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L-CATCH GUARANTEED PURSUIT TIME

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Abstract We consider a simple motion differential game of one pursuer and one evader. The dynamic equation of the pursuer and evader is describe by first order and second order differential equation respectively. Control functions of the players are subject to integral constraints. We show that pursuit is completed in l-catch sense, that is for some distance l, the difference in positions of the pursuit and evader $x(t)$ and $y(t)$ respectively is smaller than l i.e ($\|y(t) - x(t)\| \leq l$) at some time t. We construct a formula for guaranteed pursuit time and prove that pursuit is possible at that time.

MSC: 47H09
Keywords: Optimal pursuit time; integral generalized constraint; players control functions

1. Introduction

Differential game is a mathematical theory which is concerned with problem of conflicts modelled as game problem in which the state of the players depends on time in a continuous way. Differential game is an extension of sequential game theory to the continuous time case. Differential Games evolved as a research area in Applied Mathematics due to inter field research activities in game theory, calculus of variation and optimal control theory and its birth is by Rufus Isaacs in 1951. The needs of solving combat problems

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during this era was the motive behind this development. Nowadays there are many applications in various branches of knowledge such as Economics, Engineering Designs (such as Missile Guidance, Aircrafts and drones Control, Aerial Tactics etc) Behaviorial Biology etc.
The researchers on differential games have achieved rapid development from the birth of differential games. Due to its wide applications, easy integration with other disciplines to generate new branches, differential games has many types that varies according to the different classification criteria. According to the existence of payoff function, it can be divided into quantitative and qualitative differential games; for quantitative differential games, if the sum of payoffs is zero, it is called two-person zero sum differential games, if not zero, it is called two-person non-zero sum differential games, the two-person zero-sum differential games can be divided into differential games of fixed duration, pursuit-evasion differential games and differential games of survival; according to the different information structure, differential games can be divided into perfect information, incomplete information and no information differential games; in accordance with the different types of dynamic systems in the games, differential games can be divided into partial differential games and stochastic differential games, partial differential games describe the dynamic system using partial differential equations, while random interference or observation error exists in stochastic differential games; according to the number of players, differential games can be divided into two person and many person differential games, if there is motivation for cooperation between the players, it can be called cooperative differential games, if not, it is called non-cooperative differential games [6].

In a pursuit differential game, guaranteed pursuit time is a finite time, say T, for which pursuit is completed on or before the time T. Finding or estimating this time requires construction of pursuer's strategy such that for any control of evader, the strategy ensures that pursuer wins the game. There is substantial literature on this class of problem (details can be found in the following references: [9], [12] and [20]).
Other results were published on differential game with integral constraint (see [2], [3], [7], [8], [9], [10], [12], [13], [14] [15], [16], [17], [18] and [20]). In some of these research works, players dynamics are described by ordinary differential equations. In [12], Ibragimov et al. considered a pursuit differential game of one pursuer and one evader described by infinite system of first order differential equations. A sufficient condition of completion of pursuit is obtained, strategy for the pursuer is constructed and an explicit formula for the guaranteed pursuit time is give by

$$T' = \sup T_i,\ T_i = \frac{1}{2\lambda_i}\left[\ln\left(1 + 2\lambda_i\left(\frac{|z_{i0}|}{\rho_i - \sigma_i}\right)^2\right)\right]. \quad (1.1)$$

Later on Ibragimov et al. extended the guaranteed pursuit time found in [12] to an optimal pursuit time. Waziri and Ibragimov [20] studied guaranteed pursuit time of a differential game described by an infinite system of first order differential equations

$$\dot{z}_k + \beta_k z_k = u_k - v_k,\ z_k(0) = z_k^0,\ k = 1, 2, \cdots, \quad (1.2)$$

where $z_k, u_k, v_k \in \mathbb{R}^n$ and $\beta_k > 0$. In the paper, pursuer tries to transfer the state of the system from an initial state to another state (i.e.,$z(t) = z^1,\ t \in [0, \theta]$), in contrast, [12] considered the case $z(t) = 0,\ t \in [0, \theta]$. Sufficient conditions were obtained that ensure completion of the game at the guaranteed pursuit time;

$$\theta = \sup\theta_k,\ \theta_k = \frac{1}{2\beta_k}\left[\ln\left(2\beta_k\left(\frac{\|z_k^0 - z_k^1 e^{\beta_k\theta_k}\|}{\rho_k - \sigma_k}\right)^2 + 1\right)\right]. \tag{1.3}$$

Adamu et al.[1] studied a simple motion pursuit differential game of many pursuers and one evader in a Hilbert space l_2. The control functions of the pursuers and evader are subject to integral and geometric constraints respectively. Duration of the game is denoted by positive number θ.

In this paper a differential game with one pursuer and one evader is considered and the control functions of the players is subject to integral constraints. Formula of a guaranteed pursuit time is constructed.

2. Statement of the problem

Consider the space $l_2 = \left\{\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots) : \sum_{k=1}^{\infty} \alpha_k^2 < \infty\right\}$, with the $||\cdot|| : l_2 \to [0, +\infty)$ defined as

$$\|\alpha\| = \left(\sum_{k=1}^{n} \alpha_k^2\right)^{\frac{1}{2}}.$$

We consider a differential game of one pursuer and one evader in the space l_2, with the dynamic equations of the players given by;

$$\begin{cases} \dot{x}(t) = u(t),\ x(0) = x^0, \\ \ddot{y}\ (t) = v(t),\ \dot{y}\ (0) = y^1\ y(0) = y^0, \end{cases} \tag{2.1}$$

where $x, x^0, u_j, y, y^0, v \in l_2$, $u = (u_1, u_2, \cdots)$ and $v = (v_1, v_2, \cdots)$ are control parameters of pursuer and evader respectively.

Definition 2.1. A measurable function $u : [0, T] \longrightarrow l_2$, ($v : [0, T] \longrightarrow l_2$) is called an admissible control of the pursuer (evader) if it satisfies the following

$$\int_0^T \|u(s)\|^2\ ds \le \rho^2,\ \left(\int_0^T \|v(t)\|^2\ dt \le \sigma^2\right), \tag{2.2}$$

respectively, where ρ and σ are given positive numbers.

If the admissible controls of the pursuer and evader are chosen, then solutions to the dynamic equation (2.1) are given by

$$x(t) = x_0 + \int_0^t u(s)ds, \tag{2.3}$$

$$y(t) = y^0 + ty^1 + \int_0^t\int_0^r v(s)dsdr. \tag{2.4}$$

It is easy to see that

$$\int_0^t\int_0^r v(s)dsdr = \int_0^t (t-s)v(s)ds. \tag{2.5}$$

We can consider an equivalent differential game with the same control functions described by

$$\begin{cases} \dot{x}(t)_j = u_j(t),\ x_j(0) = x_j^0, \\ \dot{y}\ (t) = (\theta - t)v(t),\ y(0) = y^1\theta + y^0 = y_0, \end{cases} \tag{2.6}$$

instead of (2.1). Indeed, if the evader uses an admissible control $v(t) = (v_1(t), v_2(t), ...)$, then according to (2.1), we have

$$y(t) = y^0 + \theta y^1 + \int_0^\theta \int_0^r v(s)dsdr = y^0 + \theta y^1 + \int_0^\theta (\theta - t)v(t)dt, \tag{2.7}$$

and the same result can be obtained by (2.6)

$$y(\theta) = y_0 + \int_0^\theta (\theta - t)v(t)dt = y^0 + \theta y^1 + \int_0^\theta (\theta - t)v(t)dt. \tag{2.8}$$

Definition 2.2. Pursuit is said to be completed in L-catch sense in a game if there exist strategies u of the pursuer such that for any admissible control $v(\cdot)$ of the evader, the inequality $\|y(t) - x(t)\| \leq l$ is satisfied for some $t \in [0, \theta]$.

Definition 2.3 (Guaranteed Pursuit Time, T)**.** Pursuit is said to be completed at time $T > 0$, if there exist strategies of the pursuers $U(t, v(t))$, such that for any admissible control of the evader $v(t),\ 0 \leq t \leq T,\ \|y(\tau) - x(\tau)\| \leq l$ at some $\tau,\ 0 \leq \tau \leq T$. In the sequel, the number T is called guaranteed pursuit time.

3. Main Result

This section presents the formula for guaranteed pursuit time in the game (2.1)-(2.2) in the following theorem.

Theorem 3.1. *If $2\rho \geq T$, then the time define by*

$$T := \left(\frac{|y_0 - x_0|}{\gamma - 2\sqrt{3}\rho\sigma} \right)^2, \tag{3.1}$$

is a guaranteed pursuit time in the game (2.1)-(2.2), where γ energy resource of the dummy pursuer in (3.3), θ is any time less than T.

Proof. To prove this theorem, we first introduce dummy pursuer with state variable η and motion described by the following equation.

$$\dot{\eta}\ (t) = \zeta(t),\ \eta(0) = x_0 \tag{3.2}$$

with the control function subject to

$$\left(\int_0^\theta \|\zeta(t)\|^2 dt \right)^{\frac{1}{2}} \leq \gamma = \rho + \frac{l}{\sqrt{\theta}}. \tag{3.3}$$

We Construct the dummy pursuer's strategy as follows

$$\zeta(t) = \begin{cases} \dfrac{(y_0 - x_0)}{T} + (\theta - t)v(t), & 0 \leq t \leq \theta, \\ 0, \quad t \geq \theta, \end{cases} \tag{3.4}$$

The above constructed strategy (3.4) is admissibility, the process is as follows. Using Minkowski's inequality,

$$\begin{aligned}\left(\int_0^T \|\zeta(t)\|^2\,dt\right)^{\frac{1}{2}} &= \left(\int_0^T \left\|\frac{(y_0-x_0)}{T}+(\theta-t)v(t)\right\|^2 dt\right)^{\frac{1}{2}}\\ &\le \left(\int_0^T \left\|\frac{(y_0-x_0)}{T}\right\|^2 dt\right)^{\frac{1}{2}}+\left(\int_0^T \|(\theta-t)v(t)\|^2\,dt\right)^{\frac{1}{2}}\\ &= \left(\int_0^T \left\|\frac{(y_0-x_0)}{T}\right\|^2 dt\right)^{\frac{1}{2}}+\left(\int_0^T \|(\theta-t)\|^2\,\|v(t)\|^2\,dt\right)^{\frac{1}{2}}\end{aligned} \tag{3.5}$$

it is easy to see that

$$\left(\int_0^T \|(\theta-t)\|^2\,\|v(t)\|^2\,dt\right)^{\frac{1}{2}} \le \sqrt{3}\sigma T \tag{3.6}$$

as shown below. Using integration by part

$$\int (mn)\,dt = m\int ndt - \int\left(m'\int ndt\right)dt. \tag{3.7}$$

Let

$$\begin{aligned} &m=(\theta-t)^2,\ m'=-2\,(\theta-t)\\ &n=\|v(t)\|^2,\ \int_0^T ndt=\int_0^T \|v(t)\|^2 dt\le\sigma^2\end{aligned} \tag{3.8}$$

then

$$\begin{aligned}\int_0^T \|(\theta-t)\|^2\,\|v(t)\|^2\,dt &= (\theta-t)^2\int_0^T \|v(t)\|^2 dt-\int_0^T\left(-2\,(\theta-t)\int_0^T \|v(t)\|^2 dt\right)dt\\ &\le (\theta-t)^2\,\sigma^2-\int_0^T\left(-2\,(\theta-t)\,\sigma^2\right)dt\\ &= (\theta-t)^2\,\sigma^2+2\sigma^2\int_0^T(\theta-t)\,dt\\ &= (\theta-t)^2\,\sigma^2+2\sigma^2\left(\frac{2\theta T-T^2}{2}\right),\\ &\le (\theta-t)^2\,\sigma^2+2\sigma^2\frac{T^2}{2},\ T>\theta,\end{aligned}$$

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also since $t \leq \theta$, then

$$\begin{aligned}\int_0^T \|(\theta - t)\|^2 \|v(t)\|^2 \, dt \leq& (\theta - t)^2 \sigma^2 + 2\sigma^2 \frac{T^2}{2}, \\ =& \left(\theta^2 - 2\theta t + t^2\right) \sigma^2 + \sigma^2 T^2 \\ \leq& \left(2\theta^2 - 2\theta t\right) \sigma^2 + \sigma^2 T^2 \\ \leq& 2\theta^2 \sigma^2 + \sigma^2 T^2 \\ \leq& 2T^2 \sigma^2 + \sigma^2 T^2 \\ =& 3T^2 \sigma^2.\end{aligned}$$

Therefore

$$\left(\int_0^T \|(\theta - t)\|^2 \|v(t)\|^2 \, dt\right)^{\frac{1}{2}} \leq \sqrt{3}\sigma T.$$

Looking at (2.2) and (3.6), the inequality (3.5) becomes

$$\begin{aligned}\left(\int_0^T \|\zeta(t)\|^2 \, dt\right)^{\frac{1}{2}} \leq& \left(\int_0^T \left\|\frac{(y_0 - x_0)}{T}\right\|^2 dt\right)^{\frac{1}{2}} + \sqrt{3}\sigma T \\ \leq& \frac{|y_0 - x_0|}{T} \left(\int_0^T dt\right)^{\frac{1}{2}} + \sqrt{3}\sigma T \\ =& \frac{|y_0 - x_0|}{T} (T)^{\frac{1}{2}} + \sqrt{3}\sigma T \\ =& \frac{(y_0 - x_0)}{\sqrt{\left(\frac{|y_0 - x_0|}{\gamma - 2\sqrt{3}\rho\sigma}\right)^2}} + \sqrt{3}\sigma T \\ =& \gamma - 2\sqrt{3}\rho\sigma + \sqrt{3}\sigma T. \\ \leq& \gamma - \sqrt{3}\sigma T + \sqrt{3}\sigma T \\ =& \gamma.\end{aligned}$$

Hence the strategy is admissible. Next is to show pursuit is possible when the dummy pursuer uses the time defined in Theorem 3.1 above. Solution of $\dot{\eta}\,(t)$ is given by

$$\eta(T) = x_0 + \int_0^T \zeta(s) ds$$

using the strategy (3.4) we have

$$\begin{aligned}\eta(T) =& x_0 + \int_0^T \left(\frac{y_0 - x_0}{T} + (\theta - t)v(s)\right) ds \\ =& x_0 + \int_0^T \frac{y_0 - x_0}{T} ds + \int_0^T (\theta - t)v(s)ds \\ =& x_0 + \frac{y_0 - x_0}{T} \int_0^T ds + \int_0^T (\theta - t)v(s)ds \\ =& y_0 + \int_0^T (\theta - t)v(s)ds \\ =& y(T).\end{aligned}$$

We define the strategy of the real pursuer using the strategy of the dummy pursuer as follows

$$u(t) = \frac{\rho}{\gamma}\zeta(t).$$

The strategy is indeed admissible, as shown below

$$\begin{aligned}\left(\int_0^T \|u(t)\|^2 dt\right)^{\frac{1}{2}} =& \left(\int_0^T \left\|\frac{\rho}{\gamma}\zeta(t)\right\|^2 dt\right)^{\frac{1}{2}} \\ =& \frac{\rho}{\gamma}\left(\int_0^T \|\zeta(t)\|^2 dt\right)^{\frac{1}{2}} \\ \leq& \frac{\rho}{\gamma}\gamma = \rho.\end{aligned}$$

For the completion of pursuit, we show that $\|y(\tau) - x(\tau)\| \leq l$.

$$\begin{aligned}\|y(T) - x(T)\| &= \|\eta(T) - x(T)\| \\ &= \left\|x_0 + \int_0^T \zeta(s)ds - x_0 - \int_0^T u(s)ds\right\| \\ &= \left\|\int_0^T \zeta(s)ds - \int_0^T \frac{\rho}{\gamma}\zeta(s)ds\right\|;\ u(s) = \frac{\rho}{\gamma}\zeta(s) \\ &= \left\|\int_0^T \zeta(s)ds - \frac{\rho}{\gamma}\int_0^T \zeta(s)ds\right\| \\ &= \left(1 - \frac{\rho}{\gamma}\right)\left\|\int_0^T \zeta(s)ds\right\|\end{aligned} \tag{3.9}$$

$$\leq \left(1-\frac{\rho}{\gamma}\right)\int_0^T \|\zeta(s)\|\,ds$$
$$\leq \left(1-\frac{\rho}{\gamma}\right)\left[\left(\int_0^T 1^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \|\zeta(s)\|^2\,ds\right)^{\frac{1}{2}}\right]$$
$$=\left(1-\frac{\rho}{\gamma}\right)\gamma\sqrt{T}$$
$$=(\gamma-\rho)\sqrt{T}=l. \tag{3.10}$$

Hence pursuit is completed at time T which is a guaranteed pursuit time.

4. Conclusion

We have studied a differential game of one pursuer and one evader, where the motion of the pursuer and evader is described by first order and second order differential equations respectively. Control functions of pursuers and evader are subject to integral constraints. We give a formula for a certain time and proved that it is indeed a guaranteed pursuit time in l-catch sense.

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