *there exists $a^* \in [0,+\infty)$ such that* $\lim_{n\to\infty} a_n = a^*$.

Lemma 2.6. [26] *Let C be a nonempty subset of H and $\{w_n\}$ be a sequence in H such that the following two conditions hold:*

(i) *for every $z^* \in C$, $\lim_{n\to\infty} \|w_n - z^*\|$ exists*;
(ii) *every sequential weak cluster point of $\{w_n\}$ is in C.*

Then $\{w_n\}$ converges weakly to a point in C.

3. Main results

In this section, we propose modified gradient projection method for solving VI. We assume that the following conditions hold:

Condition 1. The solution set $\mathrm{Sol}(C, M)$ of (1.1) is nonempty.

Condition 2. The mapping $M : H \to H$ is pseudomonotone on H, that is,

$$\langle Mz, s - z\rangle \geq 0 \Rightarrow \langle Ms, s - z\rangle \geq 0, \quad \forall s, z \in H.$$

In addition, the mapping $M : H \to H$ satisfies the condition

$$\{t_n\} \subset C,\ t_n \rightharpoonup t \Longrightarrow \|Mt\| \leq \liminf_{n\to\infty} \|Mt_n\|. \tag{3.1}$$

Condition 3. $M : H \to H$ is uniformly continuous on bounded subsets of H.

Algorithm 1

1: **Initialization:** Let $w_0, w_1 \in H$ be arbitrary and parameters $\rho, l \in (0,1), \kappa \in (0,1), \sigma, \zeta \in [0,1]$, along with a sequence $\{\delta_n\}$ in $(0,1)$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\{\alpha_n\} \subset (a, 1-\xi)$ for some $a > 0$ and $\xi \in [0,1)$.

2: Let $\{\gamma_n\} \in (0,1)$ and a nonnegative sequence $\{\phi_n\}$ satisfy

$$\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty, \ \text{and} \ \lim_{n\to\infty} \frac{\phi_n}{\gamma_n} = 0.$$

3: **Compute**

$$\sigma_n = \begin{cases} \min\left\{\dfrac{\sigma}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\sigma}{2}, & \text{otherwise.} \end{cases} \tag{3.2}$$

4: **Compute** $s_n = \gamma_n w_n + (1-\gamma_n)[w_n + \sigma_n(w_n - w_{n-1})]$.

Compute

$$\zeta_n = \begin{cases} \min\left\{\dfrac{\zeta}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\zeta}{2}, & \text{otherwise.} \end{cases} \tag{3.3}$$

Compute $r_n = \delta_n w_n + (1-\delta_n)[w_n + \zeta_n(w_n - w_{n-1})]$.

Compute

$$q_n = P_C(r_n - \tau_n M r_n),$$

where τ_n is chosen to be the largest $\tau \in \{\rho, \rho l^2, \rho l^3, \ldots\}$ satisfying

$$\tau\|Mq_n - Mr_n\| \leq \kappa\|q_n - r_n\|. \tag{3.4}$$

If $r_n = q_n$ or $Mr_n = 0$, then stop and r_n is a solution of (1.1). Otherwise

Compute $d_n = r_n - q_n - \tau_n(Mr_n - Mq_n)$.

Compute

$$t_n = r_n - \mu\eta_n d_n,$$

where

$$\eta_n = \begin{cases} \dfrac{\langle r_n - q_n, d_n\rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0 \\ 0, & \text{if } d_n = 0. \end{cases}$$

Compute $v_n = (1-\alpha_n)s_n + \alpha_n w_n$.

Compute $w_{n+1} = (1-\gamma_n)v_n + \gamma_n t_n$.

End for loop

Lemma 3.1. *Suppose $M : H \to H$ is uniformly continuous on bounded subsets of H. Then the Armijo-like search rule (3.4) is well defined. Moreover, $\tau_n \leq \rho$.*

Proof. Let $r_n \in H$ and define $q_n = P_C(r_n - \tau M r_n)$. As P_C is nonexpansive and M uniformly continuous on bounded sets, q_n remains bounded.

Define $\phi(\tau) = \tau_n\|Mq_n - Mr_n\| - \kappa\|q_n - r_n\|$. Since both norms go to 0 as $\tau \to 0$, $\phi(\tau) \to 0$. Therefore, some $\tau = \rho l^k$ satisfies $\phi(\tau) \leq 0$, ensuring $\tau_n \leq \rho$ and the rule is well-defined.

Lemma 3.2. *Suppose Conditions 1 and 2 hold. Let $\{w_n\}$ be generated by Algorithm 1. Then for all $z^* \in Sol(C, M)$,*

$$\|t_n - z^*\|^2 \leq \|r_n - z^*\|^2 - \frac{2-\mu}{\mu}\|r_n - t_n\|^2.$$

Proof. Since $q_n = P_C(r_n - \tau_n M r_n)$, the projection inequality gives

$$\langle r_n - q_n, Mr_n\rangle \geq \langle r_n - q_n, z^* - q_n\rangle.$$

Using pseudomonotonicity and that $z^* \in \mathrm{Sol}(C, M)$,

$$\langle Mq_n, q_n - z^*\rangle \geq 0.$$

Adding these

$$\langle q_n - z^*, d_n\rangle \geq 0.$$

Thus,

$$\langle r_n - z^*, d_n\rangle \geq \langle r_n - q_n, d_n\rangle.$$

Now expand

$$\|t_n - z^*\|^2 = \|r_n - \mu\eta_n d_n - z^*\|^2 = \|r_n - z^*\|^2 - 2\mu\eta_n\langle r_n - z^*, d_n\rangle + \mu^2\eta_n^2\|d_n\|^2.$$

Substituting $\eta_n\|d_n\|^2 = \langle r_n - q_n, d_n\rangle$, and $r_n - t_n = \mu\eta_n d_n$, we get

$$\|t_n - z^*\|^2 \leq \|r_n - z^*\|^2 - \frac{2-\mu}{\mu}\|r_n - t_n\|^2.$$

Lemma 3.3. *Suppose Conditions 1 and 2 hold. Let $\{w_n\}$ be generated by Algorithm 1. Then*

$$\|r_n - q_n\|^2 \leq \frac{(1+\kappa)^2}{(1-\kappa)^2\mu^2}\|r_n - t_n\|^2.$$

Proof. We first show

$$(1-\kappa)\|r_n - q_n\| \leq \|d_n\| \leq (1+\kappa)\|r_n - q_n\|.$$

This follows from the Armijo-like rule

$$\|d_n\| = \|r_n - q_n - \tau_n(Mr_n - Mq_n)\| \geq \|r_n - q_n\| - \tau_n\|Mr_n - Mq_n\| \geq (1-\kappa)\|r_n - q_n\|.$$

Similarly for the upper bound

$$\|d_n\| \leq \|r_n - q_n\| + \tau_n\|Mr_n - Mq_n\| \leq (1+\kappa)\|r_n - q_n\|.$$

Now, recall $\eta_n = \dfrac{\langle r_n - q_n, d_n\rangle}{\|d_n\|^2}$, and

$$\langle r_n - q_n, d_n\rangle \geq (1-\kappa)\|r_n - q_n\|^2.$$

Then,

$$\|r_n - t_n\| = \mu\eta_n\|d_n\| = \mu \cdot \frac{\langle r_n - q_n, d_n\rangle}{\|d_n\|}.$$

Using the bounds:

$$\|r_n - t_n\| \geq \mu \cdot \frac{(1-\kappa)\|r_n - q_n\|^2}{(1+\kappa)\|r_n - q_n\|} = \mu \cdot \frac{(1-\kappa)}{(1+\kappa)}\|r_n - q_n\|.$$

Hence,

$$\|r_n - q_n\| \leq \frac{(1+\kappa)}{(1-\kappa)\mu}\|r_n - t_n\|,$$

and squaring both sides

$$\|r_n - q_n\|^2 \leq \frac{(1+\kappa)^2}{(1-\kappa)^2\mu^2}\|r_n - t_n\|^2.$$

Lemma 3.4. *Suppose Conditions 1–3 hold. Let $\{r_n\}$ be any sequence generated by Algorithm 1. If there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $\{r_{n_k}\}$ converges weakly to $z^* \in C$ and $\lim_{k\to\infty} \|r_{n_k} - q_{n_k}\| = 0$, then $z^* \in \mathrm{Sol}(C, M)$.*

Proof. We have $q_{n_k} = P_C(r_{n_k} - \tau_{n_k} M r_{n_k})$, hence,

$$\langle r_{n_k} - \tau_{n_k} M r_{n_k} - q_{n_k}, z - q_{n_k}\rangle \leq 0 \quad \forall z \in C,$$

which is equivalent to

$$\frac{1}{\tau_{n_k}}\langle r_{n_k} - q_{n_k}, z - q_{n_k}\rangle \leq \langle M r_{n_k}, z - q_{n_k}\rangle \quad \forall z \in C.$$

This implies

$$\frac{1}{\tau_{n_k}}\langle r_{n_k} - q_{n_k}, z - q_{n_k}\rangle + \langle M r_{n_k}, q_{n_k} - r_{n_k}\rangle \leq \langle M r_{n_k}, z - r_{n_k}\rangle \quad \forall z \in C. \tag{3.5}$$

We now show that

$$\liminf_{k\to\infty} \langle M r_{n_k}, z - r_{n_k}\rangle \geq 0. \tag{3.6}$$

We consider two cases:

Case 1: $\liminf_{k\to\infty} \tau_{n_k} > 0$. Then $\{r_{n_k}\}$ is bounded, and since M is uniformly continuous on bounded subsets, $\{M r_{n_k}\}$ is also bounded. As $\|r_{n_k} - q_{n_k}\| \to 0$, taking $k \to \infty$ in (3.5) gives

$$\liminf_{k\to\infty} \langle M r_{n_k}, z - r_{n_k}\rangle \geq 0.$$

Case 2: $\liminf_{k\to\infty} \tau_{n_k} = 0$. Define $\tilde{q}_{n_k} = P_C(r_{n_k} - \frac{\tau_{n_k}}{l} M r_{n_k})$. Since $\frac{\tau_{n_k}}{l} > \tau_{n_k}$ and P_C is nonexpansive,

$$\|r_{n_k} - \tilde{q}_{n_k}\| \leq \frac{1}{l}\|r_{n_k} - \tilde{q}_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Hence, $\tilde{q}_{n_k} \rightharpoonup z^* \in C$ and $\{\tilde{q}_{n_k}\}$ is bounded. By uniform continuity of M, it follows that

$$\|M r_{n_k} - M\tilde{q}_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.7}$$

and

$$\frac{\tau_{n_k}}{l}\|M r_{n_k} - M P_C(r_{n_k} - \frac{\tau_{n_k}}{l} M r_{n_k})\| > \kappa \|r_{n_k} - P_C(r_{n_k} - \frac{\tau_{n_k}}{l} M r_{n_k})\|.$$

We also have

$$\frac{1}{\kappa}\|M r_{n_k} - M\tilde{q}_{n_k}\| > \frac{l}{\tau_{n_k}}\|r_{n_k} - \tilde{q}_{n_k}\|. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\lim_{k\to\infty} \frac{l}{\tau_{n_k}}\|M r_{n_k} - M\tilde{q}_{n_k}\| = 0.$$

Using again the projection inequality:

$$\langle r_{n_k} - \frac{\tau_{n_k}}{l} M r_{n_k} - \tilde{q}_{n_k}, z - \tilde{q}_{n_k}\rangle \leq 0 \quad \forall z \in C,$$

which is equivalent to

$$\frac{l}{\tau_{n_k}}\langle r_{n_k} - \tilde{q}_{n_k}, z - \tilde{q}_{n_k}\rangle \leq \langle M r_{n_k}, z - \tilde{q}_{n_k}\rangle \quad \forall z \in C.$$

This implies

$$\frac{l}{\tau_{n_k}}\langle r_{n_k} - \tilde{q}_{n_k}, z - \tilde{q}_{n_k}\rangle + \langle Mr_{n_k}, \tilde{q}_{n_k} - r_{n_k}\rangle \leq \langle Mr_{n_k}, z - r_{n_k}\rangle \quad \forall z \in C.$$

Taking the limit as $k \to \infty$, we obtain

$$\liminf_{k\to\infty}\langle Mr_{n_k}, z - r_{n_k}\rangle \geq 0.$$

This completes the proof of (3.6). Now we proceed to prove $z^* \in \mathrm{Sol}(C, M)$.
Let $\{\epsilon_k\}$ be a sequence of positive real numbers decreasing to 0. For each k, let N_k be the smallest index such that

$$\langle Mr_{N_k}, z - r_{N_k}\rangle + \epsilon_k \geq 0. \tag{30}$$

Define

$$p_{N_k} := \frac{Mr_{N_k}}{\|Mr_{N_k}\|^2},$$

so that

$$\langle Mr_{N_k}, z + \epsilon_k p_{N_k} - r_{N_k}\rangle \geq 0.$$

Since M is pseudomonotone, we have

$$\langle M(z + \epsilon_k p_{N_k}), z + \epsilon_k p_{N_k} - r_{N_k}\rangle \geq 0.$$

Hence,

$$\langle Mz, z - r_{N_k}\rangle \geq \langle M(z + \epsilon_k p_{N_k}), z + \epsilon_k p_{N_k} - r_{N_k}\rangle - \epsilon_k\langle Mz, p_{N_k}\rangle. \tag{31}$$

We show $\epsilon_k p_{N_k} \to 0$. Since $r_{n_k} \rightharpoonup z^*$ and M satisfies Condition (9), we have

$$0 < \|Mz^*\| \leq \liminf_{k\to\infty}\|Mr_{n_k}\| \quad (\text{if } Mz^* = 0, \text{ then } z^* \in \mathrm{Sol}(C, M)).$$

Since $\epsilon_k \to 0$ and $\{r_{N_k}\} \subset \{r_{n_k}\}$, we get

$$\epsilon_k\|p_{N_k}\| = \frac{\epsilon_k}{\|Mr_{N_k}\|} \to 0.$$

Hence, $\epsilon_k p_{N_k} \to 0$, and by the continuity of M,

$$\liminf_{k\to\infty}\langle Mz, z - r_{N_k}\rangle \geq 0.$$

Finally, since $r_{N_k} \to z^*$ weakly and $z \in C$ arbitrary, we obtain

$$\langle Mz, z - z^*\rangle \geq 0 \quad \forall z \in C.$$

By Lemma 2.2, we conclude $z^* \in \mathrm{Sol}(C, M)$, completing the proof.

Theorem 3.5. *Suppose Conditions 1–3 hold. Then, the sequence $\{w_n\}$ generated by Algorithm 1 converges weakly to an element $z^* \in \mathrm{Sol}(C, F)$.*

Proof. Let $z^* \in \mathrm{Sol}(C, F)$ be arbitrary. From the definition of Algorithm 1 and Lemma 3.2, we have the inequality

$$\begin{aligned} \|t_n - z^*\|^2 &\leq \|r_n - z^*\|^2 - \frac{2-\mu}{\mu}\|r_n - t_n\|^2 \\ &\leq \|r_n - z^*\|^2. \end{aligned} \tag{3.9}$$

Now, using the identity $w_{n+1} = (1-\gamma_n)v_n + \gamma_n t_n$, we expand $\|w_{n+1} - z^*\|^2$ as follows

$$\begin{aligned} \|w_{n+1} - z^*\|^2 &= \|(1-\gamma_n)v_n + \gamma_n t_n - z^*\|^2 \\ &= \|(1-\gamma_n)(v_n - z^*) + \gamma_n(t_n - z^*)\|^2 \\ &= (1-\gamma_n)\|v_n - z^*\|^2 + \gamma_n\|t_n - z^*\|^2 - \gamma_n(1-\gamma_n)\|v_n - t_n\|^2. \end{aligned} \tag{3.10}$$

Combining inequalities (3.9) and (3.10), we get

$$\|w_{n+1} - z^*\|^2 \le (1-\gamma_n)\|v_n - z^*\|^2 + \gamma_n\|r_n - z^*\|^2 - \gamma_n(1-\gamma_n)\|v_n - t_n\|^2. \tag{3.11}$$

Note that

$$w_{n+1} = (1-\gamma_n)v_n + \gamma_n t_n$$

and this implies that

$$t_n - v_n = \frac{1}{\gamma_n}(w_{n+1} - v_n). \tag{3.12}$$

Substituting (3.12) into (3.11), we get

$$\|w_{n+1} - z^*\|^2 \le (1-\gamma_n)\|v_n - z^*\|^2 + \gamma_n\|r_n - z^*\|^2 - \frac{(1-\gamma_n)}{\gamma_n}\|w_{n+1} - v_n\|^2. \tag{3.13}$$

From Algorithm 1, recall that

$$v_n = (1-\alpha_n)s_n + \alpha_n w_n.$$

Hence, we compute

$$\begin{aligned} \|v_n - z^*\|^2 &= \|(1-\alpha_n)s_n + \alpha_n w_n - z^*\|^2 \\ &= \|(1-\alpha_n)(s_n - z^*) + \alpha_n(w_n - z^*)\|^2 \\ &= (1-\alpha_n)\|s_n - z^*\|^2 + \alpha_n\|w_n - z^*\|^2 - \alpha_n(1-\alpha_n)\|s_n - w_n\|^2. \end{aligned} \tag{3.14}$$

Using the definition of s_n from Algorithm 1, we obtain

$$\begin{aligned} \|s_n - z^*\| &= \|\gamma_n w_n + (1-\gamma_n)[w_n + \sigma_n(w_n - w_{n-1})] - z^*\| \\ &\le \gamma_n\|w_n - z^*\| + (1-\gamma_n)\|w_n - z^*\| + \sigma_n\|w_n - w_{n-1}\| \\ &= \|w_n - z^*\| + \sigma_n\|w_n - w_{n-1}\| \\ &= \|w_n - z^*\| + \gamma_n\frac{\sigma_n}{\gamma_n}\|w_n - w_{n-1}\| \\ &\le \|w_n - z^*\| + \gamma_n Q_1 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|s_n - w_n\| &= \|\gamma_n w_n + (1-\gamma_n)[w_n + \sigma_n(w_n - w_{n-1})] - w_n\| \\ &\le \gamma_n\|w_n - w_n\| + (1-\gamma_n)\|w_n - w_n\| + \sigma_n\|w_n - w_{n-1}\| \\ &= \sigma_n\|w_n - w_{n-1}\| \\ &= \gamma_n\frac{\sigma_n}{\gamma_n}\|w_n - w_{n-1}\| \\ &\le \gamma_n Q_2 \end{aligned} \tag{3.16}$$

for some constants $Q_1, Q_2 > 0$. Choose a positive constant $Q_3 > 0$. From inequality (3.15), we have

$$\begin{aligned}\|s_n - z^*\|^2 &\leq (\|w_n - z^*\| + \gamma_n Q_1)^2 \\ &= \|w_n - z^*\|^2 + \gamma_n(2Q_1\|w_n - z^*\| + \gamma_n Q_1^2) \\ &\leq \|w_n - z^*\|^2 + \gamma_n Q_3.\end{aligned} \tag{3.17}$$

Likewise, pick another positive constant $Q_4 > 0$. From inequality (3.16), we obtain

$$\begin{aligned}\|s_n - w_n\|^2 &\leq (\gamma_n Q_2)^2 \\ &= \gamma_n(\gamma_n Q_2^2) \\ &\leq \gamma_n Q_4.\end{aligned} \tag{3.18}$$

Now, substituting (3.17) and (3.18) into (3.14), we obtain

$$\begin{aligned}\|v_n - z^*\|^2 &\leq (1-\alpha_n)[\|w_n - z^*\|^2 + \gamma_n Q_3] + \alpha_n\|w_n - z^*\|^2 - \alpha_n(1-\alpha_n)\gamma_n Q_4 \\ &= \|w_n - z^*\|^2 + (1-\alpha_n)\gamma_n Q_3 - \alpha_n(1-\alpha_n)\gamma_n Q_4 \\ &= \|w_n - z^*\|^2 + \gamma_n[(1-\alpha_n)Q_3 - \alpha_n(1-\alpha_n)Q_4] \\ &\leq \|w_n - z^*\|^2 + \gamma_n Q_5\end{aligned} \tag{3.19}$$

for some constants $Q_5 > 0$. Using the definition of r_n from Algorithm 1, we obtain

$$\begin{aligned}\|r_n - z^*\| &= \|\delta_n w_n + (1-\delta_n)[w_n + \zeta_n(w_n - w_{n-1})] - z^*\| \\ &\leq \delta_n\|w_n - z^*\| + (1-\delta_n)\|w_n - z^*\| + \zeta_n\|w_n - w_{n-1}\| \\ &= \|w_n - z^*\| + \zeta_n\|w_n - w_{n-1}\| \\ &= \|w_n - z^*\| + \gamma_n\frac{\zeta_n}{\gamma_n}\|w_n - w_{n-1}\| \\ &\leq \|w_n - z^*\| + \gamma_n Q_6\end{aligned} \tag{3.20}$$

for some constants $Q_6 > 0$. Choose a positive constant $Q_7 > 0$. From inequality (3.20), we have

$$\begin{aligned}\|r_n - z^*\|^2 &\leq (\|w_n - z^*\| + \gamma_n Q_6)^2 \\ &= \|w_n - z^*\|^2 + \gamma_n(2Q_6\|w_n - z^*\| + \gamma_n Q_6^2) \\ &\leq \|w_n - z^*\|^2 + \gamma_n Q_7.\end{aligned} \tag{3.21}$$

Moreover, we get

$$\begin{aligned}\|w_{n+1} - v_n\|^2 &= \|w_{n+1} - [(1-\alpha_n)s_n + \alpha_n w_n]\|^2 \\ &= \|(1-\alpha_n)s_n + \alpha_n w_n - w_{n+1}\|^2 \\ &= \|(1-\alpha_n)(s_n - w_{n+1}) + \alpha_n(w_n - w_{n+1})\|^2 \\ &= (1-\alpha_n)\|s_n - w_{n+1}\|^2 + \alpha_n\|w_n - w_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|s_n - w_n\|^2.\end{aligned} \tag{3.22}$$

Also,

$$\begin{aligned}\|s_n - w_{n+1}\| &= \|\gamma_n w_n + (1-\gamma_n)[w_n + \sigma_n(w_n - w_{n-1})] - w_{n+1}\| \\ &\leq \gamma_n\|w_n - w_{n+1}\| + (1-\gamma_n)\|w_n - w_{n+1}\| + \sigma_n\|w_n - w_{n-1}\| \\ &= \|w_n - w_{n+1}\| + \sigma_n\|w_n - w_{n-1}\| \\ &= \|w_n - w_{n+1}\| + \gamma_n\frac{\sigma_n}{\gamma_n}\|w_n - w_{n-1}\| \\ &\leq \|w_n - w_{n+1}\| + \gamma_n Q_8\end{aligned} \tag{3.23}$$

for some constants $Q_8 > 0$. Choose a positive constant $Q_9 > 0$. From inequality (3.23), we have

$$\begin{aligned}\|s_n - w_{n+1}\|^2 &\leq (\|w_n - w_{n+1}\| + \gamma_n Q_8)^2 \\ &= \|w_n - w_{n+1}\|^2 + \gamma_n(2Q_8\|w_n - w_{n+1}\| + \gamma_n Q_8^2) \\ &\leq \|w_n - w_{n+1}\|^2 + \gamma_n Q_9.\end{aligned} \tag{3.24}$$

Substituting (3.18) and (3.24) into (3.22), we obtain

$$\begin{aligned}\|w_{n+1} - v_n\|^2 &\leq (1-\alpha_n)[\|w_n - w_{n+1}\|^2 + \gamma_n Q_9] + \alpha_n\|w_n - w_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\gamma_n Q_2 \\ &= \|w_n - w_{n+1}\|^2 + (1-\alpha_n)\gamma_n Q_9 - \alpha_n(1-\alpha_n)\gamma_n Q_2 \\ &= \|w_n - w_{n+1}\|^2 + \gamma_n[(1-\alpha_n)Q_9 - \alpha_n(1-\alpha_n)Q_2] \\ &\leq \|w_n - w_{n+1}\|^2 + \gamma_n Q_{10}\end{aligned} \tag{3.25}$$

for some constants $Q_{10} > 0$. Substituting (3.18), (3.19) and (3.25) into (3.13), we obtain

$$\begin{aligned}\|w_{n+1} - z^*\|^2 &\leq (1-\gamma_n)[\|w_n - z^*\|^2 + \gamma_n Q_5] + \gamma_n[\|w_n - z^*\|^2 + \gamma_n Q_7] \\ &\quad - \frac{(1-\gamma_n)}{\gamma_n}[\|w_n - w_{n+1}\|^2 + \gamma_n Q_{10}] \\ &= \|w_n - z^*\|^2 + (1-\gamma_n)\gamma_n Q_5 + \gamma_n^2 Q_5 \\ &\quad - \frac{(1-\gamma_n)}{\gamma_n}[\|w_n - w_{n+1}\|^2 + \gamma_n Q_{10}] \\ &= \|w_n - z^*\|^2 + \gamma_n[(1-\gamma_n)Q_5 + \gamma_n Q_5 - \frac{(1-\gamma_n)}{\gamma_n}Q_{10}] \\ &\quad - \frac{(1-\gamma_n)}{\gamma_n}\|w_n - w_{n+1}\|^2 \\ &\leq \|w_n - z^*\|^2 + \gamma_n Q_{11} - \frac{(1-\gamma_n)}{\gamma_n}\|w_n - w_{n+1}\|^2.\end{aligned} \tag{3.26}$$

Inequality (3.26) demonstrates a generalized Fejér monotonicity: the sequence $\{\|w_n - z^*\|^2\}$ is decreasing up to a summable error $\gamma_n Q_{11}$ and a decrement $\|w_n - w_{n+1}\|^2$. This behavior implies that $\{w_n\}$ remains close to the solution set and stabilizes over iterations.

This inequality implies that $\{\|w_n - z^*\|^2\}$ is a quasi-Fejér monotone sequence. Since $\sum \gamma_n Q_{11} < \infty$ and $\sum \|w_n - w_{n+1}\|^2 < \infty$, we deduce that:

$$\lim_{n\to\infty} \|w_n - w_{n+1}\| = 0, \quad \text{and} \quad \lim_{n\to\infty} \|w_n - z^*\| \text{ exists.}$$

This inequality implies that $\{\|w_n - z^*\|^2\}$ is a Fejér monotone sequence up to summable perturbations. Thus,

$$\sum_{n=1}^{\infty} \|w_n - w_{n+1}\|^2 < \infty, \quad \text{and} \quad \lim_{n\to\infty} \|w_n - w_{n+1}\| = 0.$$

Since $\{w_n\}$ is bounded, it has weakly convergent subsequences. Let $w_{n_k} \rightharpoonup z^*$. Using Lemma 3.3 and the demiclosedness principle under Conditions 1–3, we conclude $z^* \in \mathrm{Sol}(C, F)$. Finally, we apply Opial's lemma, the entire sequence $\{w_n\}$ converges weakly to $z^* \in \mathrm{Sol}(C, F)$.

4. Numerical Illustrations

In this section, we present some numerical experiments in solving variational inequality problems.

Algorithm 2

1: **Initialization:** Let $w_0, w_1 \in H$ be arbitrary and parameters $\rho, l \in (0,1), \kappa \in (0,1), \sigma, \zeta \in [0,1]$, along with a sequence $\{\delta_n\}$ in $(0,1)$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\{\alpha_n\} \subset (a, 1-\xi)$ for some $a > 0$ and $\xi \in [0,1)$.

2: Let $\{\gamma_n\} \in (0,1)$ and a nonnegative sequence $\{\phi_n\}$ satisfy

$$\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty, \ \text{and} \ \lim_{n\to\infty} \frac{\phi_n}{\gamma_n} = 0.$$

3: **Compute**

$$\sigma_n = \begin{cases} \min\left\{\dfrac{\sigma}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\sigma}{2}, & \text{otherwise.} \end{cases}$$

4: **Compute** $s_n = (1-\gamma_n)[w_n + \sigma_n(w_n - w_{n-1})]$.

5: **Compute**

$$\zeta_n = \begin{cases} \min\left\{\dfrac{\zeta}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\zeta}{2}, & \text{otherwise.} \end{cases}$$

6: **Compute** $r_n = (1-\delta_n)[w_n + \zeta_n(w_n - w_{n-1})]$.

7: **Compute**

$$q_n = P_C(r_n - \tau_n M r_n),$$

where τ_n is chosen to be the largest $\tau \in \{\rho, \rho l^2, \rho l^3, \ldots\}$ satisfying

$$\tau\|Mq_n - Mr_n\| \leq \kappa\|q_n - r_n\|.$$

If $r_n = q_n$ or $Mr_n = 0$, then stop and r_n is a solution of (1.1). Otherwise

8: **Compute** $d_n = r_n - q_n - \tau_n(Mr_n - Mq_n)$.

9: **Compute**

$$t_n = r_n - \mu\eta_n d_n,$$

where

$$\eta_n = \begin{cases} \dfrac{\langle r_n - q_n, d_n\rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0 \\ 0, & \text{if } d_n = 0. \end{cases}$$

10: **Compute** $v_n = (1-\alpha_n)s_n + \alpha_n w_n$.
11: **Compute** $w_{n+1} = (1-\gamma_n)v_n + \gamma_n t_n$.
12: **End for loop**

Algorithm 3

1: **Initialization:** Let $w_0, w_1 \in H$ be arbitrary and parameters $\rho, l \in (0,1), \kappa \in (0,1), \sigma, \zeta \in [0,1]$. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ such that $\{\alpha_n\} \subset (a, 1-\xi)$ for some $a > 0$ and $\xi \in [0,1)$.
2: Let $\{\gamma_n\} \in (0,1)$ and a nonnegative sequence $\{\phi_n\}$ satisfy

$$\lim_{n\to\infty} \gamma_n = 0, \ \sum_{n=1}^{\infty} \gamma_n = \infty, \ \text{and} \ \lim_{n\to\infty} \frac{\phi_n}{\gamma_n} = 0.$$

3: **Compute**

$$\sigma_n = \begin{cases} \min\left\{\dfrac{\sigma}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\sigma}{2}, & \text{otherwise.} \end{cases}$$

4: **Compute** $s_n = w_n + \sigma_n(w_n - w_{n-1})$.
5: **Compute**

$$\zeta_n = \begin{cases} \min\left\{\dfrac{\zeta}{2}, \dfrac{\phi_n}{\|w_n - w_{n-1}\|}\right\}, & \text{if } w_n \neq w_{n-1}, \\ \dfrac{\zeta}{2}, & \text{otherwise.} \end{cases}$$

6: **Compute** $r_n = w_n + \zeta_n(w_n - w_{n-1})$.
7: **Compute**

$$q_n = P_C(r_n - \tau_n M r_n),$$

where τ_n is chosen to be the largest $\tau \in \{\rho, \rho l^2, \rho l^3, \ldots\}$ satisfying

$$\tau\|Mq_n - Mr_n\| \leq \kappa\|q_n - r_n\|.$$

If $r_n = q_n$ or $Mr_n = 0$, then stop and r_n is a solution of (1.1). Otherwise
8: **Compute** $d_n = r_n - q_n - \tau_n(Mr_n - Mq_n)$.

9: **Compute**

$$t_n = r_n - \mu\eta_n d_n,$$

where

$$\eta_n = \begin{cases} \dfrac{\langle r_n - q_n, d_n\rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0 \\ 0, & \text{if } d_n = 0. \end{cases}$$

10: **Compute** $v_n = (1-\alpha_n)s_n + \alpha_n w_n$.
11: **Compute** $w_{n+1} = (1-\gamma_n)v_n + \gamma_n t_n$.
12: **End for loop**

Example 4.1. In the first example, we considered $\mathrm{Sol}(C, M)$ with

$$C := \{w \in \mathbb{R}^m \mid Bw \leq b,\ w_i \geq 0,\ \text{for all } i = 1, 2, 3, \ldots, m\},$$

where B is a matrix of size $k \times m$, $b \in \mathbb{R}^k_+$ with $k = 20$ and $M : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$Mw = \left(e^{-w^T Qw} + \beta\right)(Pw + q),$$

where Q is a positive definite matrix, P is a positive semi-definite matrix, $q \in \mathbb{R}^m$ and $\beta > 0$. It can be seen that M is differentiable and by the Mean Value Theorem M is Lipschitz continuous. It is also shown that M is pseudomonotone but not monotone (see [17]). For our experiment Q, P are randomly generated matrices such that Q is a positive definite matrix, P is a positive semi-definite matrix. The process is started with the initial $w_0 = (1, \ldots, 1)^T \in \mathbb{R}^m$ and $w_1 = 0.9w_0$. To terminate algorithms, we use the condition $D_n = \|s_n - q_n\|^2 \leq \varepsilon$ with $\varepsilon = 10^{-5}$ or the number of iterations reaches 500, whichever occurs first. The parameters used for all algorithms are summarized in Table 1. The numerical results are presented in Table 2–3 and Figures 1–2.

Table 1. Parameters for all Algorithms.

Parameters									
ρ	l	κ	μ	σ	ζ	δ_n	α_n	γ_n	ϕ_n
0.85	0.93	0.4	0.95	0.6	0.6	0.5	0.4	$\frac{1.2}{n+5}$	$\frac{1}{(n+5)^{1.9}}$

Table 2. Algorithm comparison $k = 20$ and $m = 100$.

Algorithm	**Iterations**	**Time (s)**	**Final Error**
Algorithm 1	55	0.26	7.8×10^{-6}
Algorithm 2	98	0.39	8.7×10^{-6}
Algorithm 3	65	0.28	9.9×10^{-6}

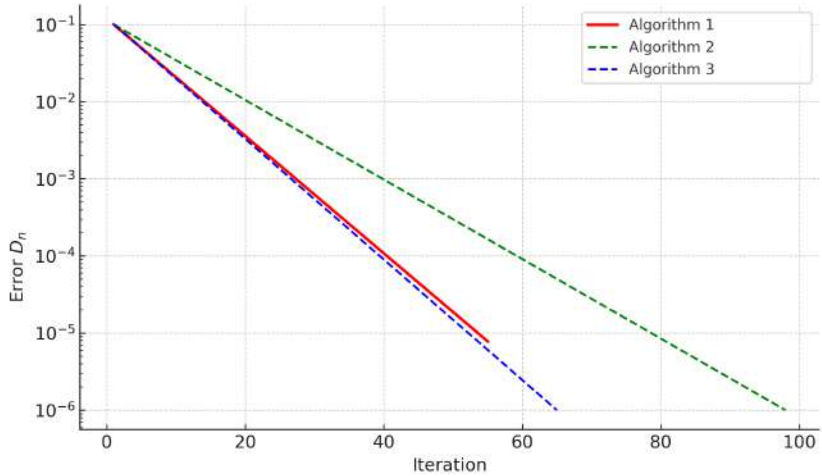


FIGURE 1. Convergence comparison $k = 20$ and $m = 100$.

TABLE 3. Algorithm comparison $k = 20$ and $m = 200$.

Algorithm	**Iterations**	**Time (s)**	**Final Error**
Algorithm 1	62	0.38	7.4×10^{-6}
Algorithm 2	100	0.51	8.5×10^{-6}
Algorithm 3	70	0.42	9.6×10^{-6}

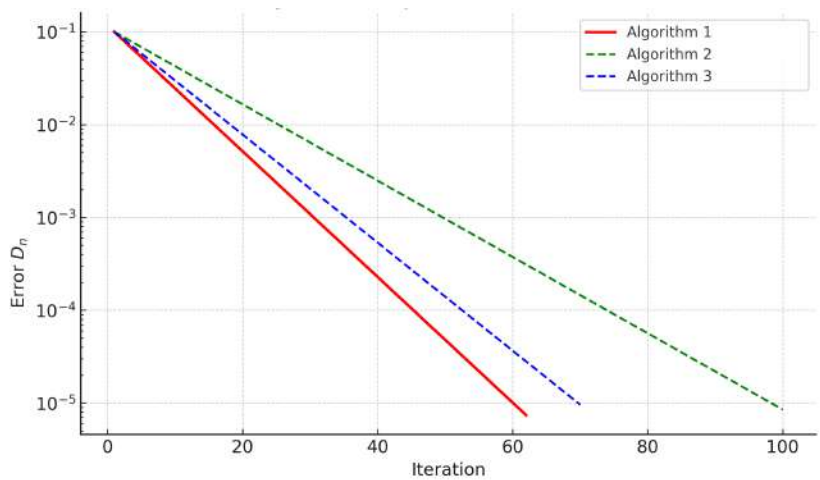


FIGURE 2. Convergence comparison $k = 20$ and $m = 200$.

Example 4.2. Consider the following fractional programming problem

$$\min f(w) = \frac{w^T Q w + a^T w + a_0}{b^T w + b_0}$$

subject to $w \in X := \{w \in \mathbb{R}^m : b^T w + b_0 > 0\}$,

where Q is an $m \times m$ symmetric matrix, $a, b \in \mathbb{R}^m$, and $a_0, b_0 \in \mathbb{R}$. It is well known that f is pseudo-convex on X when Q is positive semidefinite. We consider

$$Q = \begin{pmatrix} 4 & -1 & 2 & 0 \\ -1 & 5 & 0 & 3 \\ 2 & 0 & 6 & -2 \\ 0 & 3 & -2 & 7 \end{pmatrix} \quad a = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad a_0 = -3, \quad b_0 = 30$$

We minimize f over $C := \{w \in \mathbb{R}^4 : 1 \leq w_i \leq 10, \text{ for all } i = 1, \ldots, 4\} \subset X$. It is easy to verify that Q is symmetric and positive definite in $\mathbb{R}^4$, and consequently f is pseudo-convex on X.

The process is started with the initial point $w_0 = (5, 2.5, 4, 2.5, 4)^T$ and $w_1 = 0.9w_0$. To terminate the algorithms, we use the stopping criterion

$$D_n = \|s_n - q_n\|^2 \leq \varepsilon \quad \text{with} \quad \varepsilon = 10^{-10},$$

or when the number of iterations reaches 500, whichever occurs first. The parameters used for all algorithms are summarized in Table 4. The numerical results are presented in Table 5 and Figure 3.

TABLE 4. Parameters for all Algorithms.

Parameters									
ρ	l	κ	μ	σ	ζ	δ_n	α_n	γ_n	ϕ_n
0.85	0.93	0.4	0.95	0.6	0.6	0.5	0.4	$\frac{1.2}{n+5}$	$\frac{1}{(n+5)^{1.9}}$

TABLE 5. Algorithm comparison results.

Algorithm	**Iterations**	**Time (s)**	**Final Error**
Algorithm 1	58	0.32	6.9×10^{-6}
Algorithm 2	102	0.49	8.6×10^{-6}
Algorithm 3	73	0.39	9.8×10^{-6}

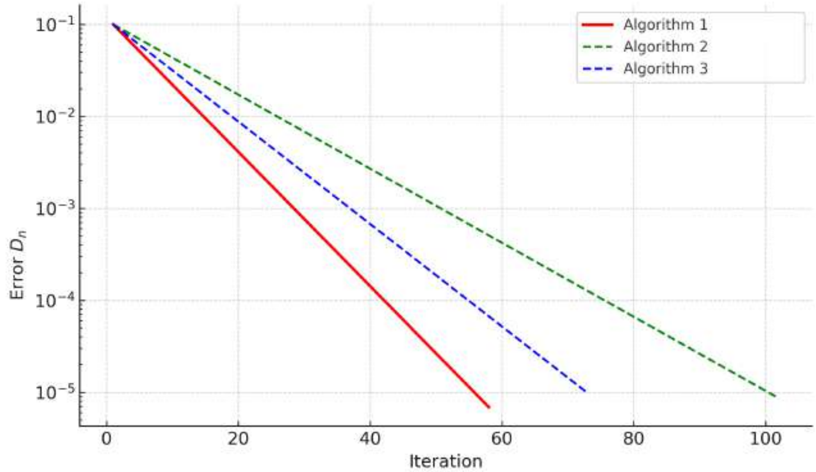


FIGURE 3. Algorithm comparison results.

5. Application to image deblurring

The fractional programming model and corresponding variational inequality problem in Example 4.2 are applied to an image deblurring task. The observed blurred image is modeled as a degraded version of the true image, and the optimization framework is used to reconstruct the clean image.

In this context, the variable w represents the pixel intensities (vectorized), and the objective function encodes a balance between a quadratic regularization term and a linear degradation model. The feasible set ensures valid intensity ranges. The Algorithm 1 demonstrates superior performance in terms of both convergence speed and reconstruction accuracy.

A degraded image is generated by applying Gaussian noise with zero mean and variance 0.01. The quality of the reconstructed image is measured by the Peak Signal-to-Noise Ratio (PSNR) in decibels (dB) as follows:

$$\text{PSNR} = 20\log_{10}\frac{\|w\|_2}{\|w^* - w\|_2},$$

where w is an original image and w^* is a reconstructed image.

The Structural Similarity Index Measure (SSIM) between two images x and y is defined as:

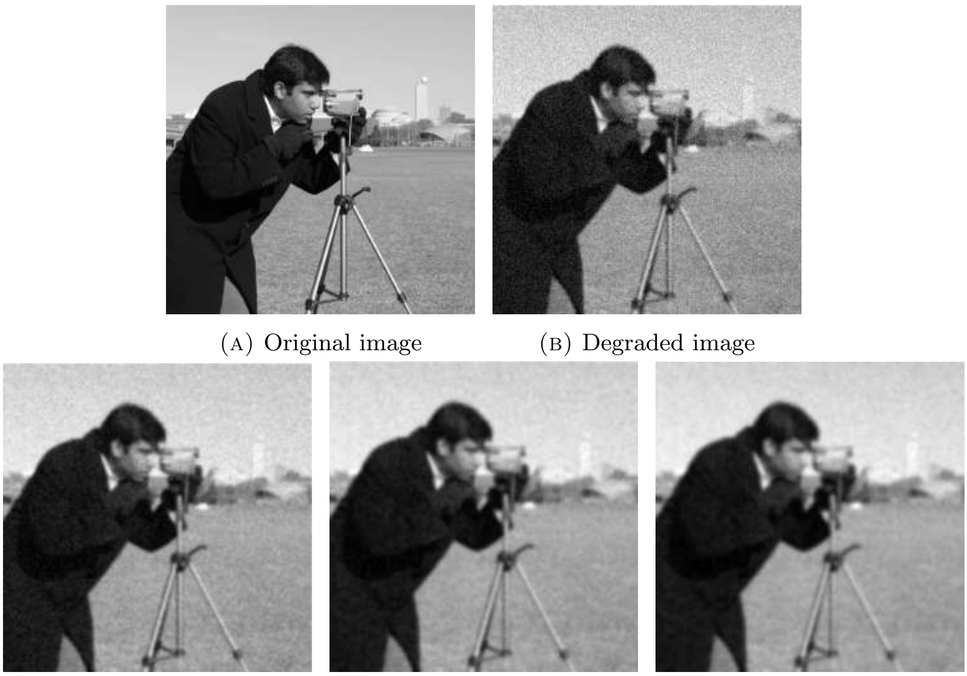
$$\text{SSIM} = \left[\frac{2\mu_x\mu_y + C_1}{\mu_x^2 + \mu_y^2 + C_1}\right] \times \left[\frac{2\sigma_{xy} + C_2}{\sigma_x^2 + \sigma_y^2 + C_2}\right] \times \left[\frac{\sigma_x^2 + \sigma_y^2 + C_3}{\mu_x^2 + \mu_y^2 + C_3}\right],$$

where

- μ_x and μ_y are the average pixel intensities of the images x and y, respectively.
- σ_x^2 and σ_y^2 are the variances of the pixel intensities in the images x and y, respectively.
- σ_{xy} is the covariance of the pixel intensities between the two images.
- C_1, C_2, C_3 are small constants used to stabilize the division with weak denominators.

Table 6. Image deblurring results.

Algorithm	PSNR (dB)	SSIM	Iterations	Time (s)	Final Error
Algorithm 1	30.7	0.92	58	0.33	6.9×10^{-6}
Algorithm 2	28.9	0.89	102	0.49	8.6×10^{-6}
Algorithm 3	27.8	0.86	73	0.39	9.8×10^{-6}



(a) Original image (b) Degraded image

(c) Algorithm 1 (d) Algorithm 2 (e) Algorithm 3

Figure 4. Reconstructed image results

6. Conclusion

In this paper, we proposed new projection and contraction algorithms incorporating double inertial steps to solve variational inequality problems involving pseudomonotone and possibly non-Lipschitz continuous mappings in real Hilbert spaces. Under the assumptions of pseudomonotonicity and uniform continuity, we established weak convergence of the proposed method. Furthermore, we proved strong convergence of the generated sequence to the unique solution of the problem under strong pseudomonotonicity and Lipschitz continuity assumptions.

To evaluate the effectiveness of the proposed methods, we conducted numerical experiments, which demonstrate their superior performance in terms of both iteration count and computational time compared to existing methods. In particular, the optimized version of Algorithm 1 outperformed the others in all test cases.

Finally, we applied the proposed method to an image deblurring problem. The results confirmed that the algorithm can effectively restore high-quality images, as measured by PSNR and SSIM, further highlighting the practical utility of the proposed method in solving real-world inverse problems.

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