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An Inertial Forward-Backward Algorithm with Moving Point Projection Method for Solving Monotone Inclusion Problems

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Abstract In this paper, we propose and analyze an inertial forward-backward algorithm integrated with a moving point projection method to solve the inclusion problem for the sum of two monotone operators in real Hilbert spaces. By leveraging the properties of the resolvent of a maximally monotone operator, the inverse strongly monotone operator, and carefully controlled scalar parameters alongside the novel moving point projection method, we derive a sequence of nonempty, closed, and convex sets containing the set of zeros of the sum of the two monotone operators. These constructed sets enable the generation of new iterative elements through the proposed moving point projection method, ensuring the resulting sequence remains well-defined. We prove that the sequence generated by this method converges weakly to a zero of the sum of the two monotone operators. This moving point projection method introduces a new approach that enhances convergence efficiency compared to existing methods.

Furthermore, to demonstrate the practical utility of the proposed method, we conduct numerical experiments to compare the convergence rates of our proposed method against existing conventional methods. The results illustrate the computational advantages and improved performance of our approach.

MSC: 47H04, 47H05, 47H09, 47H10, 90C25

Keywords: Metric projection; inclusion problem; inertial method; forward-backward algorithm; nonexpansive mapping

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1. Introduction

Throughout this work, the set of all natural numbers, the set of all real numbers, the k-dimensional Euclidean space ($k \in \mathbb{N}$), the set of all $m \times n$ real matrices ($m, n \in \mathbb{N}$), and the identity mapping are denoted by the following notations: $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^k$, $\mathbb{R}^{m\times n}$, and I, respectively.

Let $U : H \to H$ be an operator and $V : H \to 2^H$ be a set-valued operator, where H is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. Then, the inclusion problem can be expressed as follows:

$$\text{find} \quad z^* \in H \quad \text{such that} \quad 0 \in (U+V)z^*. \tag{1.1}$$

The set of all solutions of (1.1) is denoted by $(U+V)^{-1} = \{z \in H \mid 0 \in (U+V)z\}$.

The inclusion problem (1.1) holds significant attention as it has implications for several real-world applications, for example, convex optimization problems, variational inequality problems, fixed point problems, image reconstruction, image denoising, signal processing, machine learning, optimal control, and medical imaging, etc. See, for instance [1–9]. On the other hand, recall that the representation of the *fixed-point problem* for the operator $T : H \to H$ is as follows:

$$\text{find} \quad z^* \in H \quad \text{such that} \quad z^* \in F(T), \tag{1.2}$$

where $F(T) := \{ z \in H \mid Tz = z \}$. The close relationship between (1.1) and (1.2) can be seen from Lemma 2.7 (1) in the next section.

Given a real Hilbert space H with its inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$, let $U : H \to H$ be a single-valued operator and $V : H \to 2^H$ be a multi-valued operator. Then U is called:

(1) *firmly nonexpansive* if
$\|Ux - Uy\|^2 \le \langle Ux - Uy, x - y\rangle, \quad \forall x, y \in H,$
or equivalently, if
$\|Ux - Uy\|^2 \le \|x - y\|^2 - \|(I-U)x - (I-U)y\|^2, \quad \forall x, y \in H;$

(2) *Lipschitz* or *Lipschitz continuous* if there exists a constant $L \ge 0$ such that
$\|Ux - Uy\| \le L\|x - y\|, \quad \forall x, y \in H.$ In particular, U is called a *nonexpansive operator* if $L = 1$.

The graph of V is denoted by $G(V) := \{(x, \tilde{x}) \in H \times H \mid \tilde{x} \in Vx\}$ and the set of all zeros of V is written by $V^{-1}(0) = \{z \in H \mid 0 \in Vz\}$. Then, V is said to be:

(1) *monotone* if
$\langle x - y, \tilde{x} - \tilde{y}\rangle \ge 0, \quad \forall\, (x,\tilde{x}), (y,\tilde{y}) \in G(V)$ (It can be reduced to
$\langle x - y, Vx - Vy\rangle \ge 0, \quad \forall x, y \in H$ if V is single-value.);

(2) *$\alpha-$inverse strongly monotone* (or *$\alpha-$cocoercive*) if there is $\alpha > 0$ such that
$\langle x - y, \tilde{x} - \tilde{y}\rangle \ge \alpha\|\tilde{x} - \tilde{y}\|^2, \quad \forall\, (x,\tilde{x}), (y,\tilde{y}) \in G(V)$ (It can be reduced to
$\langle x - y, Vx - Vy\rangle \ge \alpha\|Vx - Vy\|^2, \quad \forall x, y \in H$ if V is single-value.);

(3) *maximally monotone* if V is monotone and $G(V)$ is not properly contained in any graph of other multi-valued monotone operator, that is, if $\hat{V} : H \to 2^H$ is a multi-valued monotone operator such that $G(V) \subseteq G(\hat{V})$, then $G(V) = G(\hat{V})$.

Recall that $J_r^V = (I + rV)^{-1}$ for some $r > 0$ is called the *resolvent* of the multi-valued operator $V : H \to 2^H$. It is well known that if V is maximally monotone and $r > 0$, then $J_r^V : H \to H$ is single-valued and firmly nonexpansive and $D(J_r^V) = H$ ($D(J_r^V)$ is the domain of J_r^V). For further information, see [10–14].

One commonly used technique to solve (1.1) is the well-known forward-backward splitting method (FBSM), which was introduced as follows by Lions and Mercier [4] and Passty [7]:

$$x_{n+1} = (I + r_n V)^{-1}(I - r_n U)x_n. \tag{1.3}$$

Under appropriate assumptions of $\{r_n\}$, they proved that (1.3) converges weakly to a solution of (1.1).

In 2001, Alvarez and Attouch [15] modified the proximal point method for maximally monotone operators by using the heavy ball technique which was introduced in [16, 17]. This algorithm, known as the inertial proximal point algorithm, has the following manner:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n V)^{-1} y_n. \end{cases} \tag{1.4}$$

It was shown that if $\{\theta_n\} \subseteq [0, 1)$ and $\{r_n\}$ is non-decreasing with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty, \tag{1.5}$$

and under some restriction of θ_n in (1.5), that is, $\theta_n < \frac{1}{3}$ for all $n \in \mathbb{N}$, then (1.4) weakly converges to a zero point of V.

The following inertial proximal point method was recently presented by Moudafi and Oliny [6] to solve (1.1):

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n V)^{-1}(y_n - r_n U x_n). \end{cases} \tag{1.6}$$

They proved that (1.6) converges weakly to a solution of (1.1) provided that $r_n < \frac{2}{L}$ with L is the Lipschitz constant of U and the condition (1.5) holds. Since the operator U is evaluated at the point x_n, it can be noticed that the algorithm (1.6) cannot take the form of a forward-backward splitting algorithm.

Recently, Lorenz and Pock [5] proposed the following inertial forward-backward algorithm for monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n V)^{-1}(I - r_n U)y_n. \end{cases} \tag{1.7}$$

It is noted that (1.7) is different from (1.6) due to y_n was mapped by U in (1.7); it is different from (1.6). Although the method with the inertial term given above has weak convergence, norm convergence is preferred over weak convergence in certain applied areas [18].

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then, for any $x_0 \in H$ there exists a unique $z^* \in C$ such that $\|x - z^*\| = \inf_{z \in C} \|x - z\|$ and if we define $P_C : H \to C$ by $P_C x = z^*$ for all $x \in H$, then P_C is called the *metric projection* or the *nearest point projection* of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e., $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$. Moreover,

$$\langle x - z^*, z - z^*\rangle = \langle x - P_C x, z - P_C x\rangle \leq 0$$

holds for all $x \in H$ and $z \in C$, see [12, 13, 19] for more details.

In 1954, Mann [20] introduced the following algorithm which is widely recognized and significant for solving (1.2) as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \end{cases} \tag{1.8}$$

where $\{\alpha_n\} \subseteq [0,1]$. Reich [21] proved that if sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$ then Mann's algorithm (1.8) converges weakly to a fixed point of T.

In 2020, Artsawang and Ungchittrakool [2] proposed an inertial Mann-type iterative method designed to identify fixed points of nonexpansive mappings, with applications extending to monotone inclusion and image restoration. The approach is summarized as follows:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\alpha_n)\delta_n y_n + \alpha_n T \delta_n y_n + \varepsilon_n, \end{cases} \tag{1.9}$$

where $\{\theta_n\}, \{\alpha_n\}, \{\delta_n\} \subseteq [0,1]$ and $\{\varepsilon_n\} \subseteq H$ which satisfy some suitable conditions. If we set $T = J_r^V(I - rU)$, then we can apply (1.9) to solve (1.1).

On the other hand, Takahashi et al. [22], proposed their new method by applying (1.8) and the metric projection to produce the following iterative sequence which is generated by: For $C_1 = C$, $x_1 = P_{C_1} x_0$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_{n+1} = \{ z \in C_n \mid \|y_n - z\| \leq \|x_n - z\| \}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{1.10}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed and convex subset of a real Hilbert space H, $0 \leq \alpha_n \leq a < 1$, $T : C \to C$ is a nonexpansive mapping. They showed that (1.10) strongly converges to a fixed point of T. This technique is usually referred as the shrinking projection method (also see Takahashi and Nakajo [23]).

Later, Khan et al. [24], applied the idea of Takahashi et al. [22] to solve (1.1). Their procedure can be demonstrated as follows:

$$\begin{cases} x_0, x_1 \in C_1 = H \text{ are selected arbitrary}, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n x_n + (1-\alpha_n) J_{r_n}^V (I - r_n U) y_n, \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \begin{array}{r} \|z_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 \\ -2\theta_n(1-\alpha_n)\langle x_n - z, x_{n-1} - x_n \rangle \end{array} \right\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases} \tag{1.11}$$

where $U : H \to H$ is α-inverse strongly monotone operator, $V : H \to 2^H$ is a maximally monotone operator such that $(U+V)^{-1}(0) \neq \emptyset$, $0 \leq \theta_n \leq \theta < 1$, $0 \leq \alpha_n \leq 1$, $0 < r_n \leq 2\alpha$, and $J_{r_n}^V = (I + r_n V)^{-1}$. Under additional assumptions on the control of scalars, they can prove that (1.11) converges strongly to a zero point in $(U+V)^{-1}(0)$.

Inspired by the research works in this direction, we introduce and study a new idea, that is, an inertial forward-backward algorithm with moving point projection method in order to solve (1.1). We can prove that the newly developed algorithm converges weakly to a solution of (1.1) under some appropriate scalar constraints. And to see the benefits of this new approach, we have established numerical experimental examples and tested

the numerical advantage of the convergence to a solution between the new method and the traditional method. The results show the advantages of the new method over the traditional method.

2. Preliminaries

This section collects a number of useful tools that are essential for proving the main theorem within the setting of real Hilbert spaces. These tools will be used in the next section. We adopt the notation "$\rightharpoonup$" to indicate weak convergence, whereas "$\to$" will signify strong convergence in the context of this study.

Lemma 2.1 ([12]). *Let H be a real Hilbert space. Then, for all $x, y \in H$, the followings hold:*

(1) $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$;
(2) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$;
(3) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ *for all* $\lambda \in \mathbb{R}$.

Lemma 2.2 ([25]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H Given that $x, y, z \in H$ and $a \in \mathbb{R}$, the set*

$$\left\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\right\}$$

is convex and closed.

Lemma 2.3 ([23]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then the following inequality holds:*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x-y\|^2, \ \forall x \in H, \ \forall y \in C.$$

Lemma 2.4 ([26]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

The subsequent lemma plays a crucial role in establishing the proof of the main theorem. Before proceeding, we introduce the concept of the "weak subsequential limit set" for a sequence $\{x_n\}$ in a Hilbert space H. This set, denoted by $\omega_w(x_n)$, consists of all points $z \in H$ for which there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to z. Formally, $\omega_w(x_n) := \{z \in H \mid \exists \{x_{n_k}\} \subseteq \{x_n\} \text{ with } x_{n_k} \rightharpoonup z\}$.

Lemma 2.5 ([27]). *Let C be a non-empty subset of a Hilbert space H, and let $\{x_n\}$ be a sequence in H satisfying the following two criteria:*

(1) For each $u \in C$, the limit $\lim_{n\to\infty} \|x_n - u\|$ exists.
(2) Every weak subsequential limit of $\{x_n\}$ belongs to C, i.e., $\omega_w(x_n) \subseteq C$.

Under these conditions, the sequence $\{x_n\}$ is guaranteed to converge weakly to some element in C.

Lemma 2.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Proof. First, we aim to show that $F(T)$ is closed. Let $\{x_n\} \subseteq F(T)$ be such that $x_n \to x^* \in H$. Then, the triangle inequality and the nonexpansiveness of T lead to the following inequality

$$\begin{aligned}\|Tx^* - x^*\| &= \|Tx^* - Tx_n + x_n - x^*\| \leq \|Tx^* - Tx_n\| + \|x_n - x^*\| \\ &\leq 2\|x_n - x^*\| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This shows that $x^* \in F(T)$ and then $F(T)$ is closed.

Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$, $\lambda \in [0,1]$, and $z_\lambda := (1-\lambda)x + \lambda y$. Then, by using Lemma 2.1 (3) and the nonexpansiveness of T, these yield us to obtain the following finding

$$\begin{aligned}\|Tz_\lambda - z_\lambda\|^2 &= \|(1-\lambda)(Tz_\lambda - x) + \lambda(Tz_\lambda - y)\|^2 \\ &= (1-\lambda)\|Tz_\lambda - x\|^2 + \lambda\|Tz_\lambda - y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ &= (1-\lambda)\|Tz_\lambda - Tx\|^2 + \lambda\|Tz_\lambda - Ty\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ &\leq (1-\lambda)\|z_\lambda - x\|^2 + \lambda\|z_\lambda - y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ &= (1-\lambda)\|\lambda(y-x)\|^2 + \lambda\|(1-\lambda)(x-y)\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ &= \lambda^2(1-\lambda)\|x-y\|^2 + \lambda(1-\lambda)^2\|x-y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \\ &= \lambda(1-\lambda)\left(\lambda\|x-y\|^2 + (1-\lambda)\|x-y\|^2 - \|x-y\|^2\right) \\ &= \lambda(1-\lambda)\,(0) \\ &= 0.\end{aligned}$$

This shows that $z_\lambda \in F(T)$, that is, $F(T)$ is convex. This completes the proof. ∎

Next, The following notation will be used:

$$\Gamma_r^{U,V} := J_r^V(I - rU) = (I + rV)^{-1}(I - rU), \quad r > 0. \tag{2.1}$$

Lemma 2.7 ([28–30]). *Let H be a real Hilbert space. Let $U : H \to H$ be an α-inverse strongly monotone operator and $V : H \to 2^H$ a maximally monotone operator. Then we have*

(1) For $r > 0$, $F(\Gamma_r^{U,V}) = (U+V)^{-1}(0)$;
(2) For $0 < r \leq s$ and $x \in H$, $\left\|x - \Gamma_r^{U,V}x\right\| \leq 2\left\|x - \Gamma_s^{U,V}x\right\|$.

Lemma 2.8. *Let H be a real Hilbert space, $U : H \to H$ be an α-inverse strongly monotone operator, $V : H \to 2^H$ be a maximally monotone operator, and let $\Gamma_r^{U,V}$ be defined by (2.1). Then, for all $x, y \in H$, the following inequality holds*

$$\begin{aligned}\left\|\Gamma_r^{U,V}x - \Gamma_r^{U,V}y\right\|^2 \leq{}& \|x-y\|^2 - r(2\alpha - r)\|Ux - Uy\|^2 \\ &- \left\|(I - J_r^V)(I - rU)x - (I - J_r^V)(I - rU)y\right\|^2.\end{aligned}$$

Proof. First, by using Lemma 2.1 (1) and the α-inverse strongly monotone of U, these allow us to get the following inequality

$$\begin{aligned}\|(I-rU)x-(I-rU)y\|^2 &= \|(x-y)-r(Ux-Uy)\|^2\\ &= \|x-y\|^2-2r\langle x-y,Ux-Uy\rangle+r^2\|Ux-Uy\|^2\\ &\leq \|x-y\|^2-2r\alpha\|Ux-Uy\|^2+r^2\|Ux-Uy\|^2\\ &= \|x-y\|^2-r(2\alpha-r)\|Ux-Uy\|^2. \end{aligned} \tag{2.2}$$

Next, using the firmly nonexpansive of J_r^V and connecting (2.2), we obtain the following result.

$$\begin{aligned}&\left\|\Gamma_r^{U,V}x-\Gamma_r^{U,V}y\right\|^2\\ &=\left\|J_r^V(I-rU)x-J_r^V(I-rU)y\right\|^2\\ &\leq \|(I-rU)x-(I-rU)y\|^2-\left\|(I-J_r^V)(I-rU)x-(I-J_r^V)(I-rU)y\right\|^2\\ &\leq \|x-y\|^2-r(2\alpha-r)\|Ux-Uy\|^2\\ &\quad-\left\|(I-J_r^V)(I-rU)x-(I-J_r^V)(I-rU)y\right\|^2.\end{aligned}$$

This completes the proof. ∎

Remark 2.9. *From Lemma 2.8, it is obvious that if $0<r<2\alpha$, then the mapping $\Gamma_r^{U,V}:H\to H$ is a nonexpansive mapping.*

3. Main results

In this section, we introduce an inertial forward-backward algorithm incorporating the moving point projection method (I-FB-MPP Algorithm) to solve the inclusion problem (1.1). The weak convergence of the proposed algorithm is proven under appropriate scalar control conditions within the framework of real Hilbert spaces.

Lemma 3.1. *Let H be a real Hilbert space, $U:H\to H$ be an α-inverse strongly monotone operator, $V:H\to 2^H$ be a maximally monotone operator such that $(U+V)^{-1}(0)\neq\emptyset$. Then, $(U+V)^{-1}(0)\neq\emptyset$ is closed and convex.*

Proof. Since $\alpha>0$, so we can find $r\in\mathbb{R}$ such that $0<r<2\alpha$. Then, Remark 2.9 ensures that $\Gamma_r^{U,V}:H\to H$ is a nonexpansive mapping. By the virtue of Lemma 2.7 (1) guarantees that $(U+V)^{-1}(0)=F(\Gamma_r^{U,V})$. And finally, Lemma 2.6 allows us to obtain the closedness and convexity of $(U+V)^{-1}(0)$. ∎

Theorem 3.2 (I-FB-MPP Algorithm)**.** *Let H be a real Hilbert space, $U:H\to H$ be an α-inverse strongly monotone operator, $V:H\to 2^H$ be a maximally monotone operator such that $(U+V)^{-1}(0)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0,x_1\in C_1=H$ and*

$$\begin{cases} y_n=x_n+\theta_n(x_n-x_{n-1}),\\ z_n=\alpha_n x_n+(1-\alpha_n)J_{r_n}^V(I-r_nU)y_n,\\ C_{n+1}=\left\{z\in H\;\middle|\;\begin{array}{r}\|z_n-z\|^2\leq\|x_n-z\|^2+(1-\alpha_n)\theta_n^2\|x_n-x_{n-1}\|^2\\ +2\theta_n(1-\alpha_n)\langle x_n-z,x_n-x_{n-1}\rangle\end{array}\right\},\\ x_{n+1}=P_{C_{n+1}}x_n,\quad n\in\mathbb{N}, \end{cases} \tag{3.1}$$

where $J_{r_n}^V = (I + r_n V)^{-1}$, $\{\theta_n\} \subseteq [0, \theta] \subseteq [0, 1)$, *and* $\{\alpha_n\} \subseteq [0, 1]$. *Assume that the following assumptions hold:*

(1) $0 < r_n < 2\alpha$ *for all* $n \in \mathbb{N}$;
(2) $\limsup_{n\to\infty} \alpha_n < 1$;
(3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$.

Then the sequence $\{x_n\}$ *converges weakly to some point* $z^* \in (U + V)^{-1}(0)$.

For convenience and easy understanding, we will divide the proof into the following lemmas.

Lemma 3.3. *The sequence* $\{x_n\}$ *generated by* (3.1) *in Theorem 3.2 is well-defined and* $(U + V)^{-1}(0) \subseteq C_n$ *for every* $n \in \mathbb{N}$ *(that is,* $C_n \neq \emptyset$ *for every* $n \in \mathbb{N}$*).*

Proof. We will show that $P_{C_{n+1}} x_n$ is well-defined for every $n \in \mathbb{N}$ (it is equivalent to show that $\{x_n\}$ is well-defined every $n \in \mathbb{N}$). First of all, we notice that since $0 < r_n < 2\alpha$ for all $n \in \mathbb{N}$, so Remark 2.9 guarantees that $\Gamma_{r_n}^{U,V}$ are nonexpansive for all $n \in \mathbb{N}$. Furthermore, Lemma 3.1 ensures that $(U + V)^{-1}(0)$ is closed and convex.

Let $z \in (U + V)^{-1}(0)$. Then, for $n = 1$, we have

$$\begin{aligned}
&\|z_1 - z\|^2 \\
&= \left\|\alpha_1(x_1 - z) + (1 - \alpha_1)(\Gamma_{r_1}^{U,V} y_1 - z)\right\|^2 \\
&\leq \alpha_1\|x_1 - z\|^2 + (1 - \alpha_1)\left\|\Gamma_{r_1}^{U,V} y_1 - z\right\|^2 \\
&\leq \alpha_1\|x_1 - z\|^2 + (1 - \alpha_1)\|y_1 - z\|^2 \\
&= \alpha_1\|x_1 - z\|^2 + (1 - \alpha_1)\|(x_1 - z) + \theta_1(x_1 - x_0)\|^2 \\
&\leq \alpha_1\|x_1 - z\|^2 + (1 - \alpha_1)\left(\|x_1 - z\|^2 + 2\theta_1\langle x_1 - z, x_1 - x_0\rangle + \theta_1^2\|x_1 - x_0\|^2\right) \\
&= \|x_1 - z\|^2 + (1 - \alpha_1)\theta_1^2\|x_1 - x_0\|^2 + 2\theta_1(1 - \alpha_1)\langle x_1 - z, x_1 - x_0\rangle . \qquad (3.2)
\end{aligned}$$

(3.2) implies that $z \in C_2$. Therefore, $(U + V)^{-1}(0) \subseteq C_2 \neq \emptyset$ and Lemma 2.2 ensures that C_2 is closed and convex. Then, there exists a unique $x_2 \in C_2$ such that $x_2 = P_{C_2} x_1$. Next, for $k \geq 2$, suppose that $(U + V)^{-1}(0) \subseteq C_k \neq \emptyset$ and then Lemma 2.2 ensures that C_k is closed and convex and consequently there exists a unique $x_k \in C_k$ such that $x_k = P_{C_k} x_{k-1}$.

Let $z \in (U + V)^{-1}(0)$. Then, by the same way as above we get that

$$\begin{aligned}
&\|z_k - z\|^2 \\
&= \left\|\alpha_k(x_k - z) + (1 - \alpha_k)(\Gamma_{r_k}^{U,V} y_k - z)\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\left\|\Gamma_{r_k}^{U,V} y_k - z\right\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\|y_k - z\|^2 \\
&= \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\|(x_k - z) + \theta_k(x_k - x_{k-1})\|^2 \\
&\leq \alpha_k\|x_k - z\|^2 + (1 - \alpha_k)\left(\|x_k - z\|^2 + 2\theta_k\langle x_k - z, x_k - x_{k-1}\rangle + \theta_k^2\|x_k - x_{k-1}\|^2\right) \\
&= \|x_k - z\|^2 + (1 - \alpha_k)\theta_k^2\|x_k - x_{k-1}\|^2 + 2\theta_k(1 - \alpha_k)\langle x_k - z, x_k - x_{k-1}\rangle . \qquad (3.3)
\end{aligned}$$

(3.3) implies that $z \in C_{k+1}$. Therefore, $(U+V)^{-1}(0) \subseteq C_{k+1} \neq \emptyset$ and Lemma 2.2 ensures that C_{k+1} is closed and convex. Then, there exists a unique $x_{k+1} \in C_{k+1}$ such that $x_{k+1} = P_{C_{k+1}} x_k$. Therefore, by mathematical induction we can conclude that $x_{n+1} = P_{C_{n+1}} x_n$ for all $n \in \mathbb{N}$, that is, $\{x_n\}$ is well-defined and $(U+V)^{-1}(0) \subseteq C_n$ for every $n \in \mathbb{N}$. This completes the proof. ∎

Lemma 3.4. *The sequence $\{x_n\}$ generated by (3.1) in Theorem 3.2 has the following properties:*

(1) For each $\tilde{z} \in (U+V)^{-1}(0)$, the sequence $\{\|x_n - \tilde{z}\|\}$ is nonincreasing and consequently $\lim_{n\to\infty} \|x_n - \tilde{z}\|$ exists;

(2) $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$ which directly yields that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. The proof of (1). By Lemma 3.3, $(U+V)^{-1}(0) \subseteq C_{n+1}$ for every $n \in \mathbb{N}$. Then, we pick some $\tilde{z} \in (U+V)^{-1}(0)$ and by using Lemma 2.1 (1) and the virtue property of the metric projection, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_n - \tilde{z}) - (x_{n+1} - \tilde{z})\|^2 \\ &= \|x_n - \tilde{z}\|^2 - \|x_{n+1} - \tilde{z}\|^2 - 2\langle x_{n+1} - \tilde{z}, x_n - x_{n+1}\rangle \\ &\leq \|x_n - \tilde{z}\|^2 - \|x_{n+1} - \tilde{z}\|^2. \end{aligned} \tag{3.4}$$

(3.4) implies that $\|x_{n+1} - \tilde{z}\| \leq \|x_n - \tilde{z}\|$ for all $n \in \mathbb{N}$. That is, the sequence $\{\|x_n - \tilde{z}\|\}$ is nonincreasing and bounded below with 0. Then, the monotone convergence theorem ensures that $\lim_{n\to\infty} \|x_n - \tilde{z}\|$ exists.

The proof of (2). Taking summation of both sides of (3.4), this gives us the following result

$$\begin{aligned} \sum_{n=1}^{k} \|x_{n+1} - x_n\|^2 &\leq \sum_{n=1}^{k} (\|x_n - \tilde{z}\|^2 - \|x_{n+1} - \tilde{z}\|^2) = \|x_1 - \tilde{z}\|^2 - \|x_{k+1} - \tilde{z}\|^2 \\ &\leq \|x_1 - \tilde{z}\|^2, \end{aligned} \tag{3.5}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ in (3.5), we get that

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 = \lim_{k\to\infty} \sum_{n=1}^{k} \|x_{n+1} - x_n\|^2 \leq \|x_1 - \tilde{z}\|^2 < +\infty. \tag{3.6}$$

And we can directly conclude from (3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

This completes the proof. ∎

The proof of Theorem 3.2. We will show that $x_n \rightharpoonup z^* \in (U+V)^{-1}(0)$ where the sequence $\{x_n\}$ generated by (3.1) in Theorem 3.2.

Proof. First, by employing (3.7), we get that

$$\|y_n - x_n\| = \theta_n \|x_n - x_{n-1}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.8}$$

By using triangle inequality and then using (3.7) and (3.8), it yields the following result

$$\|y_n - x_{n+1}\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

The triangle inequality, (3.7), (3.9) and the fact that $x_{n+1} \in C_{n+1}$ lead to the following result

$$\begin{aligned}
&\|z_n - y_n\| \\
&\leq \|z_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\
&\leq \sqrt{\|x_n - x_{n+1}\|^2 + (1-\alpha_n)\theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n(1-\alpha_n)\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle} \\
&\quad + \|y_n - x_{n+1}\| \quad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.10}$$

On the other hand for each $z \in (U+V)^{-1}(0) = F(\Gamma_{r_n}^{U,V})$, then we have $z = \Gamma_{r_n}^{U,V} z = J_{r_n}^V(I - r_n U)z$ and it can be observed that

$$\begin{aligned}
&\left\|(I - J_{r_n}^V)(I - r_n U)y_n - (I - J_{r_n}^V)(I - r_n U)z\right\|^2 \\
&= \left\|(I - r_n U)y_n - J_{r_n}^V(I - r_n U)y_n - \left((I - r_n U)z - J_{r_n}^V(I - r_n U)z\right)\right\|^2 \\
&= \left\|y_n - r_n U y_n - \Gamma_{r_n}^{U,V} y_n - (z - r_n U z - z)\right\|^2 \\
&= \left\|y_n - r_n U y_n - \Gamma_{r_n}^{U,V} y_n + r_n U z\right\|^2.
\end{aligned} \tag{3.11}$$

Next, it is not difficult to see that Lemma 2.8 and (3.11) agree on the following inequality

$$\begin{aligned}
&\left\|\Gamma_{r_n}^{U,V} y_n - z\right\|^2 \\
&= \left\|\Gamma_{r_n}^{U,V} y_n - \Gamma_{r_n}^{U,V} z\right\|^2 \\
&\leq \|y_n - z\|^2 - r_n(2\alpha - r_n)\|U y_n - U z\|^2 \\
&\quad - \left\|(I - J_{r_n}^V)(I - r_n U)y_n - (I - J_{r_n}^V)(I - r_n U)z\right\|^2 \\
&= \|y_n - z\|^2 - r_n(2\alpha - r_n)\|U y_n - U z\|^2 - \left\|y_n - r_n U y_n - \Gamma_{r_n}^{U,V} y_n + r_n U z\right\|^2.
\end{aligned} \tag{3.12}$$

By the same step of proof only the first two line of (3.3) and together with connection by (3.12), these give us the following result

$$\begin{aligned}
&\|z_n - z\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left\|\Gamma_{r_n}^{U,V} y_n - z\right\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\left(\|y_n - z\|^2 - r_n(2\alpha - r_n)\|U y_n - U z\|^2\right. \\
&\quad \left. - \left\|y_n - r_n U y_n - \Gamma_{r_n}^{U,V} y_n + r_n U z\right\|^2\right) \\
&= \|y_n - z\|^2 + \alpha_n\left(\|x_n - z\|^2 - \|y_n - z\|^2\right) - (1-\alpha_n)r_n(2\alpha - r_n)\|U y_n - U z\|^2 \\
&\quad - (1-\alpha_n)\left\|y_n - r_n U y_n - \Gamma_{r_n}^{U,V} y_n + r_n U z\right\|^2.
\end{aligned} \tag{3.13}$$

It is not hard to see that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded sequences. And then, by changing the form of writing of (3.13) together with employing (3.8) and (3.10), we get

that

$$\begin{aligned}
&(1-\alpha_n)r_n(2\alpha - r_n)\|Uy_n - Uz\|^2 + (1-\alpha_n)\left\|y_n - r_nUy_n - \Gamma_{r_n}^{U,V}y_n + r_nUz\right\|^2 \\
&\leq \|y_n - z\|^2 - \|z_n - z\|^2 + \alpha_n\left(\|x_n - z\|^2 - \|y_n - z\|^2\right) \\
&\leq \|y_n - z_n\|\left(\|y_n - z\| + \|z_n - z\|\right) + \alpha_n\|x_n - y_n\|\left(\|x_n - z\| + \|y_n - z\|\right) \to 0 \\
&\quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.14}$$

By (3.14) and the assumptions (1) - (3), these permit us to conclude that

$$\lim_{n\to\infty}\|Uy_n - Uz\| = \lim_{n\to\infty}\left\|y_n - r_nUy_n - \Gamma_{r_n}^{U,V}y_n + r_nUz\right\| = 0. \tag{3.15}$$

The triangle inequality and (3.15) yield that

$$\begin{aligned}
\left\|y_n - \Gamma_{r_n}^{U,V}y_n\right\| &= \left\|y_n - r_nUy_n + r_nUy_n - r_nUz + r_nUz - \Gamma_{r_n}^{U,V}y_n\right\| \\
&\leq r_n\|Uy_n - Uz\| + \left\|y_n - r_nUy_n - \Gamma_{r_n}^{U,V}y_n + r_nUz\right\| \to 0 \\
&\quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.16}$$

Note that $0 < \liminf_{n\to\infty} r_n$, then there exists $N \in \mathbb{N}$ and $r > 0$ such that $r \leq r_n$ for all $n \geq N$. Applying Lemma 2.7 (2), we get that

$$\left\|y_n - \Gamma_r^{U,V}y_n\right\| \leq 2\left\|y_n - \Gamma_{r_n}^{U,V}y_n\right\|, \quad \forall n \geq N. \tag{3.17}$$

Letting $n \to \infty$ in (3.17) and then (3.16) ensures that

$$\lim_{n\to\infty}\left\|y_n - \Gamma_r^{U,V}y_n\right\| = 0. \tag{3.18}$$

Notice that

$$\begin{aligned}
\left\|x_n - \Gamma_r^{U,V}x_n\right\| &\leq \|x_n - y_n\| + \left\|y_n - \Gamma_r^{U,V}y_n\right\| + \left\|\Gamma_r^{U,V}y_n - \Gamma_r^{U,V}x_n\right\| \\
&\leq 2\|x_n - y_n\| + \left\|y_n - \Gamma_r^{U,V}y_n\right\|, \quad \forall n \geq N.
\end{aligned} \tag{3.19}$$

Letting $n \to \infty$ in (3.19) and then (3.8) and (3.18) guarantee that

$$\lim_{n\to\infty}\left\|x_n - \Gamma_r^{U,V}x_n\right\| = 0. \tag{3.20}$$

Subsequently, we will prove that

$$\omega_w(x_n) \subseteq (U+V)^{-1}(0). \tag{3.21}$$

The boundedness of $\{x_n\}$ ensures that $\omega_w(x_n) \neq \emptyset$. Let $z \in \omega_w(x_n)$. Then, $\exists\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \rightharpoonup z$. Notice that $\Gamma_r^{U,V}$ is nonexpansive, then by employing (3.20) in conjunction with Lemma 2.4, we can conclude that $z \in F(\Gamma_r^{U,V}) = (U+V)^{-1}(0)$. Therefore, (3.21) holds. Now that we have (1) from Lemma 3.4 and (3.21), we can consequently conclude through the application of Lemma 2.5 that there is a $z^* \in (U+V)^{-1}(0)$ such that $x_n \rightharpoonup z^*$. This completes the proof. ∎

4. Numerical experiments

In this section, we utilize Example 3.4 from [24] to perform numerical experiments and analyze the convergence behavior of the I-FB-MPP Algorithm (Algorithm (3.1)). Furthermore, we compare its convergence performance with that of the forward-backward algorithm incorporating the moving point projection method (abbreviated as FB-MPP Algorithm), as well as with a classical benchmark method, using different initial points.

Example 4.1. *Let $H = \mathbb{R}^2$ and let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ux = 3x + (1, 2)$ and let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Vx = 4x$ where $x \in \mathbb{R}^2$.*

It is not hard to verify that U is $\frac{1}{3}$-inverse strongly monotone and V is maximally monotone. Moreover, by a direct calculation, we have for $s > 0$

$$\begin{aligned} J_s^V(x - sUx) &= (I + sV)^{-1}(x - sUx) \\ &= \frac{1 - 3s}{1 + 4s}x - \frac{s}{1 + 4s}(1, 2) \end{aligned}$$

where $x \in \mathbb{R}^2$. It is not hard to verify that $(U + V)^{-1}(0) = \left\{\left(-\frac{1}{7}, -\frac{2}{7}\right)\right\}$. Since $\alpha = \frac{1}{3}$, we can choose $s = r_n = 0.55$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{100n+1}$ and

$$\theta_n = \begin{cases} \min\left\{\dfrac{1}{n^2 \|x_n - x_{n-1}\|}, 0.5\right\}, & \text{if } x_n \neq x_{n-1}, \\ 0.5, & \text{otherwise.} \end{cases}$$

The stopping criterion is set as $E_n = \|x_{n+1} - x_n\| < 10^{-4}$. We will use four different pairs of starting points, x_0 and x_1, in the numerical tests as follows:

(1) $x_0 = (0, 1)$ and $x_1 = (0, 1)$.
(2) $x_0 = (4, -8)$ and $x_1 = (12, -3)$.
(3) $x_0 = (-75, 50)$ and $x_1 = (-100, -150)$.
(4) $x_0 = (700, 750)$ and $x_1 = (950, 900)$.

It will be shown on the next page that Figure 1 illustrates the convergence behavior of an inertial forward-backward algorithm with moving point projection method (I-FB-MPP Algorithm (Algorithm (3.1))) of Theorem 3.2) when initiated from different starting points. To compare the speed of convergence, we use the following three algorithms: inertial forward-backward Mann's algorithm (I-FB-Mann's Algorithm) based on (1.8), a forward-backward algorithm with moving point projection method (FB-MPP Algorithm) (without an inertial term), and an inertial forward-backward algorithm with moving point projection method (I-FB-MPP Algorithm (Algorithm (3.1))) of Theorem 3.2). The numerical result is illustrated in Figure 2.

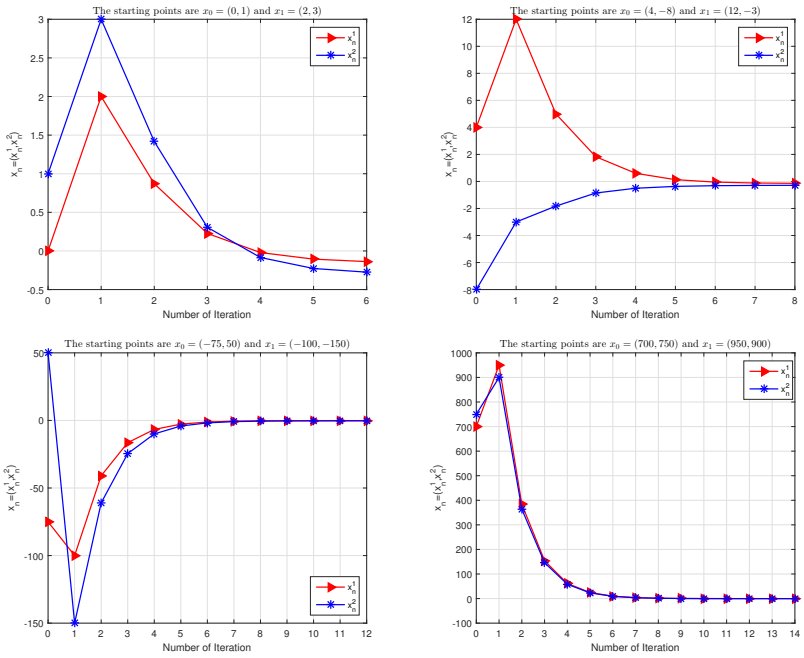


Figure 1. Convergence of $\{x_n\}$ generated by I-FB-MPP Algorithm (Algorithm (3.1))) to the solution $(-\frac{1}{7}, -\frac{2}{7})$ for different starting points

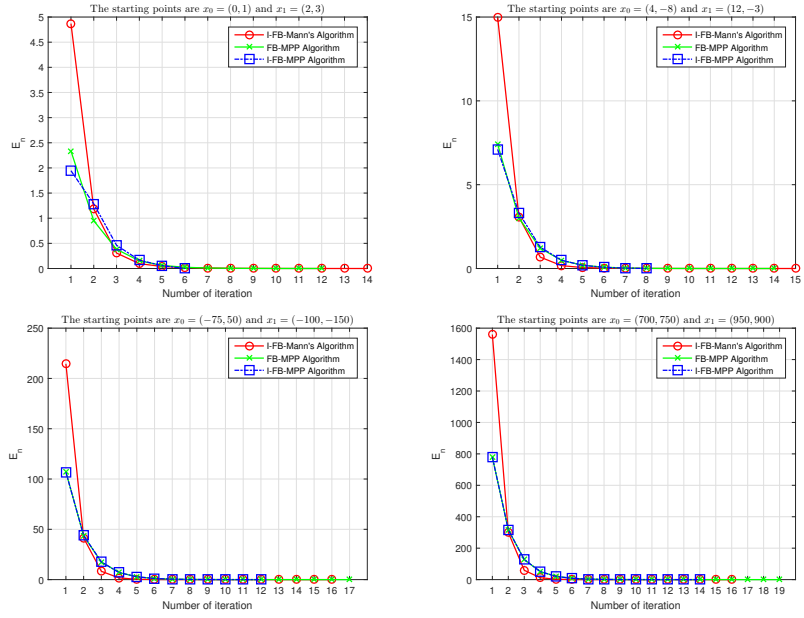


Figure 2. Convergence of error versus iteration for different starting points

We construct two real-world examples, namely Example 4.2 and Example 4.3, which are nonlinear cases. These examples demonstrate that the I-FB-MPP Algorithm (Algorithm (3.1)) of Theorem 3.2 can be applied more effectively than conventional methods.

Example 4.2. *Let $H = \mathbb{R}^2$ and let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$U\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - \sin(x_1) \\ 3x_2 + \sin(x_2) \end{bmatrix}$$

and let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Vx = 4I_2x$ where $x \in \mathbb{R}^2$.

It is not hard to verify that U is $\frac{1}{4}$-inverse strongly monotone and V is maximally monotone. Moreover, by a direct calculation, we have for $s > 0$

$$\begin{aligned} J_s^V(x - sUx) &= (I + sV)^{-1}(x - sUx) \\ &= (I_2 + 4sI_2)^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - s\begin{bmatrix} 3x_1 - \sin(x_1) \\ 3x_2 + \sin(x_2) \end{bmatrix}\right) \\ &= \frac{1}{1+4s}\begin{bmatrix} (1-3s)x_1 + s\sin(x_1) \\ (1-3s)x_2 - s\sin(x_2) \end{bmatrix} \end{aligned}$$

where $x \in \mathbb{R}^2$. It is not hard to verify that $(U+V)^{-1}(0) = \{(0,0)\}$. Since $\alpha = \frac{1}{4}$, we can choose $s = r_n = 0.49$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{100n+1}$ and

$$\theta_n = \begin{cases} \min\left\{\dfrac{1}{n^2\|x_n - x_{n-1}\|}, 0.5\right\}, & \text{if } x_n \neq x_{n-1}, \\ 0.5, & \text{otherwise.} \end{cases}$$

The stopping criterion is set as $E_n = \|x_{n+1} - x_n\| < 10^{-4}$. We will use four different pairs of starting points, x_0 and x_1, in the numerical tests as follows:

(1) $x_0 = (6, -3)$ and $x_1 = (17, 15)$.
(2) $x_0 = (1, 1)$ and $x_1 = (-100, 60)$.
(3) $x_0 = (1, -50)$ and $x_1 = (120, -2)$.
(4) $x_0 = (700, 750)$ and $x_1 = (-1950, 1900)$.

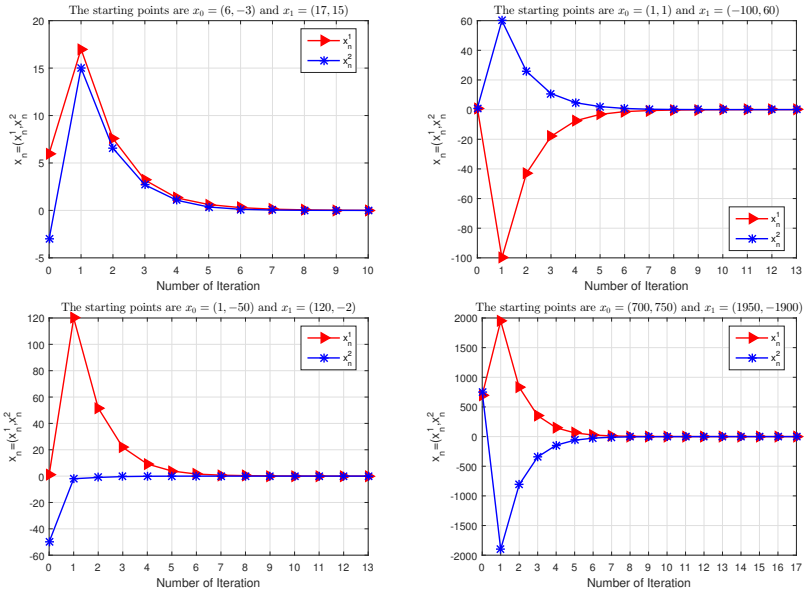


Figure 3. Convergence of $\{x_n\}$ generated by I-FB-MPP Algorithm (Algorithm (3.1))) to the solution $(0, 0)$ for different starting points

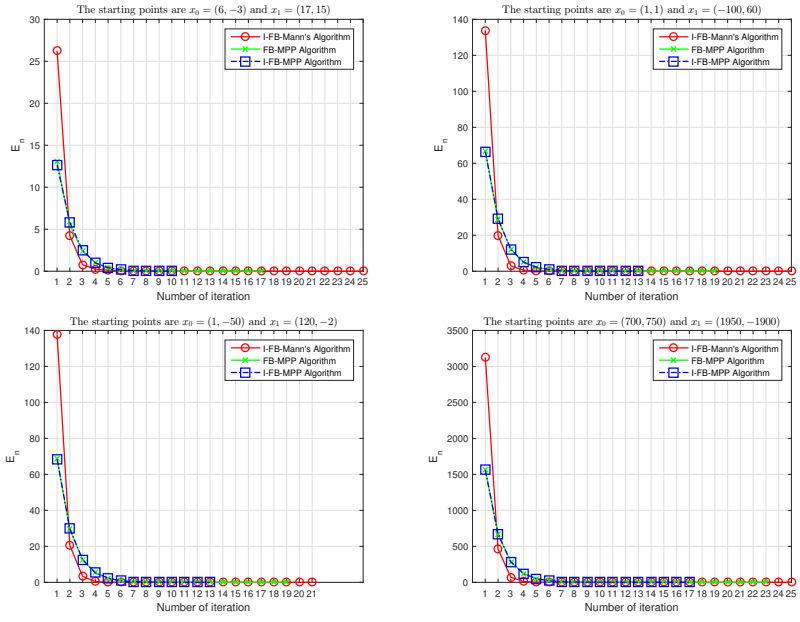


Figure 4. Convergence of error versus iteration for different starting points

Example 4.3. *Let $H = \mathbb{R}^3$ and let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by*

$$U\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + \sin(x_1 - 1) - 7 \\ 6x_2 + \cos(x_2 - 1) + 7 \\ 10x_3 - \arctan(x_3 - 0.5) - 6 \end{bmatrix}$$

and let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Vx = 2I_3x$ where $x \in \mathbb{R}^3$.

It is not hard to verify that U is $\frac{1}{11}$-inverse strongly monotone and V is maximally monotone. Moreover, by a direct calculation, we have for $s > 0$

$$\begin{aligned} J_s^V(x - sUx) &= (I + sV)^{-1}(x - sUx) \\ &= (I_3 + 2sI_3)^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - s\begin{bmatrix} 5x_1 + \sin(x_1 - 1) - 7 \\ 6x_2 + \cos(x_2 - 1) + 7 \\ 10x_3 - \arctan(x_3 - 0.5) - 6 \end{bmatrix}\right) \\ &= \frac{1}{1 + 2s}\begin{bmatrix} (1 - 5s)x_1 - s\sin(x_1 - 1) + 7s \\ (1 - 6s)x_2 - s\cos(x_2 - 1) - 7s \\ (1 - 10s)x_3 + s\arctan(x_3 - 0.5) + 6s \end{bmatrix} \end{aligned}$$

where $x \in \mathbb{R}^3$. It is not hard to verify that $(U + V)^{-1}(0) = \{(1, -1, 0.5)\}$. Since $\alpha = \frac{1}{11}$, we can choose $s = r_n = 0.15$ for all $n \in \mathbb{N}$. Let $\alpha_n = 0.1 + \frac{1}{100n+1}$ and

$$\theta_n = \begin{cases} \min\left\{\dfrac{1}{n^2\left\|x_n - x_{n-1}\right\|}, 0.5\right\}, & \text{if } x_n \neq x_{n-1}, \\ 0.5, & \text{otherwise.} \end{cases}$$

The stopping criterion is set as $E_n = \|x_{n+1} - x_n\| < 10^{-4}$. We will use four different pairs of starting points, x_0 and x_1, in the numerical tests as follows:

(1) $x_0 = (0, 0, 0)$ *and* $x_1 = (12, -3, 5)$.
(2) $x_0 = (4, -8, 5)$ *and* $x_1 = (-120, 80, -110)$.
(3) $x_0 = (700, 750, -700)$ *and* $x_1 = (-6, -10, 2)$.
(4) $x_0 = (810, -400, -580)$ *and* $x_1 = (950, -900, -1500)$.

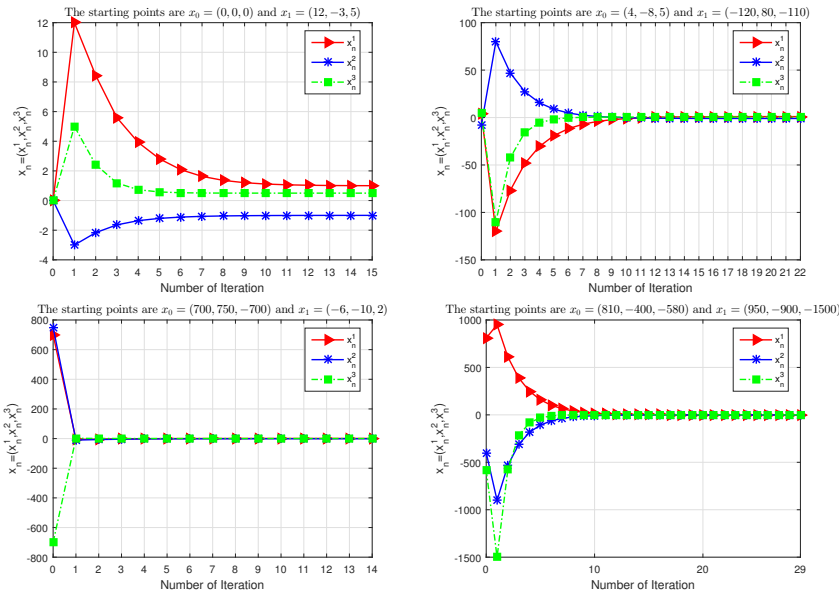


Figure 5. Convergence of $\{x_n\}$ generated by I-FB-MPP Algorithm (Algorithm (3.1))) to the solution $(1, -1, 0.5)$ for different starting points

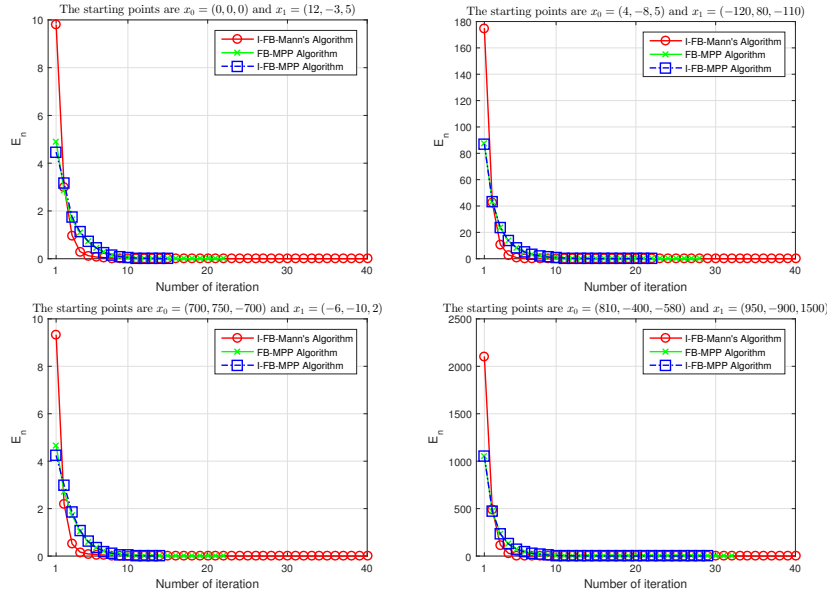


Figure 6. Convergence of error versus iteration for different starting points

Remark 4.4. *From Example 4.1, Example 4.2, and Example 4.3, these are demonstrated that the inertial forward-backward algorithm with the moving point projection method (I-FB-MPP Algorithm (Algorithm (3.1))) of Theorem 3.2) exhibits a superior convergence*

speed and requires fewer iterations to solve the inclusion problem compared to the two considered algorithms.

5. Conclusion

We developed and analyzed an inertial forward-backward algorithm incorporating the moving point projection method, referred to as the I-FB-MPP Algorithm (see Algorithm (3.1)). We demonstrated that the moving point projection method generates well-defined sequences through projections onto closed and convex sets. Furthermore, we established the weak convergence of Algorithm (3.1), as detailed in the proof of Theorem 3.2. We also compared the convergence performance of Algorithm (3.1) against classical Mann's iteration (1.8) and a forward-backward algorithm with moving point projection method (without an inertial term), illustrated by Example 4.1, Example 4.2, and Example 4.3. The numerical results presented in Figure 2, Figure 4, and Figure 6 confirm that Algorithm (3.1) achieves greater efficiency compared to these two benchmark methods.

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