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Induced Bilal Distribution: Statistical Properties with Applications to Model Precipitation and Vinyl Chloride Data

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Abstract This paper proposes a new one parameter life distribution called induced Bilal distribution for modeling some real data. The main mathematical characteristics of this distribution are derived and discussed including the cumulative distribution and probability density functions, the moments, coefficient of variation, skewness, kurtosis, mean and median deviations, Gini index, Lorenz curve, Bonferroni curve, Rényi entropy, stochastic ordering, and distribution of order statistics. Also, reliability analysis based on the odds function, survival function, reversed hazard rate function, hazard function, and the stress-strength reliability function are provided. Estimation of the model parameter is performed using the maximum likelihood method and a simulation study is supported to illustrate the estimator behaviors. Using two real data sets related to the precipitation in Minneapolis and vinyl chloride, it is shown that the new distribution outperforms some well-known important existing competitors considered in this study.

MSC: 62E05, 62E15

Keywords: Bilal distribution; Weighted distribution; Reliability analysis; Rényi entropy; Induced distribution; Distribution theory

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1. Introduction

Numerous major distributions have been established in the past century to be used as models in scientific fields. To increase the suitability of the prevalent distributions, interest in developing new classes of univariate continuous distributions has emerged recently. An existing probability model can be extended in a number of ways, producing a large variety of probability distributions. Nearly all applied fields, including biomedical research, engineering, finance, demography, environmental, and agricultural sciences, have a need for statistical analysis and data modeling. In the statistical literature, a wide range of continuous distributions have been presented for modeling lifespan data, including the exponential, Lindley, Beta, gamma, Weibull, Poisson, Gompertz, lognormal, Binomial, normal distributions and among others. There are numerous approaches for generalizing new probability distributions, comprising: mixture and transformation of existing distributions, transmutation method, non-parametric methods, power method, and exponential method. For example, Al-Omari and Dobbah [3] suggested a mixture of Shanker and gamma distributions. Imran et al. [14] proposed an extended exponential distribution. Irshad et al. [15] introduced an extended Farlie-Gumbel-Morgenstern bivariate Lindley distribution.

One approach to creating new distributions is the idea of weighted distributions. The weights are used to alter the conventional probability calculations in weighted probability distributions. For instance, rather of just counting the number of data points that make up an event, the probability of that event is determined as the sum of all the weights of the data points that make up that event. This gives modeling of complex data sets more freedom since different data points may have different effects on the overall distribution. Fisher [11] was the first to propose the concept of weighted distributions, and Patil and Rao developed it in [17]. Let X be a random variable that have the probability density function (pdf) $f(x)$ and let us assume the probability of observing $X = x$ is proportional to a weight function $w(x) \geq 0$. As a result, according to the concept of weighted probability, the density function of the observed X is given by

$$f_w(x) = \frac{w(x)f(x)}{\int_{-\infty}^{\infty} w(x)f(x)dx}.$$

The pdf provided by this equation is known as the *weighted density*, and the function $w(x) \geq 0$ is known as the *weight function*. Different circumstances lead to different weight functions, some of which have been proposed by various researchers. Mohiuddin et al. [16] suggested weighted Nwikpe distribution. Hussain and Mohammed [10] proposed weighted Zeghdoudi distribution. Ganaie and Rajagopalan [12] suggested length biased weighted new Quasi Lindley distribution. Alsmairan and Al-Omari [5] suggested weighted Suja distribution. Arnold and Nagaraja [6] investigated some properties of bivariate weighted distributions. A review of weighted distributions is given by Saghir et al. [19]. Benchiha et al. [8] proposed weighted generalized Quasi Lindley distribution. Al-Omari et al. [4] proposed length and area Bilal distributions as modifications to the Bilal distribution.

One of the modified weighted distributions forms is considered by Singh and Das [24] called Induced distribution with pdf given by

$$f_I(x) = \frac{1 - F(x)}{\mathbb{E}(X)};\ x > 0, \tag{1.1}$$

where $\mathbb{E}(X)$ is the expectation and $F(x)$ is the cumulative distribution function (cdf) of the base random variable X. Singh and Das [24] used the induced distribution idea and suggested induced Garima distribution as a modification of the Garima distribution.

Bilal distribution is a one parameter lifetime distribution offered by Abd-Elrahman [2] with a pdf given by

$$f_B(x;\theta) = \frac{6}{\theta}\left(1 - e^{-x/\theta}\right)e^{-2x/\theta}; x \geq 0, \theta > 0,$$

and cumulative distribution function (cdf) defined as

$$F_B(x;\theta) = 1 - \left(3 - 2e^{-x/\theta}\right)e^{-2x/\theta};\ x \geq 0.$$

The mean, variance, coefficient of variation, and coefficient of kurtosis of the BD are

$$\mathbb{E}_B(X) = \frac{5}{6}\theta, \mathrm{Var}_B(X) = \frac{13}{36}\theta^2, \delta_1 \approx 0.7211,\ \text{and}\ \delta_2 = \frac{1089}{169} = \approx 6.44,$$

respectively.

We also note that Bilal distribution is a particular case of the Partly-Exponential distribution introduced in Atikankul et al. [7] and investigated more in detail in Roopmok et al. [18]. A random variable X is said to have the three-parameter *Partly-Exponential* distribution if its density function is

$$f_{PE}(x) = \frac{\delta + \tau}{\tau[\delta(1-\gamma) + \tau]}\left(1 - \gamma e^{-\frac{x}{\delta}}\right)e^{-\frac{x}{\tau}}; x \geq 0,$$

where $0 \leq \gamma \leq 1, \delta > 0,$ and $0 < \tau < \delta(1+\gamma)$ are the parameters. As we can see, Bilal distribution is the Partly-Exponential distribution with $\gamma = 1, \delta = \theta,$ and $\tau = \theta/2$.

The study of the Induced Bilal (IB) distribution (the weighted form of the Bilal distribution) is the focus of this article. Not looking that it has only one parameter, it is quite flexible.

In the introduction, weighted probability distributions are defined and their significance is emphasized. The rest of the article is organized as follows. Next section provides the definition of the Induced Bilal distribution with their shapes for some selected values of the distribution parameter. Also, some probabilistic properties such are moments, moment generating function, mode of the IB distribution are presented. Section 3 explores the stochastic ordering, mean deviations and some inequality measures such as Bonferroni and Lorenz curves and Gini index. The model stress-strength reliability and Rényi entropy are demonstrated in Section 4. The fifth section discusses the reliability analysis. The distribution of order statistics is given in Section 6 and parameter estimation procedure using the methods of moments and maximum likelihood is presented in Section 7. An application to real data is investigated in Section 8. A review of the main ideas and a discussion of the prospects for IB distribution in the future are included in Section 9.

2. Induced Bilal distribution and its probabilistic properties

Using the concept of an induced distribution given in Equation (1.1), we reconnoiter Bilal distribution to introduce the pdf of the Induced Bilal distribution, denoted as $IB(\theta)$, as

$$f_{IB}(x;\theta) = \frac{6}{5\theta}\left(3e^{x/\theta} - 2\right)e^{-3x/\theta}; x \geq 0, \theta > 0.$$

Rewriting the density function of the IB distribution as $f_{IB}(x;\theta) = \frac{18}{5\theta}\left(1 - \frac{2}{3}e^{-x/\theta}\right)e^{-2x/\theta}$, we see that the Induced Bilal distribution is a particular case of the Partly-Exponential distribution with $\gamma = 2/3, \delta = \theta/2$, and $\tau = \theta$.

The pdf plots of the IB distribution are provided in Figure 1 for various values of the distribution parameter. It is clear that the IB distribution's pdf is skewed to the right, and θ is the shape parameter.

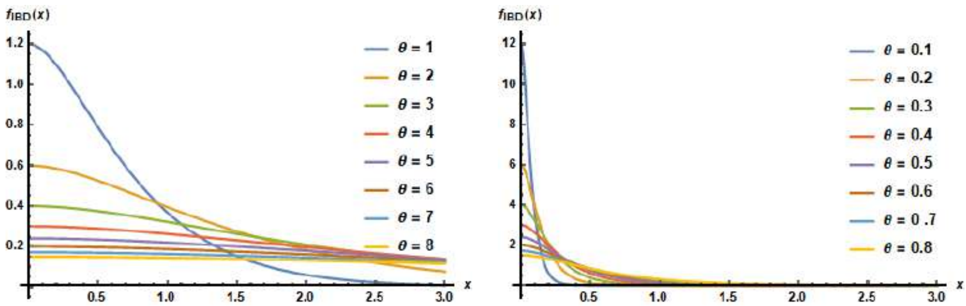

FIGURE 1. Plots of the IB distribution pdf for selected values of the distribution parameter.

The following statements follow directly from Properties 1 - 5 of Roopmok et al. [18].

(1) The cdf of the IB distribution is given by

$$F_{IB}(x;\theta) = 1 + \frac{4}{5}e^{-3x/\theta} - \frac{9}{5}e^{-2x/\theta}; x > 0, \theta > 0.$$

The shapes of the IB distribution cdf plots for $\theta = 1, 2, 3, 4, 5$ are presented in Figure 2.

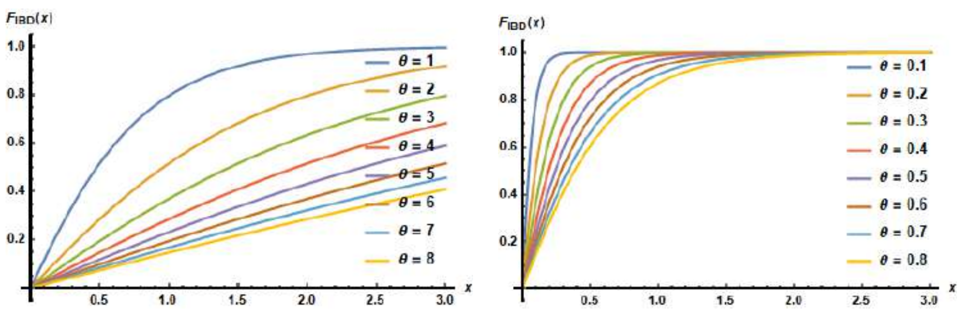

FIGURE 2. Plots of the cdf of the IB distribution for some parameter values.

(2) The moment generating function of the IB distribution is given as

$$M_{IB}(t) = \frac{6(\theta t - 5)}{5(\theta^2 t^2 - 5\theta t + 6)}; t < \frac{2}{\theta}.$$

(3) The rth moments about the origin of the IB distribution are given by

$$\mathbb{E}(X_{IB}^r) = \frac{1}{5} 6^{-r} \left(3^{r+2} - 2^{r+2}\right) \Gamma(r+1)\theta^r; \; r > -1, \theta > 0.$$

In particular, the first four IB distribution moments are given by

$$\mathbb{E}(X_{IB}) = \frac{19}{30}\theta, \qquad \mathbb{E}(X_{IB}^2) = \frac{13}{18}\theta^2,$$

$$\mathbb{E}(X_{IB}^3) = \frac{211}{180}\theta^3 \qquad \mathbb{E}(X_{IB}^4) = \frac{133}{54}\theta^4.$$

The shapes of the rth moment plots of the as a function of r for $\theta = 0.1, 0.2, \ldots, 0.8$ are presented in Figure 3. It can be noted that it is decreasing for $\theta < 0.5$ and decreasing-increasing for the other values.

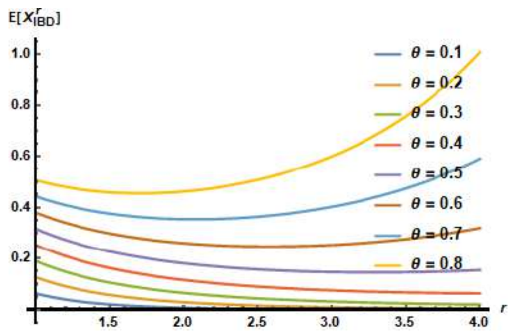

FIGURE 3. Plots of the $\mathbb{E}(X_{IB}^r)$ as a function of r.

(4) The characteristic function of the IB distribution is given as

$$\phi_{IB}(t) = \frac{6(\theta i t - 5)}{5(-\theta^2 t^2 - 5\theta i t + 6)}, t \in \mathbb{R}.$$

(5) The mode of the IB distribution is $\frac{\ln 2}{2}\theta$.

We note that the three-parameter Partly-Exponential distribution, see Atikankul et al. [7] and Roopmok et al., [18] is very general and many results cannot be derived in the closed form. Contrary to this, we can derive and establish them for the IB distribution. All results presented below are not covered in Atikankul et al. [7] and Roopmok et al. [18].

(6) The central moments of IB distribution are not presented in Roopmok et al. [18]. To derive them, we present the IB distribution density function as:

$$f_{IB}(x;\theta) = \frac{9}{5} f_{\text{Exp}}(x;\theta/2) - \frac{4}{5} f_{\text{Exp}}(x;\theta/3), x \geq 0,$$

where $f_{\text{Exp}}(x;\mu) = \frac{1}{\mu}e^{x/\mu}$ is the density function of the Exponential distribution with mean parameter $\mu > 0$.

It well known that the $n \in \mathbb{N}$ central moment of the Exponential distribution with mean parameter μ is

$$\mu_n = n!\mu^n \sum_{k=1}^{n} \frac{(-1)^k}{k!}.$$

We obtain that the nth IB distribution central moment is

$$\begin{aligned}\mathbb{E}[X_{IB} - \mathbb{E}(X_{IB})]^n &= \frac{9}{5}n!(\theta/3)^n \sum_{k=1}^{n} \frac{(-1)^k}{k!} - \frac{4}{5}n!(\theta/2)^n \sum_{k=1}^{n} \frac{(-1)^k}{k!} \\ &= \frac{n!}{5}\left(\sum_{k=1}^{n} \frac{(-1)^k}{k!}\right)\left(\frac{9}{2^n} - \frac{4}{3^n}\right)\theta^n.\end{aligned}$$

In particular, the variance and standard deviation of the IB distribution are

$$\sigma_{IB}^2 = \frac{289}{900}\theta^2, \sigma_{IB} = \frac{17}{30}\theta.$$

The coefficient of skewness β_1, coefficient of variation β_2, and coefficient of kurtosis β_3, for the IB random variable are parameter free and given by:

$$\beta_1 = \frac{8318}{4913} \approx 1.693, \beta_2 = \frac{17}{19} \approx 0.895, \beta_3 = \frac{606537}{83521} \approx 7.262.$$

(7) The quantile function of the IB distribution cannot be derived in closed form, but the following hints similar to the derivation of the median above can be used. To find the quantile function $F^{-1}(p)$ at a point $p, 0 < p < 1$, we need to solve the equation $F_{IB}(x) = p$ for x. This is equivalent to solving the equation $4e^{-3x/\theta} - 9e^{-2x/\theta} + 5(1-p) = 0$. Denoting $t = e^{-x/\theta}$, we obtain the cubic equation $4t^3 - 9t^2 + 5(1-p) = 0$. From three possible solutions of this equation we need to choose solution t_p that satisfies $0 < t_p < 1$. Then, $x_p = F^{-1}(p) = -\theta \ln t_p$.

In particular, the *median* of the IB distribution is $m = -\theta \ln a \approx 0.4795587\theta$, where $a = \frac{3}{4} - \frac{z}{8} - \frac{9}{8z} + \frac{i\sqrt{3}}{2}\left(\frac{z}{4} - \frac{9}{4z}\right) \approx 0.619039, z = (7 + 2i\sqrt{170})^{1/3}$, and $i = \sqrt{-1}$ is the imaginary unit. To show this, we solved the cubic equation $4t^3 - 9t^2 + 2.5 = 0$ with the software Maple. It has three real roots, but only one of them satisfies the constraint $0 < a < 1$.

3. Stochastic Ordering, Mean Deviations and Inequality Measures

It is possible to compare two positive continuous distributions using the stochastic ordering. For a life-time continuous random variable T, denote

(1) *the mean residual life* of T at x as $m_T(x) = \mathbb{E}[X - x|X > x]$, for $t \in \mathbb{R}$;
(2) *the density function* of T at x as $f_T(x)$;
(3) *the hazard function* of T at x as $h_T(x)$;
(4) *distribution function* of T at x as $F_T(x)$.

When compared to a random variable Y, a random variable X is smaller in

(1) *Mean residual life order* $X \leq_{MRLO} Y$, if $m_X(x) \leq m_Y(x)$ for all x;
(2) *Likelihood ratio order* $X \leq_{LRO} Y$, if $\frac{f_X(x)}{f_Y(x)}$ decreases for all x;
(3) *Hazard rate order* $X \leq_{HRO} Y$, if $h_X(x) \geq h_Y(x)$ for all x;
(4) *Stochastic order* $X \leq_{SO} Y$, if $F_X(x) \geq F_Y(x)$ for all x.

It is shown in Shaked and Shanthikumar [21] that the Likelihood ratio order is the most restrictive, namely if $X \leq_{LRO} Y$, then $X \leq_{HRO} Y$; if $X \leq_{HRO} Y$, then $X \leq_{MRLO} Y$, and if $X \leq_{HRO} Y$, then $X \leq_{SO} Y$.

Theorem 3.1. *Let $X \sim IB(\theta)$ and $Y \sim IB(\beta)$. If $\theta < \beta$, then $X \leq_{LRO} Y$, and hence the other stochastic ordering properties are also satisfied.*

Proof: The ratio of density functions is:

$$h(x) = \frac{f_{IB}(x;\theta)}{f_{IB}(x;\beta)} = \frac{\beta}{\theta}\frac{\left(3e^{x/\theta}-2\right)e^{3x/\beta}}{\left(3e^{x/\beta}-2\right)e^{3x/\theta}}.$$

We can drop the coefficient $\frac{\beta}{\theta}$ because it does not influence monotonicity of function $h(x)$. Introduce the following monotone change of variables $t = e^{x/\beta}$, then $e^{x/\theta} = t^{\beta/\theta} = t^{\tau}$, where $\tau = \beta/\theta > 1$. Because t is an increasing function of x, it is sufficient to prove that the function

$$\widetilde{h}(t) = \frac{(3t^{\tau}-2)\,t^3}{(3t-2)t^{3\tau}} = \frac{3t^{\tau+3}-2t^3}{3t^{3\tau+1}-2t^{3\tau}}, t>1, \tau>1,$$

is decreasing. For this, we need to show that its derivative is negative for $t > 1$.

$$\widetilde{h}'(t) = \frac{[3(\tau+3)t^{\tau+2}-6t^2](3t^{3\tau+1}-2t^{3\tau}) - (3t^{\tau+3}-2t^3)[3(3\tau+1)t^{3\tau}-6\tau t^{3\tau-1}]}{(3t^{3\tau+1}-2t^{3\tau})^2}.$$

For this, it is sufficient to show that the numerator is negative. After some algebra, we can derive that the numerator is negative when the function

$$g(t) = -3(\tau-1)t^{\tau+1} + (2\tau-3)t^{\tau} + (3\tau-4)t - 4(\tau-1) < 0, t > 1.$$

We note that $g(1) = -2\tau < 0$, hence, to show that $g(t)$ is negative, it is sufficient to show that $g(t)$ is decreasing for $t > 1$; that is, the derivative of function $g(t)$ is negative.

$$g'(t) = -3(\tau-1)(\tau+1)t^{\tau} + \tau(2\tau-3)t^{\tau-1} + 3\tau - 4.$$

Note that $g'(1) = -\tau^2 - 3\tau - 1 < 0$ for $\tau > 1$. Hence again, to show that $g'(t)$ is negative, it is sufficient to show that $g'(t)$ is decreasing for $t > 1$; that is, the second derivative of function $g(t)$ is negative for $t > 1$.

$$g''(t) = -3(\tau-1)(\tau+1)\tau t^{\tau-1} + (2\tau-3)t^{\tau-2}.$$

The statement that this function is negative, is equivalent to $t > \frac{2\tau-3}{3(\tau+1)}$. The last statement is true because for $\tau > 1$, we have that $t > 1 > \frac{2}{3} > \frac{2\tau-3}{3(\tau+1)}$. □

A population's spread may be determined using the notion of mean deviations. The mean deviation about the mean $D(\mu)$, where $\mu = \mathbb{E}(X) = \frac{19}{30}\theta$ is given by the following:

$$D(\mu) = \int_0^{\infty} |x-\mu| f(x)dx = 2\mu F(\mu) - 2\int_0^{\mu} x f(x)dx.$$

Note that

$$2\mu F(\mu) = \frac{19}{15}\left(1 + \frac{4}{5}e^{-19/10} - \frac{9}{5}e^{-19/15}\right)\theta \approx 0.7758\theta,$$

$$\begin{aligned}2\int_0^{\mu} xf(x)dx &= 2\int_0^{19\theta/30} x\frac{6}{5\theta}\left(3e^{x/\theta} - 2\right)e^{-3x/\theta}dx \\ &= \frac{12}{5}\left(\frac{19}{36} - \frac{17}{10}e^{-19/15} + \frac{29}{45}e^{-19/10}\right)\theta \approx 0.3483\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}D(\mu) &= \frac{19}{15}\theta\left(1 + \frac{4}{5}e^{-19/10} - \frac{9}{5}e^{-19/15}\right) - \frac{12\theta}{5}\left(\frac{19}{36} - \frac{17}{10}e^{-19/15} + \frac{29}{45}e^{-19/10}\right) \\ &= \theta\left(\frac{9}{5}e^{-19/15} - \frac{8}{15}e^{-19/10}\right) \approx 0.423339\theta.\end{aligned}$$

The mean deviation about the median $D(m)$, where $m = \text{median}(X) = -\theta \ln a$ (see Property 7 in Section 2) is given by the following:

$$\begin{aligned}D(m) &= \int_0^{\infty} |x - m| f(x)dx = \mu - 2\int_0^{m} xf(x)dx \\ &= \frac{\theta}{30}[36a^2(1 - 2\ln a) - 16a^3(1 - 3\ln a) - 1] \approx 0.559048\theta.\end{aligned}$$

The Bonferroni and the Lorenz curves are not only used in economics in order to study the income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. Also, Gini index is another method for measuring inequality in the distribution. They are applicable in the field of economics and also in other areas such as reliability, medicine, and demography.

- The *Bonferroni curve* is defined as $B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx, p \in [0, 1], q = F^{-1}(p)$. According to Property 7 Section 2, the quantile function $F^{-1}(p)$ does not have closed form. Because of that, we derive the Lorenz curve for the IB distribution in terms of q:

$$B_{IB}(p) = \frac{1}{19p}[19 - 27e^{-2q/\theta}(1 - 2q/\theta) + 8e^{-3q/\theta}(1 - 3q/\theta)].$$

- The *Lorenz curve* is defined as $L(p) = pB(p) = \frac{1}{\mu}\int_0^q xf(x)dx$. The Lorenz curve for the IB distribution is:

$$L_{IB}(p) = \frac{1}{19}[19 - 27e^{-2q/\theta}(1 - 2q/\theta) + 8e^{-3q/\theta}(1 - 3q/\theta)].$$

- The *Gini index* G for the IB distribution is given by $G = 1 - \frac{1}{\mu}\int_0^{\infty}(1 - F(x))^2dx$. The Gini index for IB distribution is parameter free and equals to:

$$G_{IB} = 1 - \frac{30}{19\theta}\int_0^{\infty}\left(\frac{9}{5}e^{-2x/\theta} - \frac{4}{5}e^{-3x/\theta}\right)^2 dx = \frac{439}{950} \approx 0.462105.$$

4. Stress-Strength Reliability and Rényi Entropy

Consider two independent random variables X and Y. The *stress-strength reliability* that explains the life of a component that has a random strength Y that is subjected to a random stress X is defined as

$$R = \mathbb{P}(Y < X) = \int_0^\infty \mathbb{P}(Y < X|X = x) f_X(x)dx = \int_0^\infty f_X(x)F_Y(x)dx.$$

Assuming that $X \sim IB(\alpha)$ and $Y \sim IB(\theta)$ are two independent random variables that follow the IB distribution with different parameters $\alpha > 0$ and $\theta > 0$, the stress-strength reliability is given by

$$R_{IB} = \frac{\alpha}{25}\left(\frac{97}{\alpha+\theta} - \frac{108}{3\alpha+2\theta} - \frac{72}{2\alpha+3\theta}\right).$$

Figure 4 represents three dimensions plots of the stress-strength reliability of the IB distribution for some selected parameters values.

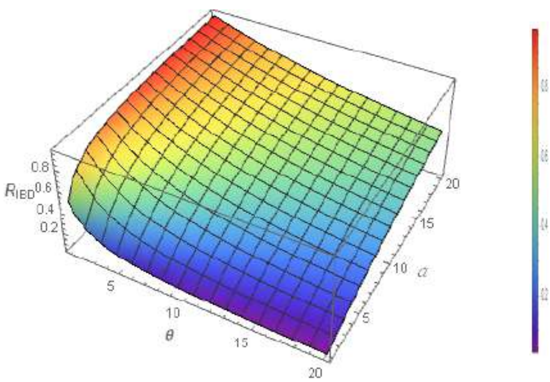

Figure 4. The stress-strength reliability of the IB distribution for some parameters values.

The Rényi entropy that is used as a measure of randomness of a system is defined as

$$RE(\beta) = \frac{1}{1-\beta}\log\left(\int_0^\infty f(x)^\beta dx\right) \beta > 0, \beta \neq 1.$$

To find the Rényi entropy for the IB distribution, we apply the famous Newton's generalized binomial theorem

$$(x+y)^\beta = \sum_{j=0}^{\infty}\binom{\beta}{j}x^{\beta-j}y^j, \text{ where } \binom{\beta}{j} = \frac{\beta(\beta-1)\dots(\beta-j+1)}{j!}, \text{ and } |x| > |y|.$$

Consider $x = 1$ and $y = -\frac{2}{3}e^{-5x/\theta}$, by the Newton's generalized binomial theorem and assuming that we can interchange the summation and integration signs (we did not investigate this problem), we have

$$\begin{aligned}
RE_{IB}(\beta) &= \frac{1}{1-\beta}\log\left[\left(\frac{18}{5\theta}\right)^{\beta}\int_0^{\infty}\left(e^{-2x/\theta}-\frac{2}{3}e^{-3x/\theta}\right)^{\beta}dx\right] \\
&= \frac{1}{1-\beta}\log\left[\left(\frac{18}{5\theta}\right)^{\beta}\int_0^{\infty}e^{-2\beta x/\theta}\left(1-\frac{2}{3}e^{-x/\theta}\right)^{\beta}dx\right] \\
&= \frac{1}{1-\beta}\log\left[\left(\frac{18}{5\theta}\right)^{\beta}\int_0^{\infty}e^{-2\beta x/\theta}\sum_{j=0}^{\infty}\binom{\beta}{j}(-1)^j\left(\frac{2}{3}\right)^j e^{-jx/\theta}dx\right] \\
&= \frac{1}{1-\beta}\log\left[\left(\frac{18}{5\theta}\right)^{\beta}\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}\left(\frac{2}{3}\right)^j\int_0^{\infty}e^{-(2\beta+j)x/\theta}dx\right] \\
&= \frac{1}{1-\beta}\log\left[\left(\frac{18}{5\theta}\right)^{\beta}\theta\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}\left(\frac{2}{3}\right)^j\frac{1}{2\beta+j}\right].
\end{aligned}$$

Some Rényi entropy values for a few chosen IB distribution parameter values are shown in Table 1. Table 1 revealers that the Rényi entropy values are all positive and depend on the parameters values. They are increasing as θ increases for fixed values of β and decreasing as β increases for fixed values of θ.

Table 1. Rényi entropy values for some selected IB distribution parameters values.

θ	$\beta=2$	$\beta=4$	$\beta=6$	β	$\theta=2$	$\theta=4$	$\theta=6$
1	0.2957	0.1254	0.0519	2	0.9889	1.6820	2.0875
1.1	0.3910	0.2207	0.1472	3	0.8818	1.5749	1.9804
1.2	0.4780	0.3077	0.2342	4	0.8185	1.5117	1.9172
1.3	0.5581	0.3878	0.3143	5	0.7760	1.4691	1.8746
1.4	0.6322	0.4619	0.3884	6	0.7450	1.4382	1.8437
1.5	0.7012	0.5309	0.4574	7	0.7214	1.4145	1.8200
1.6	0.7657	0.5954	0.5219	8	0.7026	1.3957	1.8012
1.7	0.8263	0.6560	0.5825	9	0.6872	1.3804	1.7859
1.8	0.8835	0.7132	0.6397	10	0.6745	1.3676	1.7731
1.9	0.9376	0.7672	0.6937	11	0.6636	1.3568	1.7622
2	0.9889	0.8185	0.7450	12	0.6543	1.3474	1.7529

5. Reliability Analysis

The study of the dependability of systems and components falls under the purview of the engineering discipline known as reliability analysis. It seeks the chances of a system or component to perform correctly and consistently throughout a given time frame. Typically, this is accomplished by simulating and examining the various failure modes and mechanisms, as well as by calculating the likelihood of failure and the time until failure. The ultimate purpose of reliability analysis is to create highly reliable systems and components, which have a low failure probability and a lengthy projected service life. Aerospace, automotive, electrical, and defense sectors among many others frequently use reliability analysis. In this section, some of reliability functions are discussed. The mean inactivity time (MIT) function, a widely used dependability metric, is also known as the mean past lifetime and the mean waiting time functions. It is utilized in numerous domains, such as reliability theory, survival analysis, and actuarial investigations. The MIT is defined as

$$\mathrm{MIT}(t) = t - \mathbb{E}[X|x \leq t] = t - \frac{\int_0^t xf(x)dx}{F(t)}, t > 0,$$

and $\mathrm{MIT}(t) = 0$ if $t \leq 0$.

For the IB distribution it is given by

$$\mathrm{MIT}_{IB}(t) = t - \frac{19\theta - 27e^{-2t/\theta}(\theta + 2t) + 8e^{-3t/\theta}(\theta + 3t)}{6(24e^{-3t/\theta} - 54e^{-2t/\theta} + 5)}.$$

In Figure 5, the MIT plots are explored for some values of the distribution parameter, which shows that the MIT is increasing function of t. Also, the MIT values are large with small values of the distribution parameter.

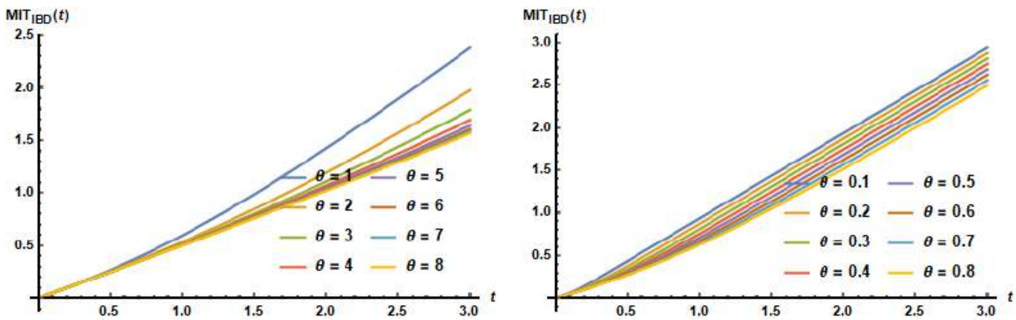

Figure 5. MIT of the IB distribution plots for some selected values of the parameter.

The probability of an item not failing before a time x is referred to as the *survival function*. It is defined as $S(x) = 1 - F(x)$, and for the IB distribution it is

$$S_{IB}(x;\theta) = \frac{9}{5}e^{-2x/\theta} - \frac{4}{5}e^{-3x/\theta}.$$

Figure 6 represents the survival function plots of the IB distribution for some selected values of the distribution parameter. It turns out that the curves are decreasing, inverse J-shaped and straight line upon the parameter value.

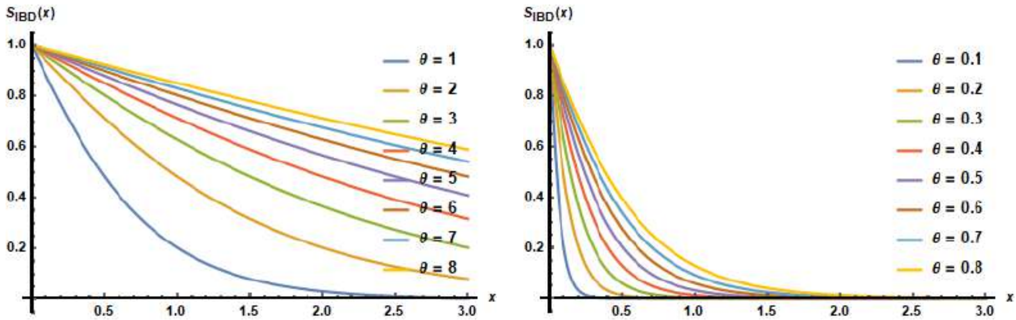

FIGURE 6. The IB distribution survival function plots for some parameter values.

The survival functions curves are decreasing skewed to the right and the distance between two curves depends on the parameter value.

The *hazard function* is critical in describing any lifetime distribution. The hazard function is given by $H(x) = \frac{f(x)}{1-F(x)}$. For IB distribution is given by

$$H_{IB}(x;\theta) = \frac{2}{\theta}\left(\frac{2}{4-9e^{x/\theta}}+1\right).$$

In Figure 7, the hazard rate function of the IB distribution plots are shown for some selection of the distribution parameter values. The hazard rate curves are increasing with large values for small values of the parameter.

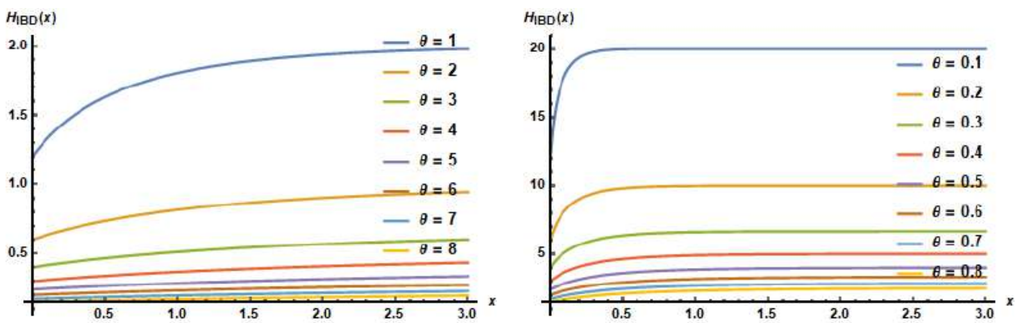

FIGURE 7. The hazard rate function of the IB distribution plots for some parameter values.

The *reversed hazard rate function* is defined as $RH(x) = \frac{f(x)}{F(x)}$ and for the IB distribution it is defined as

$$RH_{IB}(x;\theta) = \frac{6\left(3e^{x/\theta} - 2\right)e^{-3x/\theta}}{\theta\left[5 + \left(4 - 9e^{x/\theta}\right)e^{-3x/\theta}\right]}.$$

The shapes of the reversed hazard rate function of the IB distribution for some parameter values are given in Figure 8. It is revealed that the reversed hazard rate is decreasing and skewed to the right.

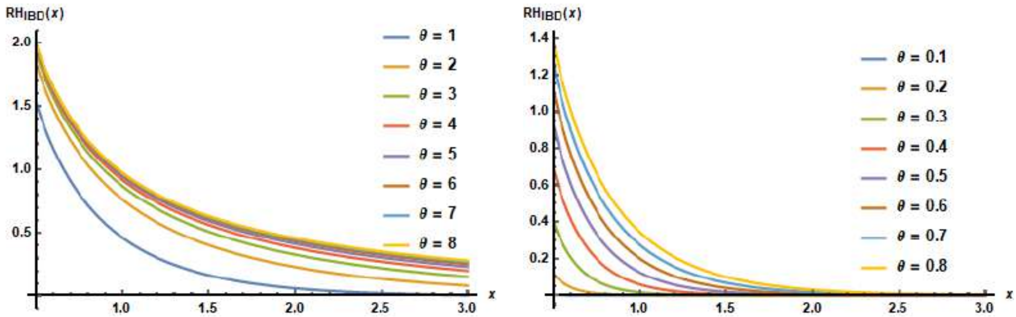

FIGURE 8. The reversed hazard function of the IB distribution plots for some parameter values.

The *odds function* of the IB distribution random variable X is defined as

$$O_{IB}(x;\theta) = \frac{F(x)}{1 - F(x)} = \frac{4 - 9e^{x/\theta} + 5e^{3x/\theta}}{9e^{x/\theta} - 4},$$

Plots of the odds functions of the IB distribution for $\theta = 1, 2, 3, 4, 5$ are presented in Figure 9. It is revealed that the odds function is increasing and its value is larger for smaller values of θ.

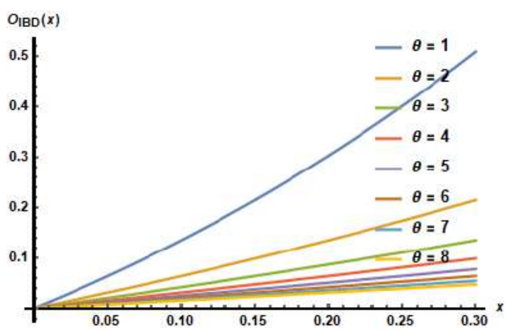

FIGURE 9. The IB distribution odds function for some parameter values.

The *cumulative hazard function* of the IB distribution is defined by

$$CH_{IB}(x;\theta) = -\log(1-F(x)) = -\log\left(\frac{1}{5}e^{-3x/\theta}\left(9e^{x/\theta}-4\right)\right).$$

The plots of the cumulative hazard function of the IB distribution for some values of the distribution parameter are presented in Figure 10. It is revealed that the cumulative hazard function is increasing and its value is larger for smaller values of θ.

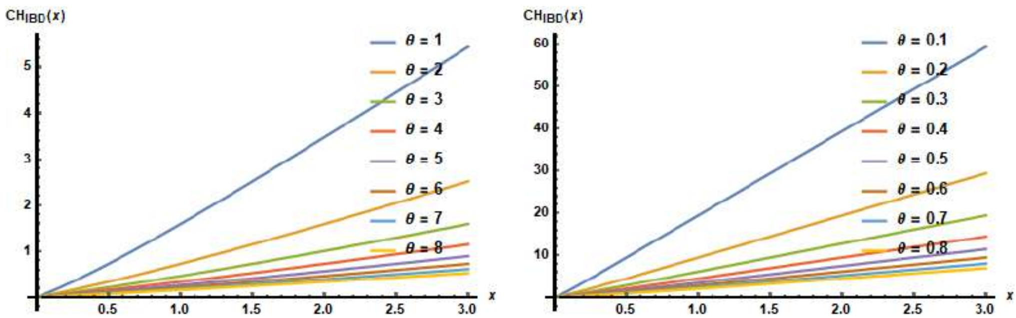

FIGURE 10. The cumulative hazard function of the IB distribution for some parameter values.

6. Order Statistics

Numerous statistical applications, such as descriptive statistics, hypothesis testing, confidence intervals, and nonparametric techniques, all use order statistics. The distribution of a sample can be summarized and described using order statistics, which can also be used to identify outliers, gauge dispersion, and define the form of the distribution. Additionally, order statistics are crucial in a variety of statistical models, including survival analysis and linear regression, as well as in applications like quality assurance and experiment design.

Consider a random sample $X_1, X_2, \ldots, X_m$ selected from a pdf $f(x)$ and cdf $F(x)$. Let $X_{(1:m)}, X_{(2:m)}, \ldots, X_{(m:m)}$ be the corresponding order statistics. It is well known that the pdf of the jth order statistic is

$$f_{(j:m)}(x) = \binom{m}{j}[F(x)]^{j-1}[1-F(x)]^{m-j}f(x); j = 1, 2, \ldots, m.$$

Substituting $f(x)$ and $F(x)$ of the IB distribution, we get

$$f_{(j:m)}^{IB}(x;\theta) = \binom{m}{j}\frac{6e^{-3(m-j+1)x/\theta}\left(3e^{x/\theta}-2\right)\left(9e^{x/\theta}-4\right)^{m-j}}{5^{m-j+1}\theta} \times \left[\frac{1}{5}e^{-3x/\theta}\left(4-9e^{x/\theta}\right)+1\right]^{j-1}.$$

Hence, the pdfs of the minimum and maximum order statistics $X_{(1:m)}$ and $X_{(m:m)}$ are:

$$f_{(1:m)}(x;\theta) = \frac{6}{5^m\theta}e^{-3x/\theta}\left(3e^{x/\theta}-2\right)\left[e^{-3x/\theta}\left(9e^{x/\theta}-4\right)\right]^{m-1},$$

$$f_{(m:m)}(x;\theta) = \frac{6}{5\theta}e^{-3x/\theta}\left(3e^{x/\theta}-2\right)\left[\frac{1}{5}e^{-3x/\theta}\left(4-9e^{x/\theta}\right)+1\right]^{m-1}.$$

To illustrate the shape of the order statistics pdf, Figures 11 and 12 represents the plots of the minimum, median and maximum order statistics with $m=5$ and 9.

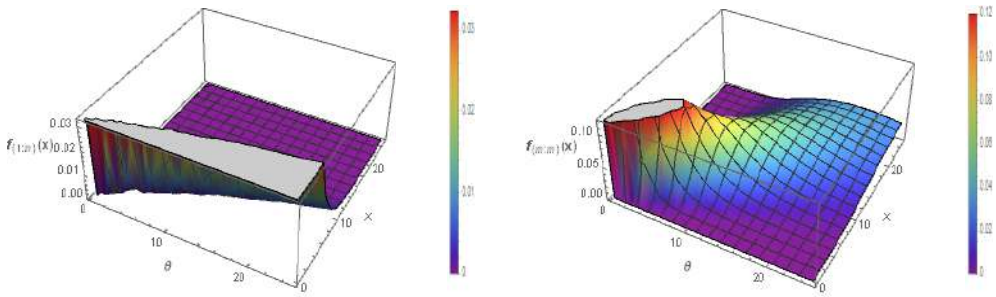

FIGURE 11. Plots of the minimum and maximum order statistics pdfs of the IB distribution for some parameter values with $m=5$.

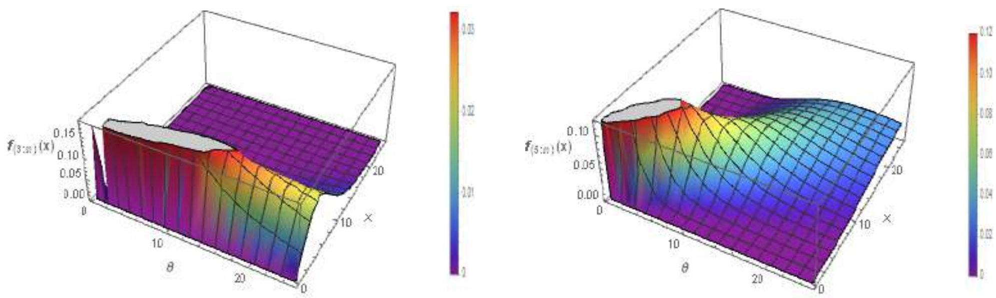

FIGURE 12. Plots of the median order statistics pdfs of the IB distribution for some parameter values with $m=5$ (left) and $m=9$ (right).

7. Parameter Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n selected from a pdf $f(x)$ and $x_1, x_2, \ldots, x_n$ be the corresponding sample values. The moment estimator of the parameter θ is obtained as a solution of the equation $\mu = \overline{X} = \frac{1}{n}\sum_{k=1}^{n} X_k$ (first theoretical moment equals to the first sample moment) and gives us $\widehat{\theta}_{IB} = \frac{19}{30}\overline{X}$. This estimator is asymptotically normal with mean θ and variance $\frac{289}{900}\theta^2$; that is,

$$\sqrt{n}(\widehat{\theta}_{IB}-\theta) \to N\left(0, \frac{289}{900}\theta^2\right)$$

in distribution as $n \to \infty$.

The likelihood function for the IB distribution is

$$L_{IB}(\theta|x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{6}{5\theta}\left(3e^{x_i/\theta} - 2\right) e^{-\frac{3x_i}{\theta}}.$$

The log-likelihood function based on a sample selected from the IB distribution is given by

$$\ell = -\frac{3n}{\theta}\overline{x} + \sum_{i=1}^{n} \log\left(3e^{\frac{x_i}{\theta}} - 2\right) + n\log\left(\frac{6}{5\theta}\right).$$

Differentiating with respect to the parameter θ, we get the non-linear equation

$$\frac{d\ell_{IB}}{d\theta} = \frac{3}{\theta^2}\left(\sum_{i=1}^{n} \frac{x_i e^{\frac{x_i}{\theta}}}{2 - 3e^{\frac{x_i}{\theta}}} + n\overline{x}\right) - \frac{n}{\theta}.$$

Since there is no exact solution to the equation, we can solve it numerically using Newton-Raphson approach. Table 2 provides some values of the MLE $\widetilde{\theta}_{IB}$ with sample sizes $n = 30, 60, 90, 180, 270, 360, 500, 700$ and $\theta = 2, 3, 4, 5, 6, 7, 8, 11$. It can be seen from Table 2 that the standard errors values are decreasing as the sample sizes are increasing.

Table 2. The MLE $\widetilde{\theta}_{IB}$ of the parameter θ.

n	θ	$\widetilde{\theta}_{IB}$	$SE(\widetilde{\theta}_{IB})$	n	θ	$\widetilde{\theta}_{IB}$	$SE(\widetilde{\theta}_{IB})$
	2	1.98120	0.32239		2	2.01781	0.10956
	3	2.97181	0.48344		3	3.02671	0.16434
	4	3.96241	0.64484		4	4.03561	0.21909
30	5	4.95301	0.80520	270	5	5.04451	0.27403
	6	5.94361	0.96758		6	6.05341	0.32869
	7	6.93421	1.12963		7	7.06232	0.38323
	8	7.92481	1.28839		8	8.07121	0.43742
	11	10.89661	1.77071		11	11.09793	0.60311
	2	1.99147	0.22915		2	2.01475	0.09476
	3	2.98720	0.34368		3	3.02213	0.14211
	4	3.98294	0.45823		4	4.02950	0.18954
60	5	4.97867	0.57330	360	5	5.03688	0.23685
	6	5.97440	0.68791		6	6.04425	0.28454
	7	6.97014	0.80162		7	7.05163	0.33133
	8	7.96587	0.91522		8	8.05901	0.37855
	11	10.95306	1.26039		11	11.08113	0.52068

	2	2.00042	0.18810		2	2.00965	0.08020
	3	3.00063	0.28212		3	3.01448	0.12030
	4	4.00084	0.37611		4	4.01931	0.16043
90	5	5.00105	0.47035	500	5	5.02413	0.20051
	6	6.00125	0.56460		6	6.02896	0.24056
	7	7.00147	0.65891		7	7.03379	0.28136
	8	8.00168	0.75261		8	8.03862	0.32065
	11	11.00230	1.02985		11	11.05310	0.44009
	2	1.99990	0.13300		2	2.00305	0.06756
	3	2.99984	0.19949		3	3.00458	0.10134
	4	3.99979	0.26604		4	4.00610	0.13507
180	5	4.99974	0.33249	700	5	5.00763	0.16898
	6	5.99968	0.39905		6	6.00916	0.20255
	7	6.99963	0.46574		7	7.01068	0.23627
	8	7.99958	0.53153		8	8.01221	0.26983
	11	10.99942	0.73144		11	11.01679	0.37303

8. Applications to Real Data

In this section, we considered some data sets to establish the applicability of the advised IB distribution. The first data set is obtained from Jamal et al. [13] and it represents 30 successive values of March precipitation (in inches) in Minneapolis/St Paul (Data IV). The data set is reported as follows:
0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.
The summary statistics for the data set are $\overline{x} = 1.6750$, SD $= 1.0006$, skewness $= 1.1447$, and kurtosis $= 1.6653$.

The second data set was considered by Bhaumik et al. [9] for small sample size. The data set consists of 34 observations on the vinyl chloride data which is obtained from clean upgrading monitoring wells in mg/L (Data III). The data set is reported as follows:
5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.
The values of $\overline{x}$, SD, skewness, and kurtosis of the vinyl chloride data are 1.8794, 1.9526, 1.6787, and 2.5345, respectively. We compare the IB distribution with the following competing models. All these distributions have two parameters $\alpha > 0$ and $\theta > 0$ and are life-time; that is, $x > 0$. Their pdfs are

- Mirra distribution (MD) (Sen et al. [20])

$$f_{MD}(x;\alpha,\theta) = \frac{\theta^3}{\alpha+\theta^2}\left(1+\frac{\alpha}{2}x^2\right)e^{-\theta x},$$

- Darna distribution (DD) (Shraa and Al-Omari [23])

$$f_{DD}(x) = \frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha + \frac{\theta^4 x^2}{2\alpha^3}\right) e^{-\theta x/\alpha},$$

- Janardan distribution (JD) (Shanker et al. [22])

$$f_{JD}(x) = \frac{\theta^2}{\alpha(\theta+\alpha^2)}(1+\alpha x)e^{-\theta x/\alpha},$$

- Two parameter Sujatha distribution (SD)

$$f_{SD}(x) = \frac{\theta^3}{\alpha\theta^2+\theta+2}\left(\alpha + x + x^2\right) e^{-\theta x}.$$

We fitted these distributions to the above two data sets, and the parameters of the distributions are estimated by using the maximum likelihood method. We compare the competitive distributions to the IB distribution using the statistical techniques provided, namely, the negative maximized log-likelihood values (LL), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Kolmogorov-Smirnov (K-S) test statistic where these measures are defined as

$$\mathrm{AIC} = -2L + 2p, \mathrm{CAIC} = -2L + \frac{2pn}{n-p-1},$$

$$\mathrm{BIC} = -2L + p\log(n), \mathrm{HQIC} = 2\log[\log n(p-2L)],$$

where p is the number of parameters and n is the sample size. The results are demonstrated in Table 3. For each criterion, the smallest values are gained by the IB distribution model, indicating the best fit among its competitive models for both data sets.

Table 3. The goodness-of-fit tests statistics for the two data sets.

Data	Model	AIC	CAIC	BIC	HQIC
First Data	IB	88.86399	89.00684	90.26518	89.31224
	MD	114.6882	115.0753	117.7409	115.7293
	DD	91.20894	91.65338	94.01133	92.10545
	JD	256.2305	256.6591	259.0985	257.1654
	SD	90.08334	90.52779	92.88574	90.97985
Second Data	IB	114.0304	114.1554	115.5567	114.5509
	MD	114.6882	115.0753	117.7409	115.7293
	Bilal	122.3611	122.4861	123.8875	122.8816
	DD	114.7601	115.1472	117.8128	115.8011
	JD	114.9053	115.2924	117.958	115.9464
	SD	119.1984	119.5855	122.2511	120.2394

The graphical method used to determine the hazard rate function of the data sets is the Total Time on Test (TTT). As in Aarset [1], convex, concave, convex-then-concave, and concave-then-convex empirical TTT plots correspond to decreasing, increasing, bathtub shape, and upside-down bathtub shape for the corresponding hazard function, respectively. Figures 13 represent the density with histogram, TTT, box, P-P and quantile-quantile (Q-Q) plots for the precipitation and vinyl chloride data sets. These figures support the superiority of IB distribution.

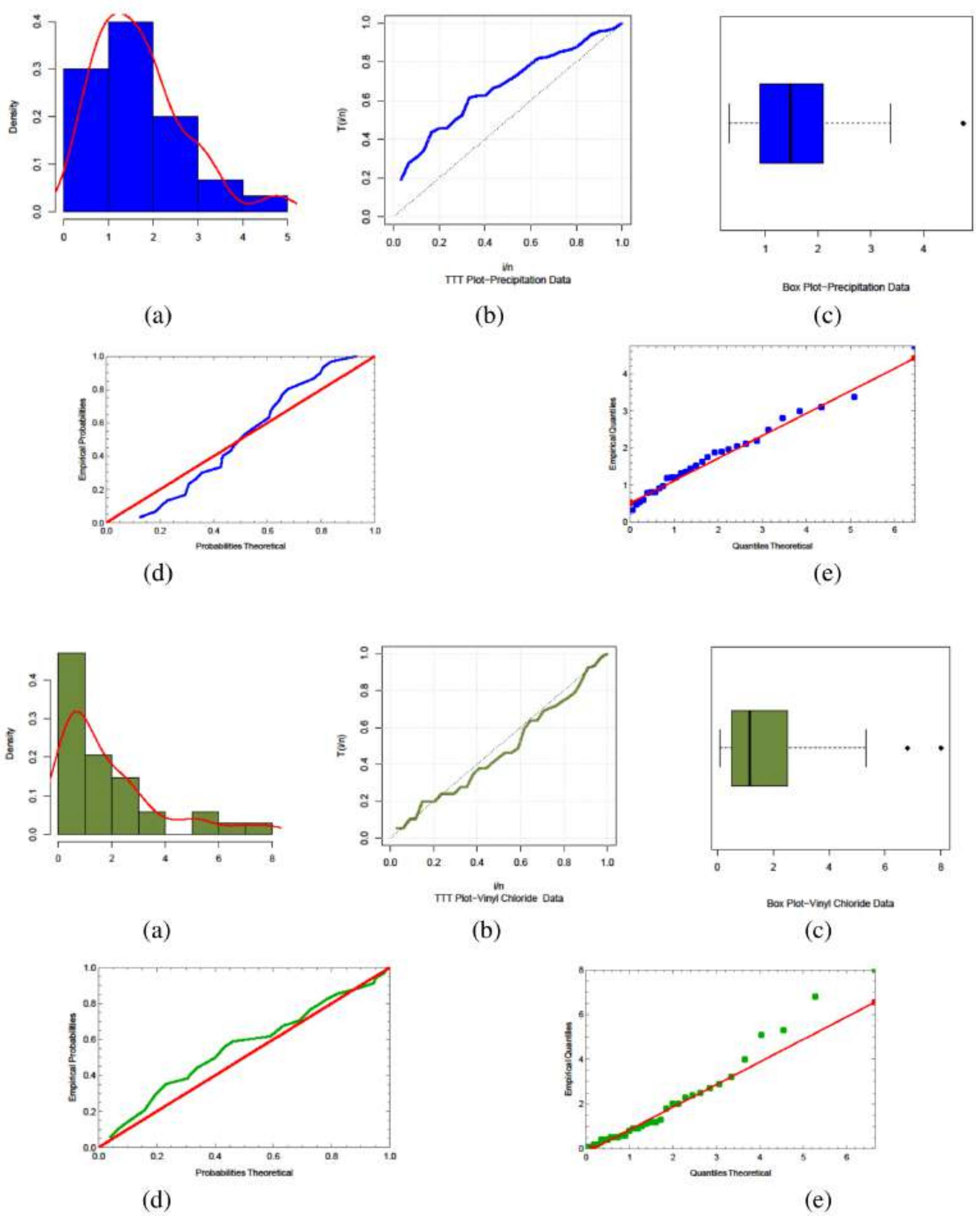

FIGURE 13. (a) the density with histogram, (b) TTT, (c) box plot, (d) P-P plot, and (e) Q-Q plot. Left: for the precipitation data, right: for the vinyl chloride data.

9. Conclusion

In this study, a novel one-parameter life distribution known as the Induced Bilal distribution is proposed. The moments, coefficient of variation, skewness, kurtosis, mean and median deviations, Gini index, Lorenz curve, Bonferroni curve, odds function, survival function, reversed hazard rate function, hazard function, Rényi entropy, stochastic ordering, and distribution of order statistics are the main mathematical characteristics of this distribution that are derived and discussed. The stress-strength reliability is additionally derived and demonstrated. The maximum likelihood method is used to estimate the model parameter. It is demonstrated using actual data sets that the new distribution performs better than various established distributions and well-known competitors.

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Declarations

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Data Availability Statement
The data sets used in this study are included within the article.

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