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A CAPUTO PROPORTIONAL FRACTIONAL DIFFERENTIAL EQUATION WITH MULTI-POINT BOUNDARY CONDITION

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Abstract In this article, inspired by the proportional fractional derivative, we investigate the uniqueness of the solutions for generalized Caputo proportional differential equation with multi-point boundary condition. We convert the formulated problem into an equivalent integral equation with the support of the Green's function. The results we obtain generalize many other existing results when compared with the Caputo type classical fractional differential equations.

MSC: 26A33, 34A12, 34A43, 34D20

Keywords: existence and uniqueness, Caputo proportional derivatives, fixed point theorems, nonlocal condition

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1. Introduction

One of the attractive area of Mathematics is the fractional calculus that generalized the classical calculus with strong connection to the integral and derivatives of discretionary order. References to this growing area of specialization can be found in [1, 3–6]. What makes it interesting in this area of research is its multi fraction operators that can be plump for the best operator lifting to fit into the model under investigation. Some researches have gone beyond that and discovered new operators which enriched the repository of fractional calculus. Among these operators we mention the new operators presented in [7–13].

In 2015 [14], the authors proposed what they called conformable derivative; a derivative that permits differentiation of non-integer order. This work by followed by the work of Abdeljawad [15] who presented the left and right derivatives of this kind. Nonetheless, uncommonly these operators do not produce the function itself when the order is 0. Anderson et al. [16, 17] modified the conformable derivative so that it gives the function itself when the order is 0 and the derivative of the function if the order is 1 and they called them proportional derivatives. These local derivatives were after that used by Jarad et al. [18] to generate a new type of nonlocal fractional operators with two parameters that brings out some known fractional derivatives as particular cases.

Beefing from the tools of the fixed point theory, many authors have studied some qualitative properties of fractional integral or differential equations with focus on the existence and uniqueness of solutions to such equations. In [19], the authors studied the existence and uniqueness for a problem involving Hilfer fractional derivative. In [20], the authors investigated some fractional integro-differential equations involving ψ-Hilfer fractional derivative. In [21, 22], the authors studied the existence of solutions to boundary value problems involving fractional derivatives. Other similar works can be found in [23–29, 31]

Motivated and inspired by the aforementioned works, in this article, we consider the following proportional fractional differential equation with multi-point boundary condition of form:

$$\begin{cases} -{}^{c}\mathcal{D}^{\alpha,\rho}_{0+}u(t) = f(t,u(t)); \quad t \in J = [0,1],\ 1 < \alpha \le 2,\ 0 < \rho \le 1, \\ u(0) = 0, \quad u(1) = \displaystyle\sum_{i=1}^{m-2} c_i u(\tau_i),\ m \in \mathbb{N}, \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{\alpha,\rho}_{0}u(\cdot)$ is the generalized proportional fractional derivative of Caputo type of order $(1 < \alpha \le 2)$, $c_i, \tau_i \in (0,1)$ with $\displaystyle\sum_{i=1}^{m-2} c_i u(\tau_i) < 1$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

2. Preliminaries

We provide some preliminary details, results and definitions of fractional calculus in this section which are important throughout this paper.

Definition 2.1. [5] The fractional integral of order υ with the lower limit a^+ for a function g is defined as

$$I^{\upsilon}_{0+}g(s) = \frac{1}{\Gamma(\upsilon)}\int_{0+}^{s}(s-\tau)^{\upsilon-1}g(\tau)d\tau, \quad \upsilon > 0,$$

provided the right side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ denotes the gamma function.

Definition 2.2. [5] The Caputo fractional derivative of order υ with the lower limit a^+ for a function g is defined as

$$^{C}D^{\upsilon}_{0^+}g(s) = \frac{1}{\Gamma(n-\upsilon)}\int_{0^+}^{s}(s-\tau)^{n-\upsilon-1}g^{n}(\tau)d\tau,\ n-1<\upsilon\leq n,\ n\in\mathbb{N},$$

provided the function g differentiable on $[a, +\infty)$, where $\Gamma(\cdot)$ denotes the gamma function.

Definition 2.3. [18] The left generalized proportional fractional integral of order υ and $\rho \in (0, 1]$ of a function g is defined by

$$I^{\upsilon,\rho}_{0^+}g(s) = \frac{1}{\rho^{\upsilon}\Gamma(\upsilon)}\int_{0^+}^{s}e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{\upsilon-1}g(\tau)d\tau,\ \upsilon\in\mathbb{C},\ Re(\upsilon)>0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.4. [18] The left generalized proportional fractional derivative of Caputo type of order $(n < \upsilon \leq n)$ and $\rho \in (0, 1]$ of a function g is defined by

$$^{C}D^{\upsilon,\rho}_{0^+}g(s) = \frac{1}{\rho^{n-\upsilon}\Gamma(n-\upsilon)}\int_{0^+}^{t}e^{\frac{\rho-1}{\rho}(s-\tau)}(s-\tau)^{n-\upsilon-1}(\mathbb{D}^{n,\rho}g)(\tau)d\tau, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma function and $n = [\upsilon] + 1$ and $\mathbb{D}^{\rho}g(\tau) = (1-\rho)g(\tau) + \rho g(\tau)$.

Remark 2.5. Observe that if $\rho = 1$ Definitions 2.3 and 2.4 coincides with the classical Definitions of Riemann-Liouville fractional integral and Caputo fractional derivative (see, Definitions 2.1 and 2.2).

Theorem 2.6. [18] *Let $\rho \in (0, 1]$, $p \in \mathbb{C}, Re(p) > 0$, $n = [Re(p)] + 1$ and , we have*

$$(I^{\upsilon,\rho C}_{0^+}D_{0^+}{}^{\upsilon,\rho}g)(s) = g(s) - e^{\frac{\rho-1}{\rho}(s)}\sum_{k=0}^{m}\frac{(\mathbb{D}^{k,\rho}_{0^+}g)(0)}{\rho^{k}k!}(s)^{k}. \tag{2.3}$$

Theorem 2.7. *(**Contraction Mapping Principle**)[2] Let X be a Banach space, $\mathcal{S} \subset X$ be closed and $T : \mathcal{S} \to \mathcal{S}$ a contraction mapping i.e*

$$\|Tu - T\bar{u}\| \leq k\|z - \bar{z}\|,\ \text{for all } u, \bar{u} \in \mathcal{S},\ \text{and some } k \in (0, 1).$$

Then $\mathcal{S}$ has a unique fixed point.

3. Existence and Uniqueness results

In this section, we are discussing the existence and uniqueness of the proposed problem using fixed point theorems. Before we proceed, we state and prove the following lemma which shows the equivalence between the proposed problem and the Volterra integral equation.

Lemma 3.1. *For $h \in L^1(J, \mathbb{R})$, the boundary value problem for fractional differential equation*

$$\begin{cases} ^{c}\mathcal{D}^{\alpha,\rho}_{0^+}u(t) + h(t) = 0; \quad t \in J = (0,1), \quad 1 < \alpha \leq 2, \quad 0 < \rho \leq 1, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} c_i u(\tau_i), \qquad c_i, \tau_i \in (0,1), \end{cases} \tag{3.1}$$

has a solution u of the form $u(t) = \int_0^1 G(t,s)h(s)ds$, where $G(t,s)$ is the Green function given by

$$G(t,s) = \frac{1}{\rho^\alpha\Gamma(\alpha)} \begin{cases} \frac{t^{\alpha-1}e^{\frac{(\rho-1)}{\rho}t}}{\rho(1-\lambda)}\left[e^{\frac{(\rho-1)}{\rho}(1-s)}(1-s)^{\alpha-1} - \sum_{i=1}^{m-2} c_i e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{\alpha-1}\right] \\ -e^{\frac{(\rho-1)}{\rho}(t-s)}(t-s)^{\alpha-1}; \quad s \le t,\ \tau_{i-1} < s \le \tau_i,\ i = 1,2,\cdots,m-1, \\ \frac{t^{\alpha-1}e^{\frac{(\rho-1)}{\rho}t}}{\rho(1-\lambda)}\left[e^{\frac{(\rho-1)}{\rho}(1-s)}(1-s)^{\alpha-1} - \sum_{i=1}^{m-2} c_i e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{\alpha-1}\right]; \\ t \le s, \quad \tau_{i-1} < s \le \tau_i, \quad i = 1,2,\cdots,m-1, \end{cases} \tag{3.2}$$

where $\Delta = \frac{1}{\rho}\left(e^{\frac{(\rho-1)}{\rho}} - \sum_{i=1}^{m-2} c_i\tau_i e^{\frac{(\rho-1)}{\rho}}\tau_i\right)$.

Proof. Applying the operator $\mathcal{I}_{0+}^{\alpha,\rho}$ on $-{}^c\mathcal{D}_{0+}^{\alpha,\rho}u(t) = h(t)$ and in view of Theorem 2.6, gives

$$u(t) = -\mathcal{I}_{0+}^{\alpha,\rho}h(t) + k_0 t^{\alpha-1}e^{\frac{(\rho-1)}{\rho}t} + \frac{k_1}{\rho}t^{\alpha-2}e^{\frac{(\rho-1)}{\rho}t}, \text{ where } k_0, k_1 \in \mathbb{R}. \tag{3.3}$$

The boundary condition $u(0) = 0$, yields $k_1 = 0$ and the condition $u(1) = \sum_{i=1}^{m-2} c_i u(\tau_i)$, gives $k_1 = \frac{1}{1-\lambda}\left[\mathcal{I}_{0+}^{\alpha}h(1) - \sum_{i=1}^{m-2} c_i e^{\frac{(\rho-1)}{\rho}}\tau_i \mathcal{I}_{0+}^{\alpha}h(\tau_i)\right]$ where $\lambda = \sum_{i=1}^{m-2} c_i u(\tau_i) < 1$.
Thus, equation (3.3) takes the form

$$u(t) = -\mathcal{I}_{0+}^{\alpha,\rho}h(t) + \frac{t^{\alpha-1}}{1-\lambda}\left[\mathcal{I}_{0+}^{\alpha,\rho}h(1) - \sum_{i=1}^{m-2} c_i e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}\mathcal{I}_{0+}^{\alpha,\rho}h(\tau_i)\right]. \tag{3.4}$$

Now, for $0 \le t \le \tau_i$, equation (3.4) yields

$$\begin{aligned} u(t) = \int_0^t &\left[\frac{-(t-s)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)}e^{\frac{\rho-1}{\rho}(t-s)} + \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)}e^{\frac{\rho-1}{\rho}(t-s)}\left((1-s)^{\alpha-1}\right.\right. \\ &\left.\left. - \sum_{j=1}^{m-2} c_j(\tau_j-s)^{\alpha-1}\right)\right] h(s)ds \\ &+ \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)}e^{\frac{\rho-1}{\rho}(t-s)}\int_t^{\tau_i}\left((1-s)^{\alpha-1} - \sum_{j=1}^{m-2} c_i(\tau_i-s)^{\alpha-1}\right)h(s)ds \\ &+ \sum_{i=2}^{m-2}\int_{\tau_{i-1}}^{\tau_i} e^{\frac{\rho-1}{\rho}(t-s)}(1-s)^{\alpha-1} - \sum_{j=1}^{m-2} c_j(\tau_j-s)^{\alpha-1}h(s)ds \\ &+ \int_{\tau_{i-2}}^{1} e^{\frac{\rho-1}{\rho}(t-s)}(1-s)^{\alpha-1}h(s)ds. \end{aligned}$$

Now, for $\tau_{l-1} \leq t \leq \tau_l,\ 2 \leq l \leq m-2$, equation (3.4) takes the form

$$\begin{aligned}
u(t) = &\int_0^{\tau_1} \Bigg[\frac{-(t-s)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-s)} + \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)} e^{\frac{\rho-1}{\rho}(t-s)} \Bigg((1-s)^{\alpha-1} \\
&- \sum_{j=1}^{m-2} c_j(\tau_j - s)^{\alpha-1}\Bigg)\Bigg] h(s)ds \\
&+ \sum_{i=2}^{m-2} \int_{\tau_{i-1}}^{\tau_i} \Bigg[\frac{-(t-s)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-s)} + \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)} e^{\frac{\rho-1}{\rho}(t-s)} \Bigg((1-s)^{\alpha-1} \\
&- \sum_{j=i}^{m-2} c_j(\tau_j - s)^{\alpha-1} + (1-s)^{\alpha-1}\Bigg)\Bigg] h(s)ds \\
&+ \int_{\tau_{i-1}}^{t} \Bigg[\frac{-(t-s)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-s)} + \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)} e^{\frac{\rho-1}{\rho}(t-s)} \Bigg((1-s)^{\alpha-1} \\
&- \sum_{j=1}^{m-2} c_j(\tau_j - s)^{\alpha-1}\Bigg)\Bigg] h(s)ds \\
&+ \frac{t^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)(1-\lambda)} e^{\frac{\rho-1}{\rho}(t-s)} \int_t^{\tau_l} \Bigg((1-s)^{\alpha-1} - \sum_{j=l}^{m-2} c_i(\tau_i - s)^{\alpha-1}\Bigg) h(s)ds \\
&+ \sum_{i=l+1}^{m-2} \int_{\tau_{i-1}}^{\tau_i} e^{\frac{\rho-1}{\rho}(t-s)}(1-s)^{\alpha-1} - \sum_{j=i}^{m-2} c_j(\tau_j - s)^{\alpha-1} h(s)ds \\
&+ \int_{\tau_{m-2}}^{1} e^{\frac{\rho-1}{\rho}(t-s)}(1-s)^{\alpha-1} h(s)ds.
\end{aligned}$$

Repeating the same produce as above for $\tau_{m-2} \leq t \leq 1,\ 2 \leq l \leq m-2$, the unique solution of the boundary value problem (1.1) is given by $u(t) = \int_0^1 G(t,s)h(s)ds$, where $G(t,s)$ is the Green function defined by (3.2).

Lemma 3.2. *The Green's function $G(t,s)$ obeys the given relations:*
(A_1) $G(t,s)$ is a continuous function over J;
(A_2) $\max_{t\in J} \int_0^1 G(t,s)ds \leq \Omega$, where Ω is defined by

$$\Omega = \left[\frac{1}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{\rho^{\alpha+1}\Delta\Gamma(\alpha+1)} - \frac{1}{\rho^{\alpha+1}\Delta\Gamma(\alpha+1)} \sum_{i=1}^{m-2} c_i(\tau_i)^\alpha\right]. \tag{3.5}$$

Proof. (A_1) It follows directly.
In order to prove (A_2). Since $\max\limits_{\rho\in(0,1]}\left\{e^{\frac{(\rho-1)}{\rho}(t-s)}\right\} < 1$, we have

$$\begin{aligned}\max_{t\in J}\int_0^1 G(t,s)ds &= \max_{t\in J}\bigg(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{(\rho-1)}{\rho}(t-s)}(t-s)^{\alpha-1}ds\\ &\quad+\frac{te^{\frac{(\rho-1)}{\rho}t}}{\rho^{\alpha+1}\Delta\Gamma(\alpha)}\int_0^1 e^{\frac{(\rho-1)}{\rho}(1-s)}(1-s)^{\alpha-1}ds\\ &\quad-\frac{te^{\frac{(\rho-1)}{\rho}t}}{\rho^{\alpha+1}\Delta\Gamma(\alpha)}\sum_{i=1}^{m-2}c_i\int_0^{\tau_i} e^{\frac{(\rho-1)}{\rho}(\tau_i-s)}(\tau_i-s)^{\alpha-1}ds\bigg)\\ &\le \max_{t\in J}\bigg(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{t}{\rho^{\alpha+1}\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\\ &\quad-\frac{t}{\rho^{\alpha+1}\Delta\Gamma(\alpha)}\sum_{i=1}^{m-2}c_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}ds\bigg)\\ &\le \frac{1}{\rho^\alpha\Gamma(\alpha+1)}+\frac{1}{\rho^{\alpha+1}\Delta\Gamma(\alpha+1)}-\frac{1}{\rho^{\alpha+1}\Delta\Gamma(\alpha+1)}\sum_{i=1}^{m-2}c_i(\tau_i)^\alpha.\end{aligned}$$

Hence, the desired result.

3.1. Uniqueness result.

Now, by using Banach contraction principle, we are proving the uniqueness of solution to the proposed problem (1.1). Hence we need the following hypotheses.
(H_1) Let $f : J\times\mathbb{R}\to\mathbb{R}$ be a continuous function;
(H_2) There exist a constant $L > 0$ such that for each $t\in J$ and $u,\bar{u}\in\mathbb{R}$

$$|f(t,u)-f(t,\bar{u})|\le L|u-\bar{u}|.$$

Here $X = C(J,\mathbb{R})$ denotes the Banach space with respect to the norm defined by

$$\|u\| = \max_{t\in J}\{|u(t)| : t\in J\}.$$

In view of Lemma 3.1, the boundary value problem (1.1) can be transform into an equivalent integral equation of the form:

$$u(t) = \int_0^1 G(t,s)f(t,u(s))ds,\quad t\in J, \tag{3.6}$$

and define the operator $T : X\to X$ by

$$Tu(t) = \int_0^1 G(t,s)f(t,u(s))ds,\quad t\in J. \tag{3.7}$$

Solutions of the proposed boundary value problem (1.1) means the fixed points of the operator T.

Theorem 3.3. *Let $0 < \alpha < 1$, and $0 < \rho \le 1$. Suppose that the assumptions (H_1) and (H_2) are satisfied. Then, problem (1.1) has a unique solution on J.*

Proof. Let $u, \bar{u} \in X$, then for each $t \in J$, we have

$$
\begin{aligned}
|((Tu)(t) - (T\bar{u})(t))| &\leq \int_0^1 G(t,s)|f(s,u(s)) - f(s,\bar{u}(s))|ds \\
&\leq L\|u-\bar{u}\| \int_0^1 G(t,s)ds \\
&\leq L\Omega\|u-\bar{u}\|
\end{aligned} \tag{3.8}
$$

Hence, as consequences of Banach contraction principle, problem (1.1) has a unique solution.

4. Conclusions

Fractional calculus has been very popular due to its application in real world problems. Firstly, we established the relation between generalized problem with Volterra integral equations. Utilizing the techniques of fixed point theorem, we established the existence of solutions of nonlinear generalized Caputo fractional differential equation with nonlocal boundary condition. The obtained results generalized some existing results in the literature.

Author contributions

The authors contributed equally in writing this article. All authors read and approved the final manuscripts

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