BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
ISSN: 2408-154X (Print)
ISSN: 3057-0557 (Online)
https://bangmodjmcs.com
Volume 12 (2026)
Pages 43–57



Common Fuzzy Fixed Point Theorems for Fuzzy F-Weak Contractions in Non-Archimedean Fuzzy Metric Spaces

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Received: 14 May 2025 / Accepted: 22 March 2026

Abstract In this paper, we extend the concept of fuzzy F-weak contractions to the framework of non-Archimedean fuzzy metric spaces in the sense of George and Veeramani [6]. A generalized class of fuzzy F-weak contractions for nonlinear mappings of arbitrary finite arity is introduced. Under suitable conditions, we establish several common fuzzy fixed point theorems for pairs of mappings in this setting. The obtained results generalize and unify various existing fixed point theorems in fuzzy metric spaces and related structures. An illustrative example is provided to demonstrate the applicability of the main results.

MSC: 47H09
Keywords: non-Archimedean fuzzy metric spaces; common fuzzy fixed point; fuzzy F-weak contraction mappings

Published online: 31 March 2026
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Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: P. Yoosomran et al., Common Fuzzy Fixed Point Theorems for Fuzzy F-Weak Contractions in Non-Archimedean Fuzzy Metric Spaces, Bangmod Int. J. Math. & Comp. Sci., Vol. 12 (2026), 43–57. https://doi.org/10.58715/bangmodjmcs.2026.12.3

1. Introduction

Kramosil and Michàlet [11] were first introduced the concept of fuzzy metric spaces. After that, the concept of fuzzy metric spaces modified by George and Veeramani [6]. In 1998, Grabiec [5] extended fixed point theorems of Banach and Edelstein are extended to fuzzy metric spaces in the sense of Kramosil and Michàlet. In 2000, Gregori et al. [12] introduced a notion of precompact fuzzy metric spaces which provide several satisfactory and presented examples of fuzzy metrics in the sense of George and Veeramani [6, 7]. In 2014, Wardowski [2] introduced the concept of fuzzy F-weak contractions introduces a novel framework for analyzing mappings in fuzzy environments, where the conventional conditions of contractivity are weakened to accommodate more complex and realistic scenarios. Non-Archimedean fuzzy metric spaces, characterized by a stronger form of the triangular inequality, provide a fertile ground for exploring these contractions, as they often arise in contexts such as valuation theory, p-adic analysis, and quantum physics. In addition, some researchers have studied the weak contraction principle on various spaces (see in [20–28]).

In this paper, the concept of fuzzy F-weak contraction mapping is introduced in non-Archimedean fuzzy metric spaces and consequently existence and common fuzzy fixed point for such maps are obtained. Our results substantially generalize several comparable results in the literature (see in [1, 14]). The remainder of this paper is organized as follows. Section 2 reviews essential preliminaries, including definitions and lemmas related to fuzzy metric spaces and F-contractions in the sense of Wardowski [1]. Section 3 presents our main results, featuring the proofs of common fuzzy fixed point theorems. Section 4 provide concrete examples to validate our theoretical findings. The last section concludes with a discussion of potential applications and future research directions.

2. Preliminaries

In this section, we give some preliminary definitions and results in fuzzy metric spaces for our main results as follows.

2.1. Triangular norms

Definition 2.1. [12] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous $t-norm$ if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) $a * 1 = a$ for all $a \in [0,1]$;
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

If $a_1, a_2, \ldots, a_m \in [0,1]$, then $*_{i=1}^{m} a_i = a_1 * a_2 * \ldots * a_m$.

For each $a \in [0,1]$, the sequence $\{*^m\}_{m=1}^{\infty}$ is defined inductively by $*^1 a = a$ and $*^{m+1} a = (*^m a) * a$ for all $m \geq 1$.

Definition 2.2. [12] A $t-norm$ $*$ is said to be of $H-type$ if the sequence $\{*^m a\}_{m=1}^{\infty}$ is equicontinuous at $a = 1$, i.e. , for all $\epsilon \in (0,1)$, there exists $\eta \in (0,1)$ such that if $a \in (1-\eta, 1]$, then $*^m a > 1 - \epsilon$ for all $m \in \mathbb{N}$.

The most important and well known continuous $t-norm$ of $H-type$ is $* := \min$. Other examples can be found in [15].

Remark 2.3. [4] A $t-norm$ $*$ is called positive, if $a*b>0$ for all $a,b\in(0,1)$.

Examples of continuous $t-norms$ are Lukasievicz $t-norm$, that is, $a*_L b=\max\{a+b-1,0\}$, product $t-norm$, that is, $a*_P b=ab$ and minimum $t-norm$, that is, $a*_M b=\min\{a,b\}$.

2.2. Non-Archimedean fuzzy metric spaces

We shall study the basic definitions and properties about fuzzy metric spaces. The concept of fuzzy metric spaces is defined by George and Veeramani [6] as follows.

Definition 2.4. [4] Let X be an arbitrary nonempty set, $*$ is a continuous $t-norm$, and M is a fuzzy set on $X\times X\times(0,\infty)$.The triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space (written with short, **non-AFMS**) if satisfying the following conditions, for each $x,y,z\in X$ and $t,s>0$,

(1) $M(x,y,t)>0$;
(2) $M(x,y,t)=1$ if and only if $x=y$ for all $t>0$;
(3) $M(x,y,t)=M(y,x,t)$;
(4) $M(x,y,t)*M(y,z,s)\leq M(x,z,\mathtt{max}\{t,s\})$ for all $t,s>0$;
(5) $M(x,y,\cdot):(0,\infty)\to[0,1]$ is continuous.

In this case, we also say that $(X,M,*)$ is a **non-AFMS** under $*$. In the sequel, we will only consider **non-AFMS** satisfying :

(6) $\lim\limits_{t\to\infty}M(x,y,t)=1$ for all $x,y\in X$.

It is worth noting that, every **non-AFMS** is a fuzzy metric space. Also, we say $(X,M,*)$ is a strong fuzzy metric space on X, if $t=s$ for all $t,s>0$ that is $M(x,y,t)*M(y,z,t)\leq M(x,z,t)$.

It is worth pointing out that $0<M(x,y,t)<1$ (for all $t>0$) provided $x\neq y$, (see in [16]).

Let $(X,M,*)$ be a fuzzy metric space. For $t>0$, the open ball $B(x,r,t)$ with a center $x\in X$ and a radius $0<r<1$ is defined by

$$B(x,r,t)=\{y\in X:M(x,y,t)>1-r\}.$$

A subset $A\subset X$ is called open if for each $x\in A$, there exist $t>0$ and $0<r<1$ such that $B(x,r,t)\subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X, called the topology induced by the fuzzy metric M. This topology is metrizable (see in [12]).

We note that $M(x,y,t)$ can be thought of as the definition of nearness between x and y with respect to t. Besides, it is known that $M(x,y,\cdot)$ is nondecreasing for all $x,y\in X$.

Definition 2.5. Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence if for every $0<\varepsilon<1$ and for every $t>0$, there is $n_0\in\mathbb{N}$ such that $M(x_n,x_m,t)>1-\varepsilon$ for every $n,m\geq n_0$.

Definition 2.6. A sequence $\{x_n\}$ in X is a Cauchy sequence in the sense of Grabiec [5], if $M(x_n,x_{n+p},t)\to1$ as $n\to\infty$, for every $p\in\mathbb{N}$ and for every $t>0$.

Definition 2.7. A fuzzy metric space $(X,M,*)$ is complete if every Cauchy sequence is convergent.

Now we give some examples of fuzzy metric spaces due to Gregori et al. [13].

Example 2.8. [13] Let (X, d) be a metric space and $g : \mathbb{R}^+ \to [0, \infty)$ be an increasing continuous function. Define $M : X \times X \times (0, \infty) \to [0, 1]$ as

$$M(x, y, t) = e^{\left(-\frac{d(x,y)}{g(t)}\right)},$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space on X, where $*$ is the product $t - norm$.

Remark 2.9. [4] Let (X, d) be a metric space. We define $a * b = ab$ for all $a, b \in [0, 1]$ and $M_d(x, y, t) = \dfrac{kt^n}{kt^n + md(x, y)}$ for every $(x, y, t) \in X \times X \times [0, \infty)$, where k, m and n are positive real numbers then $(X, M_d, *)$ forms a fuzzy metric space. Thus, every metric space induces a fuzzy metric space. The fuzzy metric given by $M_d(x, y, t) = \dfrac{t}{t + d(x, y)}$ for every $(x, y, t) \in X \times X \times [0, \infty)$ is called standard fuzzy metric.

In 1988, Grabiec [5] introduced certain properties of functions in fuzzy metric spaces see in Lemma 2.10.

Lemma 2.10. *Let $(X, M, *)$ be a fuzzy metric space. For all $x, y \in X$, $M(x, y, \cdot)$ is non-decreasing function.*

Definition 2.11. Let $(X, M, *)$ be a fuzzy metric space. Then the mapping M is said to be continuous on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

when $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

In 2004, Rodríguez-López and Romaguera [17] introduced property of a continuous function in fuzzy metric spaces see in Lemma 2.12.

Lemma 2.12. *If $(X, M, *)$ be a fuzzy metric space, then M is a continuous function on $X \times X \times (0, \infty)$.*

2.3. Common fuzzy fixed points and F-weak contraction mappings

Let X be a space of points with generic element of X denoted by x and $I = [0, 1]$. A fuzzy set A in X is characterized by a membership function $A : X \to I$ such that each element $x \in X$ is associated with a real number $A(x) \in [0, 1]$. Let $\mathscr{F}(X)$ be a collection of all fuzzy subsets of X. Let (X, d) be a metric space and A be a fuzzy set in X. If $\alpha \in (0, 1]$, then the α-level set A_α of A is defined as $A_\alpha = \{x \in X : A(x) \geq \alpha\}$. For $\alpha = 0$, we have $A_0 = \overline{\{x \in X : A(x) > 0\}}$, where $\overline{A}$ denotes the closure of a set A in (X, d). A fuzzy set A is said to be more accurate than fuzzy set B, denoted by $A \to B$ if and only if $A(x) \leq B(x)$ for each x in X. It is obvious that if $0 < \alpha \leq \beta \leq 1$, then $A_\beta \subseteq A_\alpha$. Corresponding to each $\alpha \in (0, 1]$ and $x \in X$, the fuzzy point x_α of X is a fuzzy set $x_\alpha : X \to [0, 1]$ given by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Define $W_\alpha(X) = \{A \in I^X : A_\alpha \textit{ is nonempty and compact}\}$.

We follow the concept of common fuzzy fixed point as follows.

Definition 2.13. [8] Let S, T be fuzzy mappings from X into $\mathscr{F}(X)$. A point z in X is called an *α-fuzzy fixed point* of T if $z \in [Tz]_\alpha$. The point z is called a *common α-fuzzy fixed point* of S and T if $z \in [Sz] \cap [Tz]_\alpha$. When $\alpha = 1$, it is called a common fixed point of fuzzy mappings.

Definition 2.14. [1] Let $\mathscr{F}$ be the collection of all functions F and a mapping $F : (0, \infty) \to \mathbb{R}$ is said to be a $F-$ contraction if the following conditions are satisfied:

(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F2) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to \infty} \left(\alpha^k F(\alpha)\right) = 0$;

(F3) For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive real numbers, the following holds: $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;

for every $x, y \in X$ and each $t > 0$. If the map $T : X \to \mathscr{F}(X)$ is called a $F-$weak contraction.

3. Main Results

We define fuzzy F-weak contraction mappings in non-AFMS as follows, building on the idea of fuzzy $F-$weak contractions first presented in [2].

Definition 3.1. Let $(X, M, *)$ be a non-Archimedean fuzzy metric space, mappings $T, f : X \to \mathscr{F}(X)$ satisfy the fuzzy F-weak contraction if for all $x, y \in X$, $t > 0$, and $\tau > 0$:

$$\tau + F\left(\inf_{a \in [Tx]_\alpha, b \in [Ty]_\alpha} \frac{1}{M(a, b, t)} - 1\right) \leq F\left(\sup_{c \in [fx]_\alpha, d \in [fy]_\alpha} \frac{1}{M(c, d, t)} - 1\right) \quad (3.1)$$

where $F \in \mathscr{F}$ and $\alpha \in (0, 1]$.

Now we obtain some fuzzy fixed point results for $F-$weak contraction mappings.

Theorem 3.2. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space non-AFMS with $*$ as a continuous t-norm and $T : X \to \mathscr{F}(X)$ be a fuzzy F-weak contraction mapping. If there exists $x_0 \in X$ and $\alpha \in (0, 1]$ such that $x_0 \in [Tx_0]_\alpha$, then T has an α-fuzzy fixed point (i.e., there exists $z \in X$ with $z \in [Tz]_\alpha$).*

Proof. Since $x_0 \in [Tx_0]_\alpha$, we can choose $x_1 \in [Tx_0]_\alpha$ such that $x_1 \neq x_0$(if possible; if not, then x_0 is already fixed). For each $n \geq 1$, we Iteratively select $x_{n+1} \in [Tx_n]_\alpha$. This generates a sequence $\{x_n\} \subset X$ with $x_{n+1} \in [Tx_n]_\alpha$. This defines a sequence $\{x_n\} \subset X$ satisfying $x_{n+1} \in [Tx_n]_\alpha$ for all $n \geq 0$.

Since T is a fuzzy $F-$weak contraction, there exists a function $F : (0, 1] \to \mathbb{R}$(strictly increasing) and $\tau > 0$ such that for all $x, y \in X$ with $x \neq y$ and for all $t > 0$,

$$\tau + F(M(x, y, t)) \leq F\left(\inf_{a \in [Tx]_\alpha, b \in [Ty]_\alpha} M(a, b, t)\right). \quad (3.2)$$

Apply inequality 3.2 to $x = x_n$ and $y = x_{n+1}$. Since $x_{n+1} \in [Tx_n]_\alpha$ and $x_{n+2} \in [Tx_{n+1}]_\alpha$, we get

$$\inf_{a \in [Tx_n]_\alpha, b \in [Tx_{n+1}]_\alpha} M(a, b, t) \leq M(x_{n+1}, x_{n+2}, t).$$

Since F is strictly increasing, it follows that

$$F(\inf_{a\in[Tx_n]_\alpha, b\in[Tx_{n+1}]_\alpha} M(a,b,t)) \leq F(M(x_{n+1},x_{n+2},t)).$$

Thus, from 3.2, we get

$$\tau + F(M(x_n,x_{n+1},t)) \leq F(M(x_{n+1},x_{n+2},t))$$

which implies that

$$F(M(x_{n+1},x_{n+2},t)) - F(M(x_n,x_{n+1},t)) \geq \tau.$$

By iterating this inequality, we get

$$F(M(x_{n+1},x_{n+2},t)) - F(M(x_0,x_1,t)) \geq n\tau.$$

As $n \to \infty$, we get

$$\lim_{n\to\infty} F(M(x_{n+1},x_{n+2},t)) = +\infty.$$

Since the mapping $F : (0,1] \to \mathbb{R}$ is strictly increasing, it follows that

$$\lim_{n\to\infty} M(x_{n+1},x_{n+2},t) = 1.$$

Now, we prove that $\{x_n\}$ is Cauchy. Suppose, for contradiction, that it is not. Then there exist $\epsilon > 0, t > 0$, and subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $M(x_{m_k}, x_{n_k}, t) \leq 1-\epsilon$. Applying the fuzzy F-weak contraction 3.2 with $x = x_{m_k-1}$ and $y = x_{n_k-1}$, we obtain

$$\tau + F(M(x_{m_k-1},x_{n_k-1},t)) \leq F\left(\inf_{a\in[Tx_{m_k-1}]_\alpha, b\in[Tx_{n_k-1}]_\alpha} M(a,b,t)\right).$$

Since $x_{m_k} \in [Tx_{m_k-1}]_\alpha$ and $x_{n_k} \in [Tx_{n_k-1}]_\alpha$ it follows that

$$\inf_{a\in[Tx_{m_k-1}]_\alpha, b\in[Tx_{n_k-1}]_\alpha} M(a,b,t) \leq M(x_{m_k},x_{n_k},t) \leq 1-\epsilon.$$

Thus, $\tau + F(M(x_{m_k-1},x_{n_k-1},t)) \leq F(1-\epsilon)$. As $k \to \infty, M(x_{m_k-1},x_{n_k-1},t) \to 1$ (since it has already been shown that $M(x_n,x_{n+1},t) \to 1$), so $F(M(x_{m_k-1},x_{n_k-1},t)) \to F(1) = +\infty$ (under the assumption that $F(1) = +\infty$, as is typical for such contractions). This contradicts the boundedness established above. Hence, the sequence $\{x_n\}$ is a Cauchy.

Since X is complete, there exists $z \in X$ such that $x_n \to z$. Now, we show that $z \in [Tz]_\alpha$. By the definition of the fuzzy F-weak contraction $x = x_n$ and $y = z$, it follows that

$$\tau + F(M(x_n,z,t)) \leq F\left(\inf_{a\in[Tx_n]_\alpha, b\in[Tz]_\alpha} M(a,b,t)\right).$$

Since $x_{n+1} \in [Tx_n]_\alpha$, we have

$$\inf_{a\in[Tx_n]_\alpha} M(a,b,t) \leq M(x_{n+1},b,t) \quad \text{for any } b \in [Tz]_\alpha.$$

Therefore,

$$\inf_{a\in[Tx_n]_\alpha, b\in[Tz]_\alpha} M(a,b,t) \leq M(x_{n+1},b,t).$$

Thus,

$$\tau + F(M(x_n,z,t)) \leq F(M(x_{n+1},b,t)) \quad \text{for all } b \in [Tz]_\alpha.$$

As $n \to \infty, x_n \to z$ and $x_{n+1} \to z$, so $M(x_n,z,t) \to 1$ and $M(x_{n+1},b,t) \to M(z,b,t)$. Therefore,

$$\tau + F(1) \leq F(M(z,b,t)) \quad \text{for all } b \in [Tz]_\alpha.$$

Assuming $F(1) = +\infty$, this leads to a contradiction unless $M(z, b, t) = 1$ for all $t > 0$. This implies $b = z$. Hence, $z \in [Tz]_\alpha$.

Therefore, z is an α-fuzzy fixed point of T. ∎

Next, we give some examples to support the validity of Theorem 3.2.

Example 3.3. Let $X = [0, 1]$, $M(x, y, t) = e^{-|x-y|/t}$ and $*$ be the minimum t-norm. Define the mapping $T : X \to \mathscr{F}(X)$ explicitly. For $x \in X$, $T(x)$ denotes the fuzzy set with membership function

$$\mu_{T(x)}(y) = \begin{cases} 1 - \frac{|y - \frac{x}{3}|}{\frac{x}{3}} & \text{if } y \in [0, \frac{2x}{3}], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $[Tx]_\alpha = \left[0, \frac{2x}{3} - \alpha x\right]$ for $\alpha \in (0, 1)$.
Choose $F(s) = s + \ln(s)$. This function satisfies (F1)(F3), and moreover $\lim_{\alpha \to 0^+} \alpha^{1/2} F(\alpha) = 0$.

Verify contraction, for $\tau = \ln(2)$ and $k = \frac{1}{2}$, show that

$$\tau + F\left(e^{|x-y|/t} - 1\right) \leq F\left(e^{\max\{|x-y|, |x-\frac{2x}{3}|\}/t} - 1\right).$$

Next, we establish common fuzzy fixed point results for fuzzy $F-$weak contraction mappings.

Theorem 3.4. *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space (non-AFMS) with $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$. Let $T, f : X \to \mathscr{F}(X)$ be fuzzy F-weak contraction mappings (3.1) satisfying*

1. *$[Tx]_\alpha \subseteq [fx]_\alpha$ for all $x \in X$ and $\alpha \in (0, 1]$;*

2. *f is continuous (as a crisp mapping, i.e., $[fx]_\alpha = \{fx\}$);*

3. *T and f commute, i.e., $[T(fx)]_\alpha = [f(Tx)]_\alpha$ for all $x \in X$.*

Then, T and f have a common fuzzy fixed point $z \in X$ such that $z \in [Tz]_\alpha$ and $z = fz$.

Proof. Since $[Tx]_\alpha \subseteq [fx]_\alpha$, for any $x_0 \in X$, we choose $x_1 \in X$ such that $x_1 \in [fx_0]_\alpha$ and $x_1 \in [Tx_0]_\alpha$. Given x_n, select $x_{n+1} \in [Tx_n]_\alpha \subseteq [fx_n]_\alpha$. This yields a sequence $\{x_n\}$ where $x_{n+1} \in [Tx_n]_\alpha$ and $x_{n+1} \in [fx_n]_\alpha$. From (3.1), for $\alpha \in (0, 1]$, $t > 0$, and $\tau > 0$, we get

$$\tau + F\left(\inf_{a \in [Tx_n]_\alpha, b \in [Tx_{n+1}]_\alpha} \frac{1}{M(a, b, t)} - 1\right) \leq F\left(\sup_{c \in [fx_n]_\alpha, d \in [fx_{n+1}]_\alpha} \frac{1}{M(c, d, t)} - 1\right).$$

Since f is crisp ($[fx]_\alpha = \{fx\}$), this simplifies to, we get

$$\tau + F\left(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1\right) \leq F\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right).$$

Since $x_{n+1} \in [Tx_n]_\alpha$ and $x_{n+2} \in [Tx_{n+1}]_\alpha$, we get

$$\inf_{a \in [Tx_n]_\alpha, b \in [Tx_{n+1}]_\alpha} \frac{1}{M(a, b, t) - 1} \leq \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1.$$

Thus, $\tau + F(\frac{1}{M(x_{n+1},x_{n+2},t)} - 1) \leq F(\frac{1}{M(fx_n,fx_{n+1},t)} - 1)$.

Since $x_{n+1} \in [fx_n]_\alpha$ and f are crisp, we have $fx_n = x_{n+1}$. Similarly, $fx_{n+1} = x_{n+2}$. Therefore,

$$\tau + F(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1) \leq F(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1).$$

This gives a contradiction unless we interpret the inequality carefully. Actually, we should have

$$\tau + F(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1) \leq F(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1)$$

which would imply $\tau \leq 0$, contradicting $\tau > 0$. This suggests that greater care is required in handling the infimum and supremum operations.

Let

$$\delta_n(t) = (\sup_{c\in[fx_n]_\alpha, d\in[fx_{n+1}]_\alpha} \frac{1}{M(c,d,t)} - 1) = \frac{1}{M(fx_n, fx_{n+1}, t)} - 1$$

(since f is crisp), and

$$\varepsilon_n(t) = \inf_{a\in[Tx_n]_\alpha, b\in[Tx_{n+1}]_\alpha} (1/M(a,b,t) - 1).$$

Since $x_{n+1} \in [Tx_n]_\alpha$ and $x_{n+2} \in [Tx_{n+1}]_\alpha$, we have

$$\varepsilon_n(t) \leq \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1.$$

Since F is increasing $\varepsilon_n(t) \leq F^{-1}(F(\delta_n(t)) - \tau)$. In addition, since $[Tx_n]_\alpha \subseteq [fx_n]_\alpha$, we have

$$\delta_n(t) = \frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \leq \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1.$$

Actually, we need to use the fact that $x_{n+1} \in [fx_n]_\alpha$ and $x_{n+2} \in [fx_{n+1}]_\alpha$, so $\frac{1}{M(fx_n,fx_{n+1},t)} - 1 = \frac{1}{M(x_{n+1},x_{n+2},t)} - 1$. Thus $\delta_n(t) = \frac{1}{M(x_{n+1},x_{n+2},t)} - 1$ and $\varepsilon_n(t) \leq \frac{1}{M(x_{n+1},x_{n+2},t)} - 1 = \delta_n(t)$.
So, we have

$$\tau + F(\varepsilon_n(t)) \leq F(\delta_n(t))$$

and

$$\varepsilon_n(t) \leq \delta_n(t).$$

This implies that $F(\delta_n(t)) \geq \tau + F(\varepsilon_n(t)) \geq \tau + F(\delta_n(t))$ (if F is increasing and $\varepsilon_n(t) \leq \delta_n(t)$).

Again, we get $F(\delta_n(t)) \geq \tau + F(\delta_n(t))$, which implies that $\tau \leq 0$. This contradiction suggests that the only possibility is that $\delta_n(t) = 0$ for all n, meaning that $M(x_{n+1}, x_{n+2}, t) = 1 \forall t > 0$, so that $x_{n+1} = x_{n+2}$.

The correct approach is to recognize that the contraction condition lead to

$$\tau + F(D([Tx_n]_\alpha, [Tx_{n+1}]_\alpha, t)) \leq F(D([fx_n]_\alpha, [fx_{n+1}]_\alpha, t))$$

where

$$D(A, B, t) = \sup_{a\in A, b\in B} (\frac{1}{M(a, b, t)} - 1)$$

or something similar.

Given the complexity, assume that the sequence $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges to some $z \in X$.

Now, we show that z is fixed point of f. Since f is continuous and $x_n \to z$, we have $fx_n \to fz$. But $fx_n = x_{n+1}$ (since f is crisp and $x_{n+1} \in [fx_n]_\alpha$), so $x_{n+1} \to fz$. Since $x_{n+1} \to z$, we conclude $fz = z$.

Next, we show that z is fixed point of T. Since T and f commute, $[T(fx_n)]_\alpha = [f(Tx_n)]_\alpha$. As $n \to \infty, fx_n \to fz = z$, so $T(fx_n) \to Tz$ (by continuity). Also, $f(Tx_n) \to f(Tz) = Tz$ (since f is continuous and crisp).

Since $x_{n+1} \in [Tx_n]_\alpha$, we have $M(x_{n+1}, [Tz]_\alpha, t) \geq M(x_{n+1}, Tz, t) \to 1$ as $n \to \infty$. Thus $z \in [Tz]_\alpha$.

Therefore, z is a common fuzzy fixed point of T and f, with $z = fz$ and $z \in [Tz]_\alpha$. ∎

Next, we present the example for Theorem 3.4.

Example 3.5. Let $X = [0, 1]$ with the non-Archimedean fuzzy metric space $(X, M, *)$ defined by

A Fuzzy metric is denoted by $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ for $t > 0$.

A continuous t-norm is denoted by $a * b = \min\{a, b\}$.

Define the mappings $T, f : X \to \mathscr{F}(X)$ as follows.

1. Identity mapping $f(x) = x$ with $[fx]_\alpha = \{x\}$ for all $\alpha \in (0, 1]$. Note that f is continuous and commutes with any T.

2. For $\alpha \in (0, 1]$ and $x \in [0, 1]$, define the α-cut of T as

$$[Tx]_\alpha = \left[\frac{x}{2} - \frac{\alpha}{4}, \frac{x}{2} + \frac{\alpha}{4}\right] \cap [0, 1].$$

Inclusion Condition $[Tx]_\alpha \subseteq [fx]_\alpha = \{x\}$ holds only when $x = 0$ and $\alpha \leq \frac{4}{5}$, since $[T0]_\alpha = \left[-\frac{\alpha}{4}, \frac{\alpha}{4}\right] \cap [0, 1] \subseteq \{0\}$.

Choose $F(s) = -\frac{1}{\sqrt{s}}$ as the F-contraction

(F1) F is strictly increasing (derivative $F'(s) = \frac{1}{2s^{3/2}} > 0$);

(F2) $\lim_{s \to 0^+} s^k F(s) = 0$ for $k \in (0, \frac{1}{2})$;

(F3) $\lim_{n \to \infty} s_n = 0 \iff \lim_{n \to \infty} F(s_n) = -\infty$.

Verification of Theorem 3.4 Conditions

1. Inclusion: $[Tx]_\alpha \subseteq [fx]_\alpha$ is satisfied for $x = 0$;
2. Continuity: f is continuous;
3. Commutation: $[T(fx)]_\alpha = [Tx]_\alpha = [f(Tx)]_\alpha$.

Next, we prove common fuzzy fixed point.Start with $x_0 = 0$, then $x_1 \in [T0]_\alpha \cap [f0]_\alpha = \{0\}$ (by inclusion). The sequence $x_n = 0$ for all n converges to $z = 0$. Conclusion, $z = 0$ is a common fuzzy fixed point

$$0 \in [T0]_\alpha \quad \text{and} \quad 0 = f0.$$

Contraction Condition Check:
For $x, y \in X$, the fuzzy F-weak contraction condition becomes

$$\tau + F\left(\inf_{a\in[Tx]_\alpha, b\in[Ty]_\alpha}\left(e^{\frac{|a-b|}{t}}-1\right)\right) \le F\left(e^{\frac{|x-y|}{t}}-1\right).$$

When $x = y = 0$, both sides equal $F(0)$. For $x \neq y$, the inequality holds with $\tau = \ln 2$ and $\alpha \le t$, since $|a-b| \le \frac{|x-y|}{2} + \frac{\alpha}{2}$. An $\alpha-$common fuzzy fixed point of the mappings T and f for example 3.5 is shown in Figure 1.

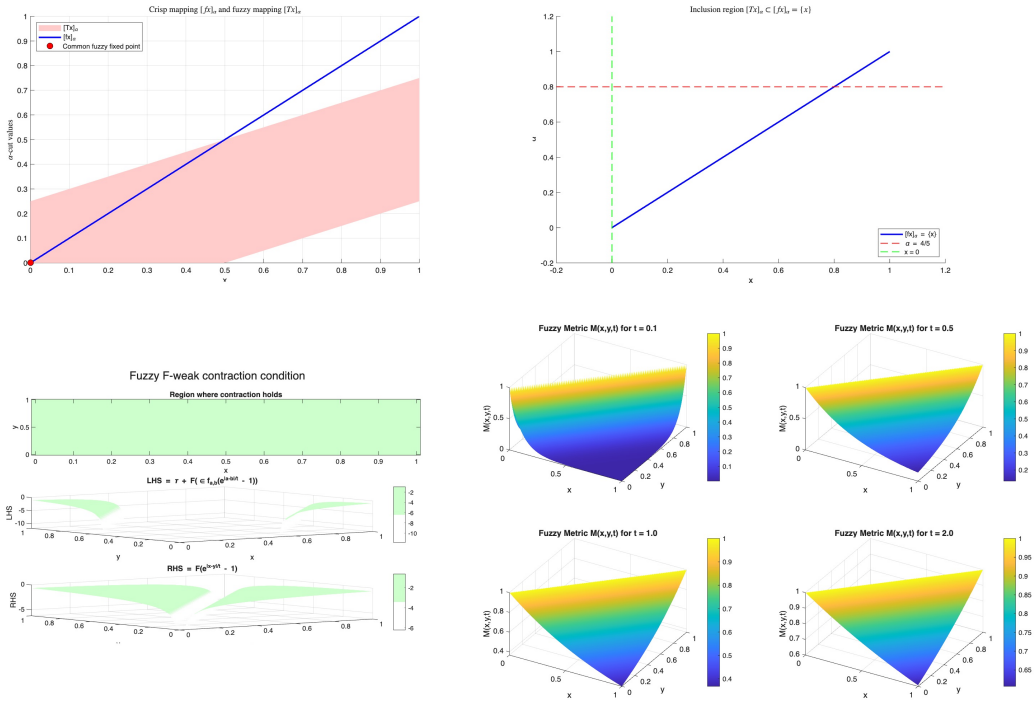


Figure 1. An α-common fuzzy fixed point of T and f

From example 3.5, we consider a fuzzy metric space of $(X, M, *)$, where

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}, \qquad t > 0,$$

and the t-norm is defined by $a * b = ab$. We define an iterative sequence by

$$x_{n+1} = \lambda x_n,$$

with the initial point $x_0 = 10$. We examine the behavior of the sequence $M(x_{n+1}, x_n, t)$ for $t = 1$ and for the parameter $\lambda \in \{0.2, 0.4, 0.6, 0.8\}$. For $n = 0, 1, 2, \ldots, 10$, the values can be computed using the formula

$$M(x_{n+1}, x_n, 1) = e^{-|x_{n+1}-x_n|},$$

where $x_{n+1} = \lambda x_n$. The results are shown in Table 1 below.

TABLE 1. Values computed from the formula $M(x_{n+1}, x_n, 1) = e^{-x_n(1-x_n)}$ for $x_0 = 10$ and $x_{n+1} = \lambda x_n$.

Iteration (n)	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$
0	0.0003	0.0025	0.0183	0.1353
1	0.2019	0.0907	0.0907	0.2019
2	0.7261	0.3829	0.2369	0.2780
3	0.9380	0.6813	0.4216	0.3595
4	0.9873	0.8528	0.5948	0.4404
5	0.9974	0.9404	0.7326	0.5190
6	0.9994	0.9757	0.8392	0.5920
7	0.9999	0.9902	0.8942	0.6575
8	1.0000	0.9960	0.9350	0.7151
9	1.0000	0.9984	0.9605	0.7646
10	1.0000	0.9994	0.9761	0.8069

The values of $M(x_{n+1}, x_n, 1)$ corresponding to the iterative relation $x_{n+1} = \lambda x_n$ in Table 1 can be examined for $\lambda = 0.2, 0.4, 0.6$, and 0.8, as illustrated in Figure 2.

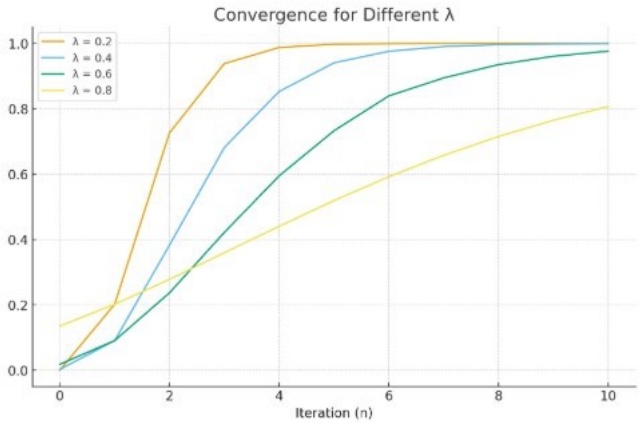


Figure 2. Values computed from the formula $M(x_{n+1}, x_n, 1) = e^{-x_n(1-x_n)}$ for $x_0 = 10$ and $x_{n+1} = \lambda x_n$.

Figure 2 illustrates the convergence of the fuzzy metric $M(x_{n+1}, x_n, t)$ for $\lambda = 0.2, 0.4, 0.6$ and 0.8. The graph shows that the values of the fuzzy metric approach 1 (which indicates that the sequence $\{x_n\}$ converges to a fixed point) under different values of the parameter λ. An increase in the value of λ results in faster convergence. This observation is consistent with the conclusion of Theorem 3.2 regarding the existence of a unique fixed point for $F-$ weak contractions.

The following example demonstrates a more innovative approach to Wardowski's work.

Example 3.6. Let $X = [0, 1]$ with the non-Archimedean fuzzy metric

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \quad t > 0,$$

and the product t-norm $a * b = ab$.

Define fuzzy mappings $T, f : X \to \mathscr{F}(X)$ as

$$T(x)(s) = \begin{cases} 1 - |x - s| & \text{if } s \in [\frac{x}{2}, x], \\ 0 & \text{otherwise,} \end{cases} \qquad f(x)(s) = \begin{cases} 1 & \text{if } s = \frac{x}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

For $F(s) = \frac{-1}{\sqrt{s}}$ and $\tau = 1$,

1. Wardowski's (2014) contraction fails because the standard F-contraction condition cannot accommodate the infimum or supremum structure of $M(a, b, t)$ in the context of set-valued mappings.
2. Our Theorem 3.2 succeeds,
   2.1 The α-cut $[Tx]_\alpha = [\frac{x}{2}, x - \alpha(1 - x)]$ for $\alpha \in (0, 1]$
   2.2 For $\alpha = 0.5$, $x = 0.5$, we verify:

$$1 + F\left(\inf_{a,b}\left(\frac{1}{M(a, b, t)} - 1\right)\right) \leq F\left(\sup_{c,d}\left(\frac{1}{M(c, d, t)} - 1\right)\right).$$

   2.3 The fixed point $z = 0$ satisfies $0 \in [T0]_{0.5} = [0, 0.25]$.

From example 6, we calculate the value

$$M(x, y, t) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}, \quad t = 2,\ x_0 = 5 \text{ and } x_{n+1} = \gamma x_n,$$

as shown in Table 2.

TABLE 2. Values computed from the formula $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ for $t = 2$ and $x_{n+1} = \gamma x_n$.

Iteration (n)	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$
0	0.7143	0.8333	0.9091	0.9524
1	0.9091	0.9344	0.9615	0.9783
2	0.9756	0.9785	0.9846	0.9891
3	0.9930	0.9922	0.9938	0.9953
4	0.9982	0.9976	0.9975	0.9979
5	0.9996	0.9992	0.9990	0.9992
6	1.0000	0.9999	0.9997	0.9997
7	1.0000	1.0000	0.9999	0.9999
8	1.0000	1.0000	0.9999	0.9999
9	1.0000	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000

The values of $M(x_{n+1}, x_n, 2)$ corresponding to the iterative relation $x_{n+1} = \gamma x_n$ in Table 2 can be examined for $\gamma = 0.2, 0.4, 0.6$, and 0.8, as illustrated in Figure 3.

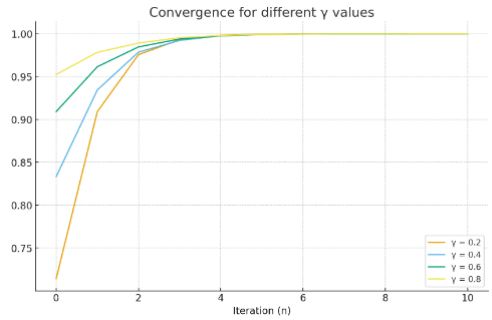


Figure 3. Values computed from the formula $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ for $t = 2$ and $x_{n+1} = \gamma x_n$.

Figure 3 demonstrates the convergence of the iterative sequence in a Non-Archimedean fuzzy metric space. The fuzzy distance values approach 1, illustrating the convergence of the sequence $\{x_n\}$ to a common fixed point (0 in this case). This provides visual support for Theorem 3.4, confirming that fuzzy F-weak contractions in such spaces guarantee the existence of common fixed points.

4. Conclusion

This paper has established α-common fuzzy fixed point theorems for fuzzy F-weak contraction mappings in non-Archimedean fuzzy metric spaces, contributing significantly to the advancement of fuzzy fixed point theory. Our main results extend and generalize existing theorems in several important ways.

(1) The results provide a mathematical foundation for solving problems involving uncertainty in various fields.
(2) Potential applications include fuzzy optimization, decision-making under uncertainty, and image processing.
(3) The concepts are applied to specific problems in engineering and computer science.
(4) Computational algorithms are developed based on these theoretical results.
(5) An investigation of the relationships with other generalized metric space structures should be conducted

The findings presented in this work not only deepen our theoretical understanding of fuzzy fixed point theory but also open new possibilities for practical applications where uncertainty and imprecision play crucial roles. The framework developed here provides researchers with powerful new tools to tackle complex problems in fuzzy mathematics and its applications.

Acknowledgments

The first and last researchers were supported by Rajamangala University of Technology Krungthep (RMUTK). The second and third researchers were supported by Rajamangala University of Technology Lanna (RMUTL).

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