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The Development Criterion for Exponential Function Projective Synchronization and Guaranteed Cost Controller of Complex Dynamical Networks Utilizing Multi-Connections and Time-Varying Delays

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Abstract This paper addresses cost-constrained synchronization control in complex dynamical networks (CDNs) with mixed time-varying delays, asymmetric couplings, and multi-connections. A novel criterion for exponential projective synchronization and guaranteed-cost control is developed using the LyapunovKrasovskii functional method, extended reciprocally convex and free-matrix-based inequalities, and LMIs. The proposed scheme reduces the cost upper bound via a new quadratic cost function. A numerical example illustrates the effectiveness of the approach.

MSC: 34A12, 34D06, 34D20, 34H05

Keywords: Guaranteed cost control; Stability; EFPS; Mixed time-varying delays

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1. Introduction

CDNs have been an interesting research topic for several decades. Many practical systems can be classified by various forms of complex networks, commonly referred to as structures consisting of nodes or vertices connected by links or edges. These networks are capable of simulating and describing a wide range of complex natural systems, including computer networks, social networks, communication networks, transportation networks, the internet, electrical power grids, cellular and metabolic networks [1–3], and more. They also display extremely complex behavior.

Problems of the stability of CDNs leading to delay of differential equations (DDEs) are found in fields such as control theory, mathematics, physics, engineering applications, science, economics, etc. In electronic engineering, time delays arise for various reasons, such as circuit integration, amplifier switching delays, communication delays, etc. Time delays can cause fluctuations and, moreover, make the network unstable. Therefore, the time delay is widely available in real-world networks, so, synchronization of time-delayed CDNs becomes a crucial and significant problem.

Over the past several decades, synchronization problems in CDNs have garnered great attention among researchers. The synchronization of CDNs, which is typical of dynamic, is a universal phenomenon. The synchronization of CDNs can explain many of the observed natural phenomena. In recent years, synchronization in complex dynamical networks (CDNs) has attracted significant attention due to its broad applicability in fields such as physics, biology, engineering, and communications. Numerous synchronization strategies have been developed, including complete synchronization (CS) [4, 5], generalized synchronization (GS) [6], exponential synchronization (ES) [7, 8] and projective synchronization (PS) [9, 10].

Exponential function projective synchronization offers a powerful way to synchronize complex dynamical systems, especially in chaotic systems or networks. The main goal is to ensure that the systems synchronize in a manner where their states remain proportional to each other, with the synchronization error decaying exponentially over time.

This synchronization approach has physical and practical applications, such as secure communications system. In chaotic communication, signals are encoded within chaotic systems to hide information, EFPS ensures that the transmitter and receiver systems remain synchronized even with parameter mismatches and disturbances. In power grid networks, CDNs can model interconnected power generators and exponential synchronization helps ensure that all generators maintain stable frequency and phase. In neural networks and brain science, brain regions can be modeled as nodes in a complex network and projective synchronization allows for scalable synchronization models (useful in brain stimulation therapies or brain-machine interfaces).

Furthermore, control methods for delayed networks have been a central topic, with approaches including feedback control [11, 12], intermittent and adaptive controls [8, 14–16], and pinning control [13–15]. However, designing controllers that ensure both stability and performance remains a challenge when mixed and time-varying delays are present. In most control practices, it is always desirable to design a control system that is not only stabilizable but also guarantees an adequate level of performance. The guaranteed cost control method, originally proposed by Chang and Peng [17], provides a framework to maintain system stability while bounding performance costs. This approach has been extended to various systems with time delays [18–20], but few works address its integration with EFPS in CDNs.

Fractional calculus and stability theory have recently enriched the analysis of complex dynamic systems. Relevant contributions include epidemic models involving fractional derivatives such as the Caputo and Atangana-Baleanu types [21–24]. For example, the work in [22, 24] models COVID-19 dynamics using nonlocal operators, offering insights into time-delay behavior. Other efforts address existence and Ulam stability in fractional boundary value problems [25], which offer theoretical tools potentially adaptable to synchronization scenarios. These studies underscore the importance of delay-sensitive analysis and support the motivation for delay-aware synchronization criteria.

Additionally, the impact of multi-connection topologies where each node can be connected through various types of delay paths has been underexplored. While some studies consider simple topological structures, few have rigorously addressed how multiple and asymmetric couplings influence network dynamics and control feasibility.

Inspired by the aforementioned conversations, this paper proposes a novel guaranteed cost control scheme for EFPS in CDNs with mixed time-varying delays and multi-connection couplings. Using a newly constructed Lyapunov-Krasovskii functional with layered integral terms and advanced matrix inequalities (e.g., the extended reciprocally convex and free-matrix-based inequalities), we derive new Linear Matrix Inequality (LMI)-based criteria ensuring exponential synchronization and cost performance. Numerical examples confirm the effectiveness of the proposed method, highlighting its potential for real-world applications with complex network dynamics. We illustrate the usefulness of the suggested criteria with a few instances in the numerical section. The main contributions of this paper are given as follows:

- The design of quadratic cost function

$$
\begin{aligned}
\mathbb{J} \;=\; & \sum_{i=1}^{N}\int_{0}^{\infty}\Big[\varepsilon_i^T(\tau)\Phi_{11i}\varepsilon_i(\tau)+\varepsilon_i^T(\tau-\eta_1(\tau))\Phi_{22i}\varepsilon_i(\tau-\eta_1(\tau)) \\
& +\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i^T(\varsigma)d\varsigma\;\Phi_{33i}\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma+\mu_i^T(\tau)\Phi_{44i}\mu_i(t)\Big]d\tau,
\end{aligned}
$$

  which is the first proposed to analyze the issue of ensuring cost control for projective synchronization of exponential functions in CDNs with asymmetric couplings and heterogeneous time-varying delays
- The construction of Lyapunov-Krasovskii functionals in the form of single integrals, double integrals and triple integrals to support the lemma.
- The application of lemma free-matrix-based integral inequality and the extended reciprocally convex matrix inequality.

In terms of LMIs, new necessary criteria are provided in order for hybrid feedback control to be guaranteed. Furthermore, we reduce their upper bound by using the new quadratic cost functions. Lastly, a numerical example is provided to demonstrate the efficacy of the suggested approach.

This article is divided into the following five sections: We provide the system description and preliminary in Section 2. In Section 3, we present the CDNs' guaranteed cost EFPS. The numerical examples are presented in Section 4. The findings are finally covered in Section 5.

2. System description and preliminary

The following complex dynamical network system consisting of N identical coupled nodes is considered:

$$\begin{aligned}
\dot{\chi}_i(\tau) &= f\left(\tau, \chi_i(\tau), \chi_i(\tau-\eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \chi_i(\varsigma)d\varsigma\right) + \omega_1 \sum_{j=1}^{N} \alpha_{ij} A_{1i} \chi_j(t) \\
&\quad + \omega_2 \sum_{j=1}^{N} \beta_{ij} A_{2i} \chi_j(\tau-\eta_1(t)) + \omega_3 \sum_{j=1}^{N} \gamma_{ij} A_{3i} \int_{\tau-\delta_1(\tau)}^{\tau} \chi_j(\varsigma)d\varsigma \qquad (2.1) \\
&\quad + \mathbb{U}_i(\tau), \qquad \tau > 0, \quad i = 1, 2, \dots, N, \\
\chi_i(\tau) &= \Theta_i(\tau), \qquad \tau \in [-\eta_{\max}, 0], \quad \eta_{\max} = \max\{\eta_{1\mathrm{M}}, \eta_2, \delta_1, \delta_2\},
\end{aligned}$$

where

- $\chi_i(\tau) = (\chi_{i1}(\tau), \chi_{i2}(\tau), \dots, \chi_{in}(\tau))^T \in \mathbb{R}^n$ represents the ith node's state vector.
- $f : \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector function that is nonlinear and smooth and continuously differentiable this explains the nodes' local dynamics and satisfies a globally Lipschitz condition.
- $\omega_1, \omega_2, \omega_3 > 0$ are real constants indicate the strength of the delayed and non-delayed couplings.
- $A_{1i}, A_{2i}, A_{3i} \in \mathbb{R}^{n\times n}$ are constant inner-coupling matrices.
- $\mathbb{U}_i(\tau) \in \mathbb{R}^m$ serves as the node i's control input.
- The time-varying delays functions $\eta_i(\tau)$ and $\delta_i(\tau)$, $i = 1, 2$ satisfy the conditions:
$$\begin{aligned}
&0 \le \eta_{1\mathrm{m}} \le \eta_1(\tau) \le \eta_{1\mathrm{M}}, \quad 0 \le \eta_2(\tau) \le \eta_2, \\
&0 \le \delta_1(\tau) \le \delta_1, \quad 0 \le \delta_2(\tau) \le \delta_2,
\end{aligned}$$
where $\eta_{1\mathrm{m}}, \eta_{1\mathrm{M}}, \eta_2, \delta_1, \delta_2$ are real constants which denote the upper bound delays.
- Defined α_{ij}, β_{ij} and γ_{ij} as follows: if there is a connection between node i and node j ($j \neq i$), then $\alpha_{ij} > 0$, $\beta_{ij} > 0$, $\gamma_{ij} > 0$, otherwise, $\alpha_{ij} = 0$, $\beta_{ij} = 0$, $\gamma_{ij} = 0$ ($j \neq i$).
- $B_{1i} = (\alpha_{ij})_{N\times N}$, $B_{2i} = (\beta_{ij})_{N\times N}$, $B_{3i} = (\gamma_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ are the coupling configuration matrices weights and topological structure for non-delayed configuration and delayed one at time τ.
- The diagonal elements of matrix B_{1i}, B_{2i} and B_{3i} are defined by
$$\alpha_{ii} = -\sum_{j=1, i\neq j}^{N} \alpha_{ij}, \quad \beta_{ii} = -\sum_{j=1, i\neq j}^{N} \beta_{ij}, \quad \gamma_{ii} = -\sum_{j=1, i\neq j}^{N} \gamma_{ij}, \quad i = 1, 2, \dots, N.$$
- A continuous starting function with vector values of $\tau \in [-\eta_{\max}, 0]$ is shown by the first condition function $\Theta_i(\tau)$. The case of $C([-\eta_{\max}, 0], \mathbb{R}^n)$ is the continuous function's Banach space with the norm
$$\|\Theta_i\| = \sup_{-\eta_{\max} \le \varsigma \le 0} \|\Theta_i(\varsigma)\|, \qquad i = 1, 2, \dots, N.$$

We always assume that there is a unique solution to (2.1) under the initial conditions.

Here are some controllers for dynamic: $\mathbb{U}_i(\tau)$ like

$$\mathbb{U}_i(\tau) = \mathbb{U}_{i1}(\tau) + \mathbb{U}_{i2}(\tau), \qquad i = 1, 2, \ldots, N, \tag{2.2}$$

where

$$\begin{aligned}
\mathbb{U}_{i1}(\tau) &= \dot{\theta}(\tau)\varsigma(\tau), \\
\mathbb{U}_{i2}(\tau) &= C_{1i}\mu_i(\tau) + C_{2i}\mu_i(\tau - \eta_2(\tau)) + C_{3i}\int_{\tau-\delta_2(t)}^{\tau} \mu_i(s)d\varsigma,
\end{aligned}$$

which C_{1i}, C_{2i}, C_{3i} are appropriate dimension matrices, $\mu_i(\tau) = \kappa_i\varepsilon_i(\tau)$ and κ_i, the constant matrix control gain is $i = 1, 2, \ldots, N$.

This vector of error is configured in the document

$$\varepsilon_i(\tau) = \chi_i(\tau) - \theta(\tau)\varsigma(\tau), \quad i = 1, \ldots, N,$$

where $\varsigma(\tau) \in \mathbb{R}^n$ might be either an equilibrium point that fulfills $\theta(\tau)$, a continuously bounded differentiable function, or

$$\begin{aligned}
\dot{\varsigma}(\tau) &= f\left(\varsigma(\tau), \varsigma(\tau - \eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{t} \varsigma(\vartheta)d\vartheta\right), \quad \forall\tau > 0, \\
\varsigma(\tau) &= \varpi(\tau), \quad \tau \in [-\eta_{\max}, 0].
\end{aligned} \tag{2.3}$$

The quadratic cost function of the associated network is defined as follows:

$$\begin{aligned}
\mathbb{J} &= \sum_{i=1}^{N} \int_0^\infty L_i\left(\tau, \varepsilon_i(\tau), \varepsilon_i(\tau - \eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma, \mu_i(\tau)\right)d\tau \\
&= \sum_{i=1}^{N} \int_0^\infty \Big[\varepsilon_i^T(\tau)\Phi_{11i}\varepsilon_i(\tau) + \varepsilon_i^T(\tau - \eta_1(\tau))\Phi_{22i}\varepsilon_i(\tau - \eta_1(\tau)) \\
&\quad + \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i^T(\varsigma)d\varsigma \; \Phi_{33i} \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma + \mu_i^T(\tau)\Phi_{44i}\mu_i(t)\Big]d\tau,
\end{aligned} \tag{2.4}$$

where $\Phi_{11i}, \Phi_{22i}, \Phi_{33i} \in \mathbb{R}^{n\times n}$ and $\Phi_{44i} \in \mathbb{R}^{m\times m}$, $i = 1, 2, \ldots, N$, are matrices that are positively definite.

After that, the error complex dynamics network is provided by replacing it with (2.1),

$$\begin{aligned}
\dot{\varepsilon}_i(\tau) &= \dot{\chi}_i(\tau) - \dot{\theta}(\tau)\varsigma(\tau) - \theta(\tau)\dot{\varsigma}(\tau) \\
&= f\left(\tau, \chi_i(\tau), \chi_i(\tau-\eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \chi_i(\varsigma), d\varsigma\right) \\
&\quad -\theta(\tau) f\left(\varsigma(\tau), \varsigma(\tau-\eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{t} \varsigma(\vartheta) d\vartheta\right) \\
&\quad +\omega_1 \sum_{j=1}^{N} \alpha_{ij} A_{1i} \varepsilon_j(\tau) + \omega_2 \sum_{j=1}^{N} \beta_{ij} A_{2i} \varepsilon_j(\tau-\eta_1(\tau)) \\
&\quad +\omega_3 \sum_{j=1}^{N} \gamma_{ij} A_{3i} \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_j(\varsigma) d\varsigma + C_{1i}\mu_i(\tau) + C_{2i}\mu_i(\tau-\eta_2(\tau)) \\
&\quad +C_{3i} \int_{\tau-\delta_2(t)}^{\tau} \mu_i(s) d\varsigma, \qquad i = 1, 2, \dots, N. \\
\varepsilon_i(\tau) &= \Theta_i(\tau) - \theta(\tau)\varpi(\tau) = \phi_i(\tau) \qquad \tau \in [-\eta_{\max}, 0], \qquad i = 1, 2, \dots, N.
\end{aligned} \tag{2.5}$$

The objectives of this paper are to design a feedback controller $\mu_i(\tau) = \kappa_i\varepsilon_i(\tau)$, $i = 1, 2, \dots, N$, such that the complex dynamical network (2.5) is exponentially stable.

The following formula is used to consider the conditional existence of The criterion for exponential function projective synchronization and guaranteed cost controller problem.

Definition 2.1. [20]. If $\mathbb{M}_i \geq 1$, $\zeta > 0$, and a function that fluctuates throughout time $\theta(\tau)$ exist, then there is exponential function projective synchronization (EFPS) for Network (2.1),

$$\lim_{\tau\to\infty} \|\varepsilon_i(\tau)\| = \lim_{\tau\to\infty} \|\chi_i(\tau) - \theta(\tau)\varsigma(\tau)\| \leq \mathbb{M}_i \|\Theta_i - \theta\varpi\| e^{-\zeta\tau}, \quad \forall\tau > 0, \quad i = 1, 2, \dots, N,$$

where $\| \bullet \|$ stands for the Euclidean vector norm.

Definition 2.2. [20]. Examining the CDNs (2.1), if rules of feedback control $\mu_i(\tau) = \kappa_i\varepsilon_i(t)$, $i = 1, 2, \dots, N$, as well as a positive number exists $\mathbb{J}^*$ in order for the error network (2.5) to be EFPS when the value (2.4) fulfills $\mathbb{J} \leq \mathbb{J}^*$, then $\mathbb{J}^*$ is a cost value that is guaranteed and $\mu_i(\tau)$, $i = 1, 2, \dots, N$ are the cost controller network's guarantees (2.1).

Lemma 2.3. [26]. *(Cauchy inequality) For any symmetric positive definite matrix $X \in M^{n\times n}$ and $a, b \in \mathbb{R}^n$ we have*

$$\pm 2a^T b \leq a^T X a + b^T X^{-1} b.$$

Lemma 2.4. [26]. *(Schur complement lemma) Given symmetric matrices $A, B, andC$ that are constant and have the proper dimensions to fulfill $A = A^T, B = B^T > 0$, then $A + C^T B^{-1} C < 0$ if and only if*

$$\begin{bmatrix} A & C^T \\ * & -B \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -B & C \\ * & A \end{bmatrix} < 0.$$

Lemma 2.5. [27]. *(Jensen's inequality) For any constant matrix $Q = Q^T > 0$ and positive numbers η_1, η_2 such that the following integrals are well defined, then*

1. $$-\int_{\tau-\eta_1}^{\tau} y(s)^T Q\, y(s)ds \leq -\frac{1}{\eta_1}\left(\int_{\tau-\eta_1}^{\tau} y(s)ds\right)^T Q \left(\int_{\tau-\eta_1}^{\tau} y(s)ds\right),$$
2. $$-\int_{-\eta_2}^{-\eta_1}\int_{\tau+s}^{\tau} y(\theta)^T Q\, y(\theta)d\theta ds \leq -\frac{2}{\eta_2^2-\eta_1^2}\left(\int_{-\eta_2}^{-\eta_1}\int_{\tau+s}^{\tau} y(\theta)d\theta ds\right)^T \times Q\left(\int_{-\eta_2}^{-\eta_1}\int_{\tau+s}^{\tau} y(\theta)d\theta ds\right).$$

Lemma 2.6. [28]. *(Free-Matrix-Based Integral Inequality) Let $y : (a, b) \mapsto \mathbb{R}^n$ be a differrentible function. For symmetric matrices $Z \in \mathbb{R}^{n\times n}$ and $N_1, N_2, N_3 \in \mathbb{R}^{3n\times 3n}$ and $M_1, M_2 \in \mathbb{R}^{3n\times n}$ such that*

$$\Pi = \begin{bmatrix} N_1 & N_2 & M_1 \\ * & N_3 & M_2 \\ * & * & Z \end{bmatrix} \geq 0,$$

then the following inequality hold:

$$-\int_a^b \dot{y}^T(t) Z \dot{y}(t)dt \leq \Upsilon^T \Omega\, \Upsilon,$$

where

$$\begin{aligned}
\Omega &= (b-a)\Big(N_1 + \frac{1}{3}N_3\Big) - Sym\Big\{M_1\Lambda_1 - M_2\Lambda_2\Big\},\\
\Lambda_1 &= r_1 - r_2\ , \quad \Lambda_2 = 2r_3 - r_1 - r_2\ ,\\
r_1 &= \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \quad r = \begin{bmatrix} 0 & I & 0 \end{bmatrix}, \quad r = \begin{bmatrix} 0 & 0 & I \end{bmatrix},\\
\Upsilon &= \begin{bmatrix} y^T(b) & y^T(a) & \frac{1}{b-a}\int_a^b y^T(t)dt \end{bmatrix}^T.
\end{aligned}$$

Lemma 2.7. [28]. *(The Extended Reciprocally Convex Matrix Inequality) The following matrix inequality is true for any matrices Y_a and Y_b, as well as symmetric matrices $R_a > 0$ and $R_b > 0$ for a real scalar $\theta \in (0, 1)$,*

$$\begin{bmatrix} \frac{1}{\theta}R_a & 0 \\ 0 & \frac{1}{1-\theta}R_b \end{bmatrix} \geq \begin{bmatrix} R_a + (1-\theta)X_a & (1-\theta)Y_a + \theta Y_b \\ * & R_b + \theta X_b \end{bmatrix},$$

where $X_a = R_a - Y_b R_b^{-1} Y_b^T$ and $X_b = R_b - Y_a^T R_a^{-1} Y_a$.

3. CDNs' guaranteed cost EFPS

With in this section, the following theorem presents a sufficient condition for the existence of the guaranteed cost control laws for the guaranteed cost EFPS of complex dynamical networks (2.1).

Let us set

$$\begin{aligned}
\|\upsilon_i\| &= \|\varepsilon_i(0)\|, \\
\|\upsilon_i\|_{cl} &= \sup_{-\eta_{\max}\leq\varsigma\leq 0}\|\varepsilon_i(\varsigma)\|, \\
\rho_i &= \lambda_{min}(P_i^{-1}), \\
\iota_i &= \lambda_{max}(P_i^{-1}), \\
\nu_i &= \lambda_{max}(P_i^{-1}T_{1i}P_i^{-1})\frac{1-e^{-2\zeta\eta_{1\mathrm{m}}}}{2\zeta} \\
&\quad +\lambda_{max}(P_i^{-1}T_{2i}P_i^{-1})\frac{1-e^{-2\zeta\eta_{1\mathrm{M}}}}{2\zeta} \\
&\quad +\eta_{1\mathrm{M}}^2\lambda_{max}(P_i^{-1}T_{3i}P_i^{-1})\frac{1-e^{-2\zeta\eta_{1\mathrm{M}}}}{2\zeta} \\
&\quad +\eta_2^2\lambda_{max}(P_i^{-1}C_1iT_{4i}^{-1}C_{1i}^TP_i^{-1})\frac{1-e^{-2\zeta\eta_2}}{8\zeta} \\
&\quad +\delta_1^2\lambda_{max}(P_i^{-1}T_{5i}P_i^{-1})\frac{1-e^{-2\zeta\delta_1}}{2\zeta} \\
&\quad +\delta_2^2\lambda_{max}(P_i^{-1}C_{1i}T_{6i}^{-1}C_{1i}^TP_i^{-1})\frac{1-e^{-2\zeta\delta_2}}{8\zeta} \\
&\quad +\eta_{1\mathrm{M}}^2\lambda_{max}(P_i^{-1}T_{7i}P_i^{-1})\frac{1-e^{-2\zeta\eta_{1\mathrm{M}}}}{2\zeta}, \\
\Upsilon_i &= \iota_i\|\upsilon_i\|^2+\nu_i\|\upsilon_i\|_{cl}^2, \\
\xi(\tau) &= \Big[\varphi(\tau),\varphi(\tau-h_1(\tau)),\varphi(\tau-\eta_{1\mathrm{m}}),\varphi(\tau-\eta_{1\mathrm{M}}),\int_{\tau-d_1(\tau)}^{\tau}\varphi(\varsigma)d\varsigma, \\
&\quad \int_{\tau-\eta_1(\tau)}^{\tau}\frac{\varphi_i(\varsigma)}{\eta_1(\tau)}d\varsigma,\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\frac{\varphi_i(\varsigma)}{(\eta_{1\mathrm{M}}-\eta_1(\tau))}d\varsigma,\dot{\varphi}(\tau),\Big], \\
\Xi(\tau) &= \Big[\varphi(\tau),\dot{\varphi}(\tau)\Big],
\end{aligned}$$

and

(1) $\Psi(\tau)=f'(\varsigma(\tau),\varsigma(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\varsigma(\xi)d\xi)\in\mathbb{R}^{n\times n}$ that is Jacobian of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ at $\varsigma(\tau)$ utilizing the derivative of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ with regard to $\chi(\tau)$,

(2) $\Psi_{\eta_1}(\tau)=f'(\varsigma(\tau),\varsigma(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\varsigma(\xi)d\xi)\in\mathbb{R}^{n\times n}$ that is Jacobian of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ at $\varsigma(\tau-\eta_1(\tau))$ using the derivative of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ with regard to $\chi(\tau-\eta_1(\tau))$,

(3) $\Psi_{\delta_1}(\tau)=f'(\varsigma(\tau),\varsigma(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\varsigma(\xi)d\xi)\in\mathbb{R}^{n\times n}$ that is Jacobian of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ at $\int_{\tau-\delta_1(\tau)}^{\tau}\varsigma(\xi)d\xi$ with the derivative of $f(\chi(\tau),\chi(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma)$ respect to $\int_{\tau-\delta_1(\tau)}^{\tau}\chi(\varsigma)d\varsigma$.

Theorem 3.1. *Examine the quadratic cost function (2.4) and close-loop error dynamical networks (2.5), $\zeta > 0$, $\Phi_{11i} > 0$, $\Phi_{22i} > 0$, $\Phi_{33i} > 0$, $\Phi_{44i} > 0$, $N_{1ji} > 0$, $N_{2ji} > 0$, $N_{3ji} > 0$, $j = 1, 2, 3$, $M_{1ji} > 0$, $M_{2ji} > 0$, $M_{3ji} > 0$ $j = 1, 2$, $T_{3a_2i} > 0$, $T_{7bi} > 0$. If symmetric positive definite matrices are present P_i, T_{1i}, T_{2i}, T_{3i}, T_{4i}, T_{5i}, T_{6i}, T_{7i}, $i = 1, 2, \ldots, N$, such that the following LMIs hold:*

$$\Lambda_{1i} = \begin{bmatrix} N_{11i} & N_{12i} & M_{11i} \\ * & N_{13i} & M_{12i} \\ * & * & T_{3a_2i} \end{bmatrix} > 0,$$

$$\Lambda_{2i} = \begin{bmatrix} N_{21i} & N_{22i} & M_{21i} \\ * & N_{23i} & M_{22i} \\ * & * & T_{3a_2i} \end{bmatrix} > 0,$$

$$\Lambda_{3i} = \begin{bmatrix} N_{31i} & N_{32i} & M_{31i} \\ * & N_{33i} & M_{32i} \\ * & * & T_{7bi} \end{bmatrix} > 0,$$

$$\Pi_{1i} = \begin{bmatrix} \Omega_{0i} + \eta_{1M}\Omega_{12i} & (g_1 - g_2)^T X_{b_2i} \\ * & -e^{2\zeta\eta_{1M}} T_{3b_2i} \end{bmatrix} < 0, \tag{3.1}$$

$$\Pi_{2i} = \begin{bmatrix} -e^{2\zeta\eta_{1M}} T_{3b_2i} & X_{b_1i}(g_2 - g_4) \\ * & \Omega_{0i} + \eta_{1M}\Omega_{11i} \end{bmatrix} < 0, \tag{3.2}$$

where

$$\begin{aligned}
\Omega_{0i} = \quad & g_1^T\Big[(\Psi + \zeta I)P_i + P_i(\Psi^T + \zeta I) - C_{1i}C_{1i}^T + \psi I\Big]g_1 \\
& - g_8 P_i g_8 + g_1^T P_i A_{1i}^T g_{11} + g_{11}^T A_1 i P_i g_1 + g_2^T P_i A_{2i}^T g_{12} \\
& + g_{12}^T A_{2i} P_i g_2 + g_5^T P_i A_{3i}^T g_{13} + g_{13}^T A_{3i} P_i g_5 + g_1^T \Psi_{\eta_1} P_i g_9 \\
& + g_9^T P_i \Psi_{\eta_1} g_1 + g_1^T \Psi_{\delta_1} P_i I g_1 + g_{10}^T P_i \Psi_{\delta_1} g_1 + g_8^T \Psi P_i g_{17} \\
& + g_{17}^T P_i \Psi g_8 + g_8^T \Psi_{\eta_1} P_i g_{18} + g_{18}^T P_i \Psi_{\eta_1} g_8 \\
& + g_8^T \Psi_{\delta_1} P_i g_{19} + g_{19}^T P_i \Psi_{\delta_1} g_8 + g_2^T (2\psi_{\eta_1} I) g_2 \\
& + g_5^T (2\psi_{\delta_1} I) g_5 - g_9^T (\Psi_{\eta_1} I) g_9 - g_{10}^T (\Psi_{\eta_1} I) g_{10} \\
& - g_{17}^T (\Psi I) r_{17} - g_{18}^T (\Psi_{\eta_1} I) g_{18} - g_{19}^T (\Psi_{\eta_1} I) g_{19} \\
& - g_{14}^T 4\eta_2^2 T_{4i} g_{14} - g_{15}^T (4e^{-2\zeta\eta_2} T_{4i}) g_{15} - g_{16}^T (4\delta_2^2 T_{6i}) g_{16} \\
& + g_1^T \left[2e^{2\zeta\eta_2} C_{2i} T_{4i} C_{2i}^T + e^{2\zeta\delta_2} C_{3i} T_{6i} C_{3i}^T\right] g_1 \\
& + g_8^T \left[2e^{2\zeta\eta_2} C_{2i} T_{4i} C_{2i}^T + e^{2\zeta\delta_2} C_{3i} T_{6i} C_{3i}^T\right] g_8 \\
& + \left[g_1^T T_{1i} g_1 - e^{-2\zeta\eta_{1m}} g_3^T T_{1i} g_3\right] + \left[g_1^T T_{2i} g_1 - e^{-2\zeta\eta_{1M}} g_4^T T_{2i} g_4\right]
\end{aligned}$$

$$
\begin{aligned}
&+g_1^T\omega_1\psi_\alpha\left(\sum_{j=1}^{N}(\alpha_{ij})^2\right)Ig_1+g_8^T\omega_1\psi_\alpha\left(\sum_{j=1}^{N}(\alpha_{ij})^2\right)Ig_8\\
&+g_1^T\omega_2\psi_\beta\left(\sum_{j=1}^{N}(\beta_{ij})^2\right)Ig_1+g_8^T\omega_2\psi_\beta\left(\sum_{j=1}^{N}(\beta_{ij})^2\right)Ig_8\\
&+g_1^T\omega_3\psi_\gamma\left(\sum_{j=1}^{N}(\gamma_{ij})^2\right)Ig_1+g_8^T\omega_3\psi_\gamma\left(\sum_{j=1}^{N}(\gamma_{ij})^2\right)Ig_8\\
&-g_{11}^T\left(\frac{\psi_\alpha}{2N_{\omega_1}}\right)Ig_{11}-g_{12}^T\left(\frac{\psi_\beta}{2N_{\omega_2}}\right)Ig_{12}-g_{13}^T\left(\frac{\psi_\gamma}{2N_{\omega_3}}\right)Ig_{13}\\
&+\eta_{1M}^2\begin{bmatrix} g_1\\ g_8\end{bmatrix}^T T_{3i}\begin{bmatrix} g_1\\ g_8\end{bmatrix}+\left[g_1^T(\delta_1^2T_{5i})g_1\right]-e^{-2\zeta\delta_1}\left[g_5^TT_{5i}g_5\right]+g_8^T\left[\frac{\eta_{1M}^2T_{7i}}{2}\right]g_8\\
&+e^{-2\zeta\eta_{1M}}\left[g_1^TT_{3ci}\ g_1\right]-e^{-2\zeta\eta_{1M}}\left[g_4^TT_{3di}\ g_4\right]+e^{-2\zeta\eta_{1M}}\left[g_2^T(T_{3di}-T_{3ci})g_2\right]\\
&-\left[(g_1-g_6)^T\left(2e^{-2\zeta\eta_{1M}}T_{7i}\right)(g_1-g_6)\right]-\left[(g_2-g_7)^T\left(2e^{-2\delta\eta_{1M}}T_{7i}\right)(g_2-g_7)\right]\\
&-e^{-2\zeta\eta_{1M}}\left[\left[g_1^T(\eta_{1M}T_{3b_{12}i}+T_{3ci})g_1\right]\right]-e^{-2\zeta\eta_{1M}}\left[\left[g_2^T(T_{3di}-T_{3ci})g_2\right]\right]\\
&-e^{-2\zeta\eta_{1M}}\left[\left[g_4^T(\eta_{1M}T_{3b_{12}i}+T_{3di})g_4\right]\right]\\
&+\ sym\left\{\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}^T\left(\eta_{1M}e^{-2\zeta\eta_{1M}}M_{11i}\right)(g_1-g_2)\right\}\\
&+\ sym\left\{\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}^T\left(\eta_{1M}e^{-2\zeta\eta_{1M}}M_{12i}\right)(2g_6-g_1-g_2)\right\}\\
&+\ sym\left\{\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}^T\left(\eta_{1M}e^{-2\zeta\eta_{1M}}M_{21i}\right)(g_2-g_4)\right\}\\
&+\ sym\left\{\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}^T\left(\eta_{1M}e^{-2\zeta\eta_{1M}}M_{22i}\right)(2g_7-g_2-g_4)\right\}\\
&+g_8^T\eta_2^2C_{1i}g_{14}+g_{14}^T\eta_2^2C_{1i}^Tg_8+g_1^TC_{1i}g_{15}+g_{15}^TC_{1i}^Tg_1+g_1^T\delta_2^2C_{1i}g_{16}\\
&+g_{16}^T\delta_2^2C_{1i}^Tg_1+g_1^TP_ig_{20}+g_{20}^TP_ig_1+g_2^TP_ig_{21}+g_{21}^TP_ig_2+g_5^TP_ig_{22}+g_{22}^TP_ig_5\\
&-g_{20}^T\Phi_{11i}^{-1}g_{20}-g_{21}^T\Phi_{22i}^{-1}g_{21}-g_{22}^T\Phi_{33i}^{-1}g_{22}+g_1^T(\frac{1}{4}C_{1i}\Phi_{44i}C_1i)g_1,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{11i} = \quad & \Big[g_6^T\big(-\eta_{1M}e^{-2\zeta\eta_{1M}}T_{3a_1i}\big)g_6\Big] + \Big[g_6^T\big(-\eta_{1M}e^{-2\zeta\eta_{1M}}T_{3b_1i}\big)g_6\Big] \\
& + \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \Big[\eta_{1M}e^{-2\zeta\eta_{1M}}\big(N_{11i} + \frac{N_{13i}}{3}\big)\Big] \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix} \\
& - \frac{e^{-2\zeta\eta_{1M}}}{\eta_{1M}} \begin{bmatrix} g_1 - g_2 \\ g_2 - g_4 \end{bmatrix}^T \begin{bmatrix} 0 & X_{b_2i} \\ * & T_{3b_2i} \end{bmatrix} \begin{bmatrix} g_1 - g_2 \\ g_2 - g_4 \end{bmatrix} \\
& + \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \Big[\eta_{1M}e^{-2\zeta\eta_{1M}}\big(N_{31i} + \frac{N_{33i}}{3}\big)\Big] \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}, \\
\Omega_{12i} = \quad & \Big[g_7^T\big(-\eta_{1M}e^{-2\zeta\eta_{1M}}T_{3a_1i}\big)g_7\Big] + \Big[g_7^T\big(-\eta_{1M}e^{-2\zeta\eta_{1M}}T_{3b_1i}\big)g_7\Big] \\
& + \begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix}^T \Big[\eta_{1M}e^{-2\zeta\eta_{1M}}\big(N_{21i} + \frac{N_{23i}}{3}\big)\Big] \begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix} \\
& - \frac{e^{-2\delta\eta_{1M}}}{\eta_{1M}} \begin{bmatrix} g_1 - g_2 \\ g_2 - g_4 \end{bmatrix}^T \begin{bmatrix} T_{3b_2i} & X_{b_1i} \\ * & 0 \end{bmatrix} \begin{bmatrix} g_1 - g_2 \\ g_2 - g_4 \end{bmatrix} \\
& + sym\left\{ \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(e^{-2\zeta\eta_{1M}}M_{31i}\big)(g_1 - g_2) \right\} \\
& + sym\left\{ \begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(e^{-2\zeta\eta_{1M}}M_{32i}\big)(2g_6 - g_1 - g_2) \right\}.
\end{aligned}
$$

Additionally, the feedback control system is

$$
\mu_i(\tau) = -\frac{1}{2}C_{1i}^T P_i^{-1}\varepsilon_i(\tau), \qquad \tau \in \mathbb{R}^+, \tag{3.3}
$$

and the following is the quadratic cost function's upper bound (2.4):

$$
\begin{aligned}
\mathbb{J} \quad < \quad & \sum_{i=1}^{N} \Big[\varepsilon_i^T(0)P_i^{-1}\varepsilon_i(0) + \int_{-\eta_{1m}}^{0} e^{2\zeta\varsigma}\varepsilon_i^T(\varsigma)P_i^{-1}T_{1i}P_i^{-1}\varepsilon_i(\varsigma)d\varsigma \\
& + \int_{-\eta_{1M}}^{0} e^{2\zeta\varsigma}\varepsilon_i^T(\varsigma)P_i^{-1}T_{2i}P_i^{-1}\varepsilon_i(\varsigma)d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
&+\eta_{1M}\int_{-\eta_{1M}}^{0}\int_{\varsigma}^{0} e^{2\zeta\vartheta}\Xi_i^T(\vartheta)P_i^{-1}T_{3i}P_i^{-1}\Xi_i(\vartheta)d\vartheta d\varsigma \\
&+\eta_2\int_{-\eta_2}^{0}\int_{\varsigma}^{0} e^{2\zeta\vartheta}\dot{\varepsilon}_i{}^T(\vartheta)P_i^{-1}C_{1i}T_{4i}^{-1}C_{1i}^TP_i^{-1}\dot{\varepsilon}_i(\vartheta)d\vartheta d\varsigma \\
&+\delta_1\int_{-\delta_1}^{0}\int_{\varsigma}^{0} e^{2\zeta\vartheta}\varepsilon_i^T(\vartheta)P_i^{-1}T_{5i}P_i^{-1}\varepsilon_i(\vartheta)d\vartheta d\varsigma \\
&+\delta_2\int_{-\delta_2}^{0}\int_{\varsigma}^{0} e^{2\zeta\vartheta}\varepsilon_i^T(\vartheta)P_i^{-1}C_{1i}T_{6i}^{-1}C_{1i}^TP_i^{-1}\varepsilon_i(\vartheta)d\vartheta d\varsigma \\
&+\int_{-\eta_{1M}}^{0}\int_{\rho}^{0}\int_{\varsigma}^{0} e^{2\zeta\vartheta}\dot{\varepsilon}_i{}^T(\vartheta)P_i^{-1}T_{7i}P_i^{-1}\dot{\varepsilon}_i(\vartheta)d\vartheta d\varsigma d\rho.
\end{aligned}
\tag{3.4}
$$

Proof. Since $f(.)$ is continuous differentiable, using the hybrid feedback control (3.3) and $Y_i = P_i^{-1}$, $\varphi_i(t) = Y_i\varepsilon_i(t)$, $i = 1, 2, \ldots, N$, it is easy to know that the error CDNs (2.5) is exprronentially stable of the linear systems with time-varying delays shown below

$$
\begin{aligned}
\dot{\varepsilon}_i(\tau) \;=\;& \Psi\varepsilon_i(\tau) + \Psi_{\eta_1}\varepsilon_i(\tau-\eta_1(\tau)) + \Psi_{\delta_1}\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma \\
&+\omega_1\sum_{j=1}^{N}\alpha_{ij}A_1\varepsilon_j(\tau) + \omega_2\sum_{j=1}^{N}\beta_{ij}A_2\varepsilon_j(\tau-\eta_1(\tau)) \\
&+\omega_3\sum_{j=1}^{N}\gamma_{ij}A_3\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_j(\varsigma)d\varsigma + C_{1i}\mu_i(\tau) + C_{2i}\mu_i(\tau-\eta_2(\tau)) \\
&+C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)d\varsigma, \qquad i = 1, 2, \ldots, N.
\end{aligned}
\tag{3.5}
$$

The Lyapunov-Krasovskii functional candidate that we construct is as follows:

$$
V(t, e_i(t)) = \sum_{k=1}^{7} V_k(t, e_i(t)), \tag{3.6}
$$

where

$$
\begin{aligned}
V_1(\tau,\varepsilon_i(\tau)) &= \sum_{i=1}^{N}\varepsilon_i^T(\tau)Y_i\varepsilon_i(\tau), \\
V_2(\tau,\varepsilon_i(\tau)) &= \sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{m}}}^{\tau} e^{2\zeta(\varsigma-\tau)}\varepsilon_i^T(\varsigma)Y_iT_{1i}Y_i\varepsilon_i(\varsigma)d\varsigma, \\
&\quad +\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau} e^{2\zeta(\varsigma-\tau)}\varepsilon_i^T(\varsigma)Y_iT_{2i}Y_i\varepsilon_i(\varsigma)d\varsigma, \\
V_3(\tau,\varepsilon_i(\tau)) &= \eta_{1\mathrm{M}}\sum_{i=1}^{N}\int_{-\eta_{1\mathrm{M}}}^{0}\int_{\tau+\varsigma}^{\tau} e^{2\zeta(\vartheta-\tau)}\Xi_i^T(\theta)Y_iT_{3i}Y_i\Xi_i(\vartheta)d\vartheta d\varsigma, \\
V_4(\tau,\varepsilon_i(\tau)) &= \eta_2\sum_{i=1}^{N}\int_{-\eta_2}^{0}\int_{\tau+\varsigma}^{\tau} e^{2\zeta(\vartheta-\tau)}\dot{\mu}_i^T(\vartheta)T_{4i}^{-1}\dot{\mu}_i(\vartheta)d\vartheta d\varsigma,
\end{aligned}
$$

$$\begin{aligned}
V_5(\tau,\varepsilon_i(\tau)) &= \delta_1\sum_{i=1}^{N}\int_{-\delta_1}^{0}\int_{\tau+\varsigma}^{\tau}e^{2\zeta(\vartheta-\tau)}\varepsilon_i^T(\vartheta)Y_iT_{5i}Y_i\varepsilon_i(\vartheta)d\vartheta d\varsigma,\\
V_6(\tau,\varepsilon_i(\tau)) &= \delta_2\sum_{i=1}^{N}\int_{-\delta_2}^{0}\int_{\tau+\varsigma}^{\tau}e^{2\delta(\vartheta-\tau)}\mu_i^T(\vartheta)T_{6i}^{-1}\mu_i(\vartheta)d\vartheta d\varsigma,\\
V_7(\tau,\varepsilon_i(\tau)) &= \sum_{i=1}^{N}\int_{\tau-\eta_{1M}}^{\tau}\int_{\rho}^{\tau}\int_{\varsigma}^{\tau}e^{2\delta(\vartheta-\tau)}\dot{\varepsilon}_i^T(\vartheta)Y_iT_{7i}Y_i\dot{\varepsilon}_i(\vartheta)d\vartheta d\varsigma d\rho.
\end{aligned}$$

Taking the derivative of $V(\tau,\varepsilon_i(\tau))$ along the solution of the system (3.5), we have the following:

$$\begin{aligned}
\dot{V}_1(\tau,\varepsilon_i(\tau)) &= 2\sum_{i=1}^{N}\varphi_i^T(\tau)\Bigg[\Psi\varepsilon_i(\tau)+\Psi_{\eta_1}\varepsilon_i(\tau-\eta_1(\tau))+\Psi_{\delta_1}\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma\\
&\quad+\omega_1\sum_{j=1}^{N}\alpha_{ij}A_1\varepsilon_j(\tau)+\omega_2\sum_{j=1}^{N}\beta_{ij}A_2\varepsilon_j(\tau-\eta_1(\tau))\\
&\quad+\omega_3\sum_{j=1}^{N}\gamma_{ij}A_3\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_j(\varsigma)d\varsigma+C_{1i}\mu_i(\tau)+C_{2i}\mu_i(\tau-\eta_2(\tau))\\
&\quad+C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)d\varsigma\Bigg]\\
&= \sum_{i=1}^{N}\varphi_i^T(\tau)\Big[(\Psi+\zeta I)P_i+P_i(\Psi^T+\zeta I)-C_{1i}C_{1i}^T\Big]\varphi_i(\tau)\\
&\quad+2\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\eta_1}P_i\varphi_i(\tau-\eta_1(\tau))\\
&\quad+2\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\delta_1}P_i\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma\\
&\quad+2\omega_1\sum_{i=1}^{N}\varphi_i^T(\tau)\sum_{j=1}^{N}\alpha_{ij}A_1P_i\varphi_j(\tau)\\
&\quad+2\omega_2\sum_{i=1}^{N}\varphi_i^T(\tau)\sum_{j=1}^{N}\beta_{ij}A_2P_i\varphi_j(\tau-\eta_1(\tau))\\
&\quad+2\omega_3\sum_{i=1}^{N}\varphi_i^T(\tau)\sum_{j=1}^{N}\gamma_{ij}A_3P_i\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_j(\varsigma)d\varsigma\\
&\quad+2\sum_{i=1}^{N}\varphi_i^T(\tau)C_{2i}\mu_i(\tau-\eta_2(\tau))\\
&\quad+2\sum_{i=1}^{N}\varphi_i^T(\tau)C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)d\varsigma-2\delta V_1(\tau,\varepsilon_i(\tau)),
\end{aligned}$$

$$
\begin{aligned}
\dot{V}_2(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\Big[\varphi_i^T(\tau)T_{1i}\varphi_i(\tau)-e^{-2\zeta\eta_{1\mathrm{m}}}\varphi_i^T(\tau-\eta_{1\mathrm{m}})T_{1i}\varphi_i(\tau-\eta_{1\mathrm{m}})\Big]\\
&\quad+\sum_{i=1}^{N}\Big[\varphi_i^T(\tau)T_{2i}\varphi_i(\tau)-e^{-2\zeta\eta_{1\mathrm{M}}}\varphi_i^T(\tau-\eta_{1\mathrm{M}})T_{2i}\varphi_i(\tau-\eta_{1\mathrm{M}})\Big]\\
&\quad-2\zeta V_2(\tau,\varepsilon_i(\tau)),\\
\dot{V}_3(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\left[\eta_{1\mathrm{M}}^2\begin{bmatrix}\varphi_i(\tau)\\ \dot{\varphi}_i(\tau)\end{bmatrix}^T T_{3i}\begin{bmatrix}\varphi_i(\tau)\\ \dot{\varphi}_i(\tau)\end{bmatrix}\right]\\
&\quad-\sum_{i=1}^{N}\left[\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\Xi_i^T(\varsigma)Y_iT_{3i}Y_i\Xi_i(\varsigma)d\varsigma\right]\\
&\quad-2\zeta V_3(t,\varepsilon_i(t)),\\
\dot{V}_4(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\left[\eta_2^2\dot{\mu}_i^T(\tau)T_{4i}^{-1}\dot{\mu}_i(\tau)-\eta_2e^{-2\zeta\eta_2}\int_{\tau-\eta_2}^{\tau}\dot{\mu}_i^T(\varsigma)T_{4i}^{-1}\dot{\mu}_i(\varsigma)d\varsigma\right]\\
&\quad-2\zeta V_4(\tau,\varepsilon_i(\tau)),\\
\dot{V}_5(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\left[\delta_1^2\varphi_i^T(\tau)T_{5i}\varphi_i(\tau)-\delta_1e^{-2\zeta\delta_1}\int_{\tau-\delta_1}^{\tau}\varphi_i^T(\varsigma)T_{5i}\varphi_i(\varsigma)d\varsigma\right]\\
&\quad-2\zeta V_5(\tau,\varepsilon_i(\tau)),\\
\dot{V}_6(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\left[\delta_2^2\mu_i^T(\tau)T_{6i}^{-1}\mu_i(\tau)-\delta_2e^{-2\zeta\delta_2}\int_{\tau-\delta_2}^{\tau}\mu_i^T(\varsigma)T_{6i}^{-1}\mu_i(\varsigma)d\varsigma\right]\\
&\quad-2\zeta V_6(\tau,\varepsilon_i(\tau)),\\
\dot{V}_7(\tau,\varepsilon_i(\tau)) &\leq \sum_{i=1}^{N}\left[\dot{\varphi}_i^T(\tau)\left(\frac{\eta_{1\mathrm{M}}^2T_{7i}}{2}\right)\dot{\varphi}_i(\tau)-e^{-2\zeta\eta_{1\mathrm{M}}}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\int_{\varsigma}^{\tau}\dot{\varphi}_i^T(\vartheta)T_{7i}\dot{\varphi}_i(\vartheta)d\vartheta d\varsigma\right]\\
&\quad-2\zeta V_7(\tau,\varepsilon_i(\tau)).
\end{aligned}
\tag{3.7}
$$

By applying lemma 2.3 and lemma 2.5, we receive

$$
\begin{aligned}
2\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\eta_1}P_i\varphi_i(\tau-\eta_1(\tau)) &\leq \frac{1}{\psi_{\eta_1}}\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\eta_1}P_iP_i\Psi_{\eta_1}^T\varphi_i(\tau)\\
&\quad+\psi_{\eta_1}\sum_{i=1}^{N}\varphi_i^T(\tau-\eta_1(\tau))\varphi_i(\tau-\eta_1(\tau)),
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
2\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\delta_1}P_i\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma &\leq \frac{1}{\psi_{\delta_1}}\sum_{i=1}^{N}\varphi_i^T(\tau)\Psi_{\delta_1}P_iP_i\Psi_{\delta_1}^T\varphi_i(\tau) \qquad (3.9)\\
&\quad+\psi_{d_1}\sum_{i=1}^{N}\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)d\varsigma\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma,
\end{aligned}
$$

$$
\begin{aligned}
&2\omega_1 \sum_{i=1}^{N} \varphi_i^T(\tau) \sum_{j=1}^{N} \alpha_{ij} A_1 P_i \varphi_j(\tau) \\
&\qquad \leq \omega_1 \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg[\psi_\alpha \varphi_i^T(\tau) (\alpha_{ij})^2 \varphi_i(\tau) \\
&\qquad\quad + \frac{1}{\psi_\alpha} \varphi_j^T(\tau) P_i A_1^T A_1 P_i \varphi_j(\tau) \Bigg] \\
&\qquad = \sum_{i=1}^{N} \Bigg[\omega_1 \psi_\alpha \Bigg(\sum_{j=1}^{N} (\alpha_{ij})^2 \Bigg) \varphi_i^T(\tau) \varphi_i(\tau) \\
&\qquad\quad + \frac{N\omega_1}{\psi_\alpha} \varphi_i^T(\tau) P_i A_1^T A_1 P_i \varphi_i(\tau) \Bigg],
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&2\omega_2 \sum_{i=1}^{N} \varphi_i^T(\tau) \sum_{j=1}^{N} \beta_{ij} A_2 P_i \varphi_j(\tau - \eta_1(\tau)) \\
&\qquad \leq \omega_2 \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg[\psi_\beta \varphi_i^T(\tau) (\beta_{ij})^2 \varphi_i(\tau) \\
&\qquad\quad + \frac{1}{\psi_\beta} \varphi_j^T(\tau - \eta_1(\tau)) P_i A_2^T A_2 P_i \varphi_j(\tau - \eta_1(\tau)) \Bigg] \\
&\qquad = \sum_{i=1}^{N} \Bigg[\omega_2 \psi_\beta \Bigg(\sum_{j=1}^{N} (\beta_{ij})^2 \Bigg) \varphi_i^T(\tau) \varphi_i(\tau) \\
&\qquad\quad + \frac{N\omega_2}{\psi_\beta} \varphi_i^T(\tau - \eta_1(\tau)) P_i A_2^T A_2 P_i \varphi_i(\tau - \eta_1(\tau)) \Bigg],
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&2\omega_3 \sum_{i=1}^{N} \varphi_i^T(\tau) \sum_{j=1}^{N} \gamma_{ij} A_3 P_i \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_j(\varsigma) d\varsigma \\
&\qquad \leq \omega_3 \sum_{i=1}^{N} \sum_{j=1}^{N} \Bigg[\psi_\gamma \varphi_i^T(\tau) (\gamma_{ij})^2 \varphi_i(\tau) \\
&\qquad\quad + \frac{1}{\psi_\gamma} \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_j^T(\varsigma) d\varsigma \ P_i A_3^T A_3 P_i \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_j(\varsigma) d\varsigma \Bigg] \\
&\qquad = \sum_{i=1}^{N} \Bigg[\omega_3 \psi_\gamma \Bigg(\sum_{j=1}^{N} (\gamma_{ij})^2 \Bigg) \varphi_i^T(\tau) \varphi_i(\tau) \\
&\qquad\quad + \frac{N\omega_3}{\psi_\gamma} \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i^T(\varsigma) d\varsigma \ P_i A_3^T A_3 P_i \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i(\varsigma) d\varsigma \Bigg],
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&2\sum_{i=1}^{N}\varphi_i^T(\tau)C_{2i}\mu_i(\tau-\eta_2(\tau))\\
&\qquad\qquad \le 3e^{2\zeta\eta_2}\sum_{i=1}^{N}\varphi_i^T(\tau)C_{2i}T_{4i}C_{2i}^T\varphi_i(\tau)\\
&\qquad\qquad\quad +\frac{e^{-2\zeta\eta_2}}{3}\sum_{i=1}^{N}\mu_i^T(\tau-\eta_2(\tau))T_{4i}^{-1}\mu_i(\tau-\eta_2(\tau)),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&2\sum_{i=1}^{N}\varphi_i^T(\tau)C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)d\varsigma\\
&\qquad\qquad \le 2e^{2\zeta\delta_2}\sum_{i=1}^{N}\varphi_i^T(\tau)C_{3i}T_{6i}C_{3i}^T\varphi_i(\tau)\\
&\qquad\qquad\quad +\frac{e^{-2\zeta\delta_2}}{2}\sum_{i=1}^{N}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i^T(\varsigma)d\varsigma\ T_{2i}^{-1}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)d\varsigma\\
&\qquad\qquad \le 2e^{2\zeta\delta_2}\sum_{i=1}^{N}\varphi_i^T(\tau)C_{3i}T_{6i}C_{3i}^T\varphi_i(\tau)\\
&\qquad\qquad\quad +\frac{\delta_2(\tau)e^{-2\zeta\delta_2}}{2}\sum_{i=1}^{N}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i^T(\varsigma)T_{6i}^{-1}\mu_i(\varsigma)d\varsigma,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&-\eta_2e^{-2\zeta\eta_2}\sum_{i=1}^{N}\int_{\tau-\eta_2}^{\tau}\dot{\mu}_i^T(\varsigma)T_{4i}^{-1}\dot{\mu}_i(\varsigma)d\varsigma\\
&\qquad\qquad \le -\eta_2(\tau)e^{-2\zeta\eta_2}\sum_{i=1}^{N}\int_{\tau-\eta_2}^{\tau}\dot{\mu}_i^T(\varsigma)T_{4i}^{-1}\dot{\mu}_i(\varsigma)d\varsigma\\
&\qquad\qquad \le \frac{1}{4}e^{-2\zeta\eta_2}\sum_{i=1}^{N}\varphi_i^T(\tau)C_{1i}T_{4i}^{-1}C_{1i}^T\varphi_i(\tau)\\
&\qquad\qquad\quad +\frac{e^{-2\zeta\eta_2}}{3}\sum_{i=1}^{N}\mu_i^T(\tau-\eta_2(\tau))T_{4i}^{-1}\mu_i(\tau-\eta_2(\tau))\\
&\qquad\qquad\quad -e^{-2\zeta\eta_2}\sum_{i=1}^{N}\mu_i^T(\tau-\eta_2(\tau))T_{4i}^{-1}\mu_i(\tau-\eta_2(\tau)),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&-\delta_1e^{-2\zeta\delta_1}\sum_{i=1}^{N}\int_{\tau-\delta_1}^{\tau}\varphi_i^T(\varsigma)\ T_{5i}\ \varphi_i(\varsigma)d\varsigma\\
&\qquad\qquad \le -\delta_1(\tau)e^{-2\zeta\delta_1}\sum_{i=1}^{N}\int_{\tau-\delta_1}^{\tau}\varphi_i^T(\varsigma)\ T_{5i}\ \varphi_i(\varsigma)d\varsigma\\
&\qquad\qquad \le -e^{-2\zeta\delta_1}\sum_{i=1}^{N}\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)d\varsigma\ T_{5i}\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma,
\end{aligned}
\tag{3.16}
$$

$$-\delta_2 e^{-2\zeta\delta_2}\sum_{i=1}^{N}\int_{\tau-\delta_2}^{\tau}\mu_i^T(\varsigma)T_{6i}^{-1}\mu_i(\varsigma)d\varsigma$$
$$\leq -\delta_2(\tau)e^{-2\zeta\delta_2}\sum_{i=1}^{N}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i^T(\varsigma)T_{6i}^{-1}\mu_i(\varsigma)d\varsigma. \tag{3.17}$$

We set

$$T_{3i}=T_{3ai}+T_{3bi}=\begin{bmatrix} T_{3a_1 i} & 0 \\ 0 & R_{3a_2 i}\end{bmatrix}+\begin{bmatrix} T_{3b_1 i} & R_{b_{12}i} \\ * & R_{3b_2 i}\end{bmatrix},\quad i=1,2,\dots,N$$

consider

$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\Xi_i^T(\varsigma)Y_iT_{3i}Y_i\Xi_i(\varsigma)d\varsigma$$
$$=-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\Xi_i^T(\varsigma)Y_iT_{3ai}Y_i\Xi_i(\varsigma)d\varsigma$$
$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\Xi_i^T(\varsigma)Y_iT_{3bi}Y_i\Xi_i(\varsigma)d\varsigma,$$

using lemma 2.5 and lemma 2.6 as $\Lambda_{1i},\Lambda_{2i}>0$, the T_{3ai} -dependent term can be estimated as

$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\Xi_i^T(\varsigma)Y_iT_{3ai}Y_i\Xi_i(\varsigma)d\varsigma$$
$$=-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)T_{3a_1 i}\ \varphi_i(\varsigma)d\varsigma$$
$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\varphi_i^T(\varsigma)T_{3a_1 i}\ \varphi_i(\varsigma)d\varsigma$$
$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\dot\varphi_i^T(\varsigma)T_{3a_2 i}\ \dot\varphi_i(\varsigma)d\varsigma$$
$$-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\dot\varphi_i^T(\varsigma)T_{3a_2 i}\ \dot\varphi_i(\varsigma)d\varsigma$$
$$\leq \eta_1(\tau)\sum_{i=1}^{N}\xi^T(\tau)\Big[\big(g_6\big)^T\big[-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}T_{3a_1 i}\big]\big(g_6\big)\Big]\xi(\tau)$$
$$+(\eta_{1\mathrm{M}}-\eta_1(\tau))\sum_{i=1}^{N}\xi^T(\tau)\Big[\big(g_7\big)^T\big[-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}T_{3a_1 i}\big]\big(g_7\big)\Big]\xi(\tau)$$

$$+\eta_1(\tau)\sum_{i=1}^{N}\ \xi^T(\tau)\left[\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}^T\left[\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}\big(N_{11i}+\frac{N_{13i}}{3}\big)\right]\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}\right]\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}^T\big(\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}M_{11i}\big)(g_1-g_2)\right\}\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1\\ g_2\\ g_6\end{bmatrix}^T\big(\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}M_{12i}\big)(2g_6-g_1-g_2)\right\}\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\ \xi^T(\tau)\left[\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}^T\left[\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}\big(N_{21i}+\frac{N_{23i}}{3}\big)\right]\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}\right]\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}^T\big(\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}M_{21i}\big)(g_2-g_4)\right\}\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_2\\ g_4\\ g_7\end{bmatrix}^T\big(\eta_{1\mathrm{M}}e^{-2\varsigma\eta_{1\mathrm{M}}}M_{22i}\big)(2g_7-g_2-g_4)\right\}\xi(\tau).$$

For $i=1,2,\ldots,N$, we get

$$\begin{aligned}
2\int_{\tau-\eta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)T_{3ci}\ \dot{\varphi}_i(\varsigma)d\varsigma &= \varphi_i^T(\tau)T_{3ci}\ \varphi_i(\tau)-\varphi_i^T(\tau-\eta_1(\tau))T_{3ci}\ \varphi_i(\tau-\eta_1(\tau)),\\
-2\int_{\tau-\eta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)T_{3ci}\ \dot{\varphi}_i(\varsigma)d\varsigma &= -\int_{\tau-\eta_1(\tau)}^{\tau}\Xi_i^T(\varsigma)Y_i\begin{bmatrix}0 & T_{3ci}\\ * & 0\end{bmatrix}Y_i\ \Xi_i(\varsigma)d\varsigma.
\end{aligned}$$

The following free weighting matrix-based zero-value term is added to $T_{3\mathrm{b}i}$-dependent term, $i=1,2,\ldots,N$,

$$\begin{aligned}
0 \;=\;& \varphi_i^T(\tau)T_{3ci}\ \varphi_i(\tau)-\varphi_i^T(\tau-\eta_1(\tau))T_{3ci}\ \varphi_i(\tau-\eta_1(\tau))\\
&-2\int_{\tau-\eta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)T_{3ci}\ \dot{\varphi}_i(\varsigma)d\varsigma\\
&+\varphi_i^T(\tau-\eta_1(\tau))T_{3di}\ \varphi_i(\tau-\eta_1(\tau))-\varphi_i^T(\tau-\eta_{1\mathrm{M}})T_{3di}\ \varphi_i(\tau-\eta_{1\mathrm{M}})\\
&-2\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\varphi_i^T(\varsigma)T_{3di}\ \dot{\varphi}_i(\varsigma)d\varsigma,
\end{aligned}$$

T_{3bi} -dependent term can be estimated as

$$\begin{aligned}
&-\eta_{1\mathrm{M}} e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_{1\mathrm{M}}}^{\tau} \Xi_i^T(\varsigma) Y_i T_{3\mathrm{b}i} Y_i \, \Xi_i(\varsigma) d\varsigma \\
= \; & -\eta_{1\mathrm{M}} e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} \Xi_i^T(\varsigma) Y_i T_{3\mathrm{b}i} Y_i \, \Xi_i(\varsigma) d\varsigma \\
& -\eta_{1\mathrm{M}} e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)} \Xi_i^T(\varsigma) Y_i T_{3\mathrm{b}i} Y_i \, \Xi_i(\varsigma) d\varsigma \\
= \; & -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} \Xi_i^T(\varsigma) Y_i \begin{bmatrix} \eta_{1\mathrm{M}} T_{3\mathrm{b}_1 i} & \eta_{1\mathrm{M}} T_{3\mathrm{b}_{12} i} \\ * & \eta_{1\mathrm{M}} T_{3\mathrm{b}_2 i} \end{bmatrix} Y_i \, \Xi_i(\varsigma) d\varsigma \\
& -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)} \Xi_i^T(\varsigma) Y_i \begin{bmatrix} \eta_{1\mathrm{M}} T_{3\mathrm{b}_1 i} & \eta_{1\mathrm{M}} T_{3\mathrm{b}_{12} i} \\ * & \eta_{1\mathrm{M}} T_{3\mathrm{b}_2 i} \end{bmatrix} Y_i \, \Xi_i(\varsigma) d\varsigma \\
= \; & e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \varphi_i^T(\tau) T_{3ci} \, \varphi_i(\tau) - e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \varphi_i^T(\tau-\eta_{1\mathrm{M}}) T_{3di} \, \varphi_i(\tau - h_{1\mathrm{M}}) \\
& + e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \varphi_i^T(\tau-\eta_1(\tau)) \big[T_{3di} - T_{3ci} \big] \varphi_i(\tau-\eta_1(\tau)) \\
& -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \left[\int_{\tau-\eta_1(\tau)}^{\tau} \Xi_i^T(\varsigma) Y_i \, \Gamma_{1i} \, Y_i \Xi_i(\varsigma) d\varsigma - \int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)} \Xi_i^T(\varsigma) Y_i \, \Gamma_{2i} \, Y_i \Xi_i(\varsigma) d\varsigma \right],
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{1i} &= \begin{bmatrix} \eta_{1\mathrm{M}} T_{3\mathrm{b}_1 i} & \eta_{1\mathrm{M}} T_{3\mathrm{b}_{12} i} + T_{3ci} \\ * & \eta_{1\mathrm{M}} T_{3\mathrm{b}_2 i} \end{bmatrix}, \\
\Gamma_{2i} &= \begin{bmatrix} \eta_{1\mathrm{M}} T_{3\mathrm{b}_1 i} & \eta_{1\mathrm{M}} T_{3\mathrm{b}_{12} i} + T_{3di} \\ * & \eta_{1\mathrm{M}} T_{3\mathrm{b}_2 i} \end{bmatrix},
\end{aligned}$$

consider

$$\begin{aligned}
& -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} \Xi_i^T(\varsigma) Y_i \, \Gamma_{1i} \, Y_i \Xi_i(\varsigma) d\varsigma \\
& \qquad = -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} \varphi_i^T(\varsigma) \big(\eta_{1\mathrm{M}} R_{\mathrm{b}_1 i}\big) \varphi_i(\varsigma) d\varsigma \\
& \qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} 2\varphi_i^T(\varsigma) \big(\eta_{1\mathrm{M}} T_{3\mathrm{b}_{12} i} + T_{3ci}\big) \dot{\varphi}_i(\varsigma) d\varsigma \\
& \qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}} \sum_{i=1}^{N} \int_{\tau-\eta_1(\tau)}^{\tau} \dot{\varphi}_i^T(\varsigma) \big(\eta_{1\mathrm{M}} T_{3\mathrm{b}_2 i}\big) \dot{\varphi}_i(\varsigma) d\varsigma,
\end{aligned}$$

and

$$
\begin{aligned}
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\Xi_i^T(\varsigma)Y_i\ \Gamma_{2i}\ Y_i\Xi_i(\varsigma)d\varsigma\\
&\qquad\qquad = -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\varphi_i^T(\varsigma)\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_1 i}\big)\varphi_i(\varsigma)d\varsigma\\
&\qquad\qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}2\varphi_i^T(\varsigma)\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12} i}+T_{3di}\big)\dot{\varphi}_i(\varsigma)d\varsigma\\
&\qquad\qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\dot{\varphi}_i^T(\varsigma)\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_2 i}\big)\dot{\varphi}_i(\varsigma)d\varsigma,
\end{aligned}
$$

using lemma 2.5 to estimate Γ_{1i}-dependent term and Γ_{2i}-dependent term yields

$$
\begin{aligned}
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\Xi_i^T(\varsigma)Y_i\ \Gamma_{1i}\ Y_i\Xi_i(\varsigma)d\varsigma\\
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\Xi_i^T(\varsigma)Y_i\ \Gamma_{2i}\ Y_i\Xi_i(\varsigma)d\varsigma\\
&\quad = -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\xi^T(\tau)\Big[\eta_1(\tau)\big[g_6^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_1 i}\big)g_6\big]\Big]\xi(\tau)\\
&\qquad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\xi^T(\tau)\Big[\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\big[g_7^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_1 i}\big)g_7\big]\Big]\xi(\tau)\\
&\qquad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\xi^T(\tau)\Big[\big[g_1^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12} i}+T_{3ci}\big)g_1\big]+\big[g_2^T\big(T_{3di}-T_{3ci}\big)g_2\big]\Big]\xi(\tau)\\
&\qquad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\xi^T(\tau)\Big[\big[g_4^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12} i}+T_{3di}\big)g_4\big]\Big]\xi(\tau)\\
&\qquad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\frac{\eta_{1\mathrm{M}}}{\eta_1(\tau)}\Big(\varphi_i(\tau)-\varphi_i(\tau-\eta_1(\tau))\Big)^T T_{3\mathrm{b}_2 i}\\
&\qquad \times\Big(\varphi_i(\tau)-\varphi_i(\tau-\eta_1(\tau))\Big)\\
&\qquad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\frac{\eta_{1\mathrm{M}}}{(\eta_{1\mathrm{M}}-\eta_1(\tau))}\Big(\varphi_i(\tau-\eta_1(\tau))-\varphi_i(\tau-\eta_{1\mathrm{M}}\Big)^T T_{3\mathrm{b}_2 i}\\
&\qquad \times\Big(\varphi_i(\tau-\eta_1(\tau))-\varphi_i(\tau-\eta_{1\mathrm{M}}\Big),
\end{aligned}
$$

by lemma 2.7, we have

$$
\begin{aligned}
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\frac{\eta_{1\mathrm{M}}}{\eta_1(\tau)}\Big(\varphi_i(\tau)-\varphi_i(\tau-\eta_1(\tau))\Big)^T T_{3\mathrm{b}_2 i}\Big(\varphi_i(\tau)-\varphi_i(\tau-\eta_1(\tau))\Big)\\
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\frac{\eta_{1\mathrm{M}}}{(\eta_{1\mathrm{M}}-\eta_1(\tau))}\Big(\varphi_i(\tau-\eta_1(\tau))-\varphi_i(\tau-\eta_{1\mathrm{M}}\Big)^T\\
&\times T_{3\mathrm{b}_2 i}\Big(\varphi_i(\tau-\eta_1(\tau))-\varphi_i(\tau-\eta_{1\mathrm{M}}\Big)\\
&\qquad\le -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\xi^T(\tau)\Big(\begin{bmatrix} g_1-g_2\\ g_2-g_4\end{bmatrix}^T\\
&\qquad\times\begin{bmatrix} T_{3\mathrm{b}_2 i}+\frac{\left(\eta_{1\mathrm{M}}-\eta_1(\tau)\right)}{\eta_{1\mathrm{M}}}X_{\mathrm{a}_1 i} & \frac{\left(\eta_{1\mathrm{M}}-\eta_1(\tau)\right)}{\eta_{1\mathrm{M}}}X_{\mathrm{b}_1 i}+\frac{\eta_1(\tau)}{\eta_{1\mathrm{M}}}X_{\mathrm{b}_2 i}\\ * & T_{3\mathrm{b}_2 i}+\frac{\eta_1(\tau)}{\eta_{1\mathrm{M}}}X_{\mathrm{a}_2 i}\end{bmatrix}\\
&\qquad\times\begin{bmatrix} g_1-g_2\\ g_2-g_4\end{bmatrix}\Big)\xi^T(\tau),
\end{aligned}
$$

where $X_{\mathrm{a}_1 i}=T_{3\mathrm{b}_2 i}-X_{\mathrm{b}_2 i}T_{3\mathrm{b}_2 i}^{-1}X_{\mathrm{b}_2 i}^T,\qquad X_{\mathrm{a}_2 i}=T_{3\mathrm{b}_2 i}-X_{\mathrm{b}_1 i}^T T_{3\mathrm{b}_2 i}X_{\mathrm{b}_1 i},$

and consider $W_{1i}-$dependent term

$$
\begin{aligned}
&-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau}\int_{\varsigma}^{\tau}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma,\\
&\qquad= -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\int_{\varsigma}^{\tau}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma\\
&\qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\int_{\tau-\eta_1(\tau)}^{\tau}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma\\
&\qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\int_{\varsigma}^{\tau-\eta_1(\tau)}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma\\
&\qquad= -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\int_{\varsigma}^{\tau}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma\\
&\qquad\quad -\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\dot\varphi_i^T(\varsigma)T_{7i}\dot\varphi_i(\varsigma)d\varsigma\\
&\qquad\quad -e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\int_{\varsigma}^{\tau-\eta_1(\tau)}\dot\varphi_i^T(\vartheta)T_{7i}\dot\varphi_i(\vartheta)d\vartheta d\varsigma.
\end{aligned}
$$

Using lemma 2.6 and as $\Lambda_{3i}>0$, we have

$$
-\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\dot\varphi_i^T(\varsigma)T_{7i}\dot\varphi_i(\varsigma)d\varsigma
$$

$$\leq \quad \eta_1(\tau)\sum_{i=1}^{N} \xi^T(\tau)\left[\left[\begin{array}{c} g_1 \\ g_2 \\ g_6 \end{array}\right]^T \left[\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\big(N_{31i}+\frac{N_{33i}}{3}\big)\right]\left[\begin{array}{c} g_1 \\ g_2 \\ g_6 \end{array}\right]\right]\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\left[\begin{array}{c} g_1 \\ g_2 \\ g_6 \end{array}\right]^T \big(e^{-2\zeta\eta_{1\mathrm{M}}}M_{31i}\big)(g_1-g_2)\right\}\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\left[\begin{array}{c} g_1 \\ g_2 \\ g_6 \end{array}\right]^T \big(e^{-2\zeta\eta_{1\mathrm{M}}}M_{32i}\big)(2g_6-g_1-g_2)\right\}\xi(\tau),$$

using lemma 2.5,

$$-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\int_{\varsigma}^{\tau}\dot{\varphi}_i^T(\vartheta)T_{7i}\dot{\varphi}_i(\vartheta)d\vartheta d\varsigma$$

$$\leq -\frac{2}{\eta_1(\tau)^2}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\int_{\varsigma}^{\tau}\dot{\varphi}_i^T(\vartheta)d\vartheta d\varsigma\big(e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\big)\int_{\tau-\eta_1(\tau)}^{\tau}\int_{\varsigma}^{\tau}\dot{\varphi}_i^T(\vartheta)d\vartheta d\varsigma$$

$$= -\frac{2}{\eta_1(\tau)^2}\sum_{i=1}^{N}\int_{\tau-\eta_1(\tau)}^{\tau}\Big[\varphi_i^T(\tau)-\varphi_i^T(\varsigma)\Big]d\varsigma\big(e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\big)\int_{\tau-\eta_1(\tau)}^{\tau}\Big[\varphi_i^T(\tau)-\varphi_i^T(\varsigma)\Big]d\varsigma$$

$$= -\sum_{i=1}^{N}\xi^T(\tau)\Big[\big(g_1-g_6\big)^T\big(2e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\big)\big(g_1-g_6\big)\Big]\xi(\tau),$$

$$-e^{-2\zeta\eta_{1\mathrm{M}}}\sum_{i=1}^{N}\int_{\tau-\eta_{1\mathrm{M}}}^{\tau-\eta_1(\tau)}\int_{\varsigma}^{\tau-\eta_1(\tau)}\dot{\varphi}_i^T(\vartheta)T_{7i}\dot{\varphi}_i(\vartheta)d\vartheta d\varsigma$$

$$\leq -\sum_{i=1}^{N}\xi^T(\tau)\Big[\big(g_2-g_7\big)^T\big(2e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\big)\big(g_3-g_7\big)\Big]\xi(\tau).$$

For $i=1,2,\ldots,N$, the following identity equations present

$$\begin{aligned} 0 \quad = \quad & \sum_{i=1}^{N}\xi^T(\tau)\Big[\dot{\varepsilon}_i(\tau)-\Psi\varepsilon_i(\tau)-\Psi_{\eta_1}e_i(\tau-\eta_1(\tau))-\Psi_{\delta_1}\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma\Big]\xi(\tau) \\ & \sum_{i=1}^{N}\xi^T(\tau)\Big[-\omega_1\sum_{j=1}^{N}\alpha_{ij}A_1\varepsilon_j(\tau)-\omega_2\sum_{j=1}^{N}\beta_{ij}A_2\varepsilon_j(\tau-\eta_1(\tau))\Big]\xi(\tau) \\ & \sum_{i=1}^{N}\xi^T(\tau)\Big[-\omega_3\sum_{j=1}^{N}\gamma_{ij}A_3\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_j(\varsigma)d\varsigma\Big]\xi(\tau) \\ & \sum_{i=1}^{N}\xi^T(\tau)\Big[-C_{1i}\mu_i(\tau)-C_{2i}\mu_i(\tau-\eta_2(\tau))-C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)\Big]\xi(\tau), \end{aligned}$$

$$
\begin{aligned}
0 \;=\; & \sum_{i=1}^{N} \xi^T(\tau)\Big[-2\dot{\varphi}_i(\tau)P\dot{\varphi}_i(\tau)+2\dot{\varphi}_i(\tau)\Psi P\varphi_i(\tau)+2\dot{\varphi}_i(\tau)\Psi_{\eta_1}P\varphi_i(\tau-\eta_1(\tau))\Big]\xi(\tau)\\
& \sum_{i=1}^{N} \xi^T(\tau)\Big[+2\dot{\varphi}_i(\tau)\Psi_{\delta_1}P\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma+2\dot{\varphi}_i(\tau)\omega_1\sum_{j=1}^{N}\alpha_{ij}A_1P\varphi_j(\tau)\Big]\xi(\tau)\\
& \sum_{i=1}^{N} \xi^T(\tau)\Big[+2\dot{\varphi}_i(\tau)\omega_2\sum_{j=1}^{N}\beta_{ij}A_2P\varphi_j(\tau-\eta_1(\tau))\Big]\xi(\tau)\\
& \sum_{i=1}^{N} \xi^T(\tau)\Big[+2\dot{\varphi}_i(\tau)\omega_3\sum_{j=1}^{N}\gamma_{ij}A_3P\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_j(\varsigma)d\varsigma\Big]\xi(\tau)+2\dot{\varphi}_i(\tau)C_{1i}\mu_i(\tau)\\
& \sum_{i=1}^{N} \xi^T(\tau)\Big[+2\dot{\varphi}_i(\tau)C_{2i}\mu_i(\tau-\eta_2(\tau))+2\dot{\varphi}_i(\tau)C_{3i}\int_{\tau-\delta_2(\tau)}^{\tau}\mu_i(\varsigma)\Big]\xi(\tau), \qquad (3.18)
\end{aligned}
$$

from (3.18), we have

$$
\begin{aligned}
& 2\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi P_i\varphi_i(\tau)\\
& \qquad \le \frac{1}{\psi}\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi P_iP_i\Psi^T\dot{\varphi}_i(\tau)\\
& \qquad\quad +\psi\sum_{i=1}^{N}\varphi_i^T(\tau)\varphi_i(\tau), \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
& 2\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi_{\eta_1}P_i\varphi_i(\tau-\eta_1(\tau))\\
& \qquad \le \frac{1}{\psi_{\eta_1}}\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi_{\eta_1}P_iP_i\Psi_{\eta_1}^T\dot{\varphi}_i(\tau)\\
& \qquad\quad +\psi_{\eta_1}\sum_{i=1}^{N}\varphi_i^T(\tau-\eta_1(\tau))\varphi_i(\tau-\eta_1(\tau)), \qquad (3.20)
\end{aligned}
$$

$$
\begin{aligned}
& 2\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi_{\delta_1}P_i\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma\\
& \qquad \le \frac{1}{\psi_{\delta_1}}\sum_{i=1}^{N}\dot{\varphi}_i^T(\tau)\Psi_{\delta_1}P_iP_i\Psi_{\delta_1}^T\dot{\varphi}_i(\tau)\\
& \qquad\quad +\psi_{\delta_1}\sum_{i=1}^{N}\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i^T(\varsigma)d\varsigma\int_{\tau-\delta_1(\tau)}^{\tau}\varphi_i(\varsigma)d\varsigma, \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
&2\omega_1 \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) \sum_{j=1}^{N} \alpha_{ij} A_1 P_i \varphi_j(\tau) \\
&\qquad \le \sum_{i=1}^{N} \Bigg[\omega_1 \psi_\alpha \Bigg(\sum_{j=1}^{N} (\alpha_{ij})^2 \Bigg) \dot{\varphi}_i^T(\tau) \dot{\varphi}_i(\tau) \\
&\qquad\quad + \frac{N\omega_1}{\psi_\alpha} \varphi_i^T(\tau) P_i A_1^T A_1 P_i \varphi_i(\tau) \Bigg],
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&2\omega_2 \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) \sum_{j=1}^{N} \beta_{ij} A_2 P_i \varphi_j(\tau - \eta_1(\tau)) \\
&\qquad \le \sum_{i=1}^{N} \Bigg[\omega_2 \psi_\beta \Bigg(\sum_{j=1}^{N} (\beta_{ij})^2 \Bigg) \dot{\varphi}_i^T(\tau) \dot{\varphi}_i(\tau) \\
&\qquad\quad + \frac{N\omega_2}{\psi_\beta} \varphi_i^T(\tau - \eta_1(\tau)) P_i A_2^T A_2 P_i \varphi_i(\tau - \eta_1(\tau)) \Bigg],
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&2\omega_3 \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) \sum_{j=1}^{N} \gamma_{ij} A_3 P_i \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_j(\varsigma) d\varsigma \\
&\qquad \le \sum_{i=1}^{N} \Bigg[\omega_3 \psi_\gamma \Bigg(\sum_{j=1}^{N} (\gamma_{ij})^2 \Bigg) \dot{\varphi}_i^T(\tau) \dot{\varphi}_i(\tau) \\
&\qquad\quad + \frac{N\omega_3}{\psi_\gamma} \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i^T(\varsigma) d\varsigma \; P_i A_3^T A_3 P_i \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i(\varsigma) d\varsigma \Bigg],
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
&2 \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) C_{2i} \mu_i(\tau - \eta_2(\tau)) \\
&\qquad \le 3e^{2\zeta\eta_2} \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) C_{2i} T_{4i} C_{2i}^T \dot{\varphi}_i(\tau) \\
&\qquad\quad + \frac{e^{-2\zeta\eta_2}}{3} \sum_{i=1}^{N} \mu_i^T(\tau - \eta_2(\tau)) T_{4i}^{-1} \mu_i(\tau - \eta_2(\tau)),
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
&2 \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) C_{3i} \int_{\tau-\delta_2(\tau)}^{\tau} \mu_i(\varsigma) d\varsigma \\
&\qquad \le 2e^{2\zeta\delta_2} \sum_{i=1}^{N} \dot{\varphi}_i^T(\tau) C_{3i} T_{6i} C_{3i}^T \dot{\varphi}_i(\tau) \\
&\qquad\quad + \frac{\delta_2(\tau) e^{-2\zeta\delta_2}}{2} \sum_{i=1}^{N} \int_{\tau-\delta_2(\tau)}^{\tau} \mu_i^T(\varsigma) T_{6i}^{-1} \mu_i(\varsigma) d\varsigma,
\end{aligned} \tag{3.26}
$$

and we let,

$$
\begin{aligned}
&\sum_{i=1}^{N} L_i\Big(\tau, \varepsilon_i(\tau), \varepsilon_i(\tau-\eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma, \mu_i(\tau)\Big) \\
&\quad = \sum_{i=1}^{N} \Big[\varepsilon_i^T(\tau)\Phi_{11i}\varepsilon_i(\tau) + \varepsilon_i^T(\tau-\eta_1(\tau))\Phi_{22i}\varepsilon_i(\tau-\eta_1(\tau)) \\
&\qquad + \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i^T(\varsigma)d\varsigma\Phi_{33i}\int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma + \mu_i^T(\tau)\Phi_{44i}\mu_i(\tau)\Big] \\
&\quad \leq \sum_{i=1}^{N} \Big[\varphi_i^T(\tau)P\Phi_{11i}P\varphi_i(\tau) + \varphi_i^T(\tau-\eta_1(\tau))P\Phi_{22i}P\varphi_i(\tau-\eta_1(\tau)) \\
&\qquad + \int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i^T(\varsigma)P\Phi_{33i}P\int_{\tau-\delta_1(\tau)}^{\tau} \varphi_i(\varsigma) + \frac{1}{4}\varphi^T(\tau)C_1\Phi_{44i}C_1\varphi_i(\tau)\Big], \qquad (3.27)
\end{aligned}
$$

from (3.7) to (3.27), we have

$$
\begin{aligned}
\dot{V}(\tau, \varepsilon_i(\tau)) \quad \leq \quad & -2\zeta V(\tau, \varepsilon_i(\tau)) \\
& + \sum_{i=1}^{N} \xi^T(\tau)g_1^T\Big[(\Psi+\zeta I)P_i + P_i(\Psi^T+\zeta I) - C_{1i}C_{1i}^T\Big]g_1\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[-(g_8^TP_ig_8) + g_8^T(\frac{1}{\psi}\Psi P_iP_i\Psi^T)g_8\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[+g_1^T(\psi I)g_1 + g_2^T(2\psi_{\eta_1}I)g_2 + g_5^T(2\psi_{\delta_1}I)g_5\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[g_1^T(\frac{1}{\psi_{\eta_1}}\Psi_{\eta_1}P_iP_i\Psi_{\eta_1}^T)g_1 + g_8^T(\frac{1}{\psi_{\eta_1}}\Psi_{\eta_1}P_iP_i\Psi_{\eta_1}^T)g_8\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[g_1^T(\frac{1}{\psi_{\delta_1}}\Psi_{\delta_1}P_iP_i\Psi_{\delta_1}^T)g_1 + g_8^T(\frac{1}{\psi_{\delta_1}}\Psi_{\delta_1}P_iP_i\Psi_{\delta_1}^T)g_8\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[g_1^TT_{1i}g_1 - e^{-2\zeta\eta_{1\mathrm{m}}}g_3^TT_{1i}g_3^T\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[g_1^TT_{2i}g_1 - e^{-2\zeta\eta_{1\mathrm{M}}}g_4^TT_{2i}g_4^T\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)g_1^T\Big(3e^{2\zeta\eta_2}C_{2i}T_{4i}C_{2i}^T + 2e^{2\zeta\delta_2}C_{3i}T_{6i}C_{3i}^T\Big)g_1\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[\frac{1}{4}e^{-2\zeta\eta_2}g_1^T(C_{1i}T_{4i}^{-1}C_{1i}^T)g_1\Big]\xi(\tau) \\
& + \sum_{i=1}^{N} \xi^T(\tau)\Big[g_1^T(\delta_1^2T_{5i})g_1 - e^{-2\zeta\delta_1}\left[g_5^TT_{5i}g_5\right]\Big]\xi(\tau)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\xi^T(\tau)\left[\frac{1}{4}\delta_2^2\ g_1^T(C_{1i}T_{6i}^{-1}C_{1i}^T)g_1\right]\xi(t)+\sum_{i=1}^{N}\xi^T(\tau)\left[g_8^T\left(\frac{\eta_{1\mathrm{M}}^2T_{7i}}{2}\right)g_8\right]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[\left(g_1-g_6\right)^T\left(2e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\right)\left(g_1-g_6\right)\right]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[\left(g_2-g_7\right)^T\left(2e^{-2\zeta\eta_{1\mathrm{M}}}T_{7i}\right)\left(g_2-g_7\right)\right]\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)g_8^T\left(3e^{2\zeta\eta_2}C_{2i}T_{4i}C_{2i}^T+2e^{2\zeta\delta_2}C_{3i}T_{6i}C_{3i}^T\right)g_8\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[g_1^T(P_i\Phi_{11i}P_i)g_1+g_2^T(P_i\Phi_{22i}P_i)g_2+g_5^T(P_i\Phi_{33i}P_i)g_5\right]\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[g_1^T(\frac{1}{4}C_{1i}\Phi_{44i}C_1i)g_1\right]\xi(\tau),\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[\left[g_1^T(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12}i}+T_{3ci})g_1\right]+\left[g_2^T(T_{3di}-T_{3ci})g_2\right]\right]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[g_4^T(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12}i}+T_{3di})g_4\right]\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[g_1^T\omega_1\psi_\alpha(\sum_{j=1}^{N}(\alpha_{ij})^2)Ig_1+g_1^T\left(2\frac{N\omega_1}{\psi_\alpha}P_iA_1^TA_1P_i\right)g_1\right]\xi(t)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[g_1^T\omega_2\psi_\beta(\sum_{j=1}^{N}(\beta_{ij})^2)Ig_1+g_2^T\left(2\frac{N\omega_2}{\psi_\beta}P_iA_2^TA_2P_i\right)g_2\right]\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[g_1^T\omega_3\psi_\gamma(\sum_{j=1}^{N}(\gamma_{ij})^2)Ig_1+g_5^T\left(2\frac{N\omega_3}{\psi_\gamma}P_iA_3^TA_3P_i\right)g_5\right]\xi^T(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\eta_{1\mathrm{M}}^2\begin{bmatrix}g_1\\g_8\end{bmatrix}^TT_{3i}\begin{bmatrix}g_1\\g_8\end{bmatrix}\xi(\tau)+\sum_{i=1}^{N}\xi^T(\tau)\left[\frac{1}{4}\eta_2^2\ g_2^T(C_{1i}T_{4i}^{-1}C_{1i}^T)g_2\right]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[\left[g_1^T(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12}i}+T_{3ci})g_1\right]+\left[g_4^T(\eta_{1\mathrm{M}}T_{3\mathrm{b}_{12}i}+T_{3di})g_4\right]\right]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\left[\left[g_2^T(T_{3di}-T_{3ci})g_2\right]\right]\xi(\tau)\\
&=\sum_{i=1}^{N}\xi^T(\tau)\left[e^{-2\zeta\eta_{1\mathrm{M}}}\left[g_1^TT_{3ci}\ g_1\right]-e^{-2\zeta\eta_{1\mathrm{M}}}\left[g_4^TT_{3di}\ g_4\right]\right]\xi(\tau)\\
&+\sum_{i=1}^{N}\xi^T(\tau)\left[e^{-2\zeta\eta_{1\mathrm{M}}}\left[g_2^T(T_{3di}-T_{3ci})g_2\right]\right]\xi(\tau)
\end{aligned}
$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}M_{11i}\big)(g_1-g_2)\right\}\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}M_{12i}\big)(2g_6-g_1-g_2)\right\}\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix}^T \big(\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}M_{21i}\big)(g_2-r_4)\right\}\xi(\tau)$$

$$+\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix}^T \big(\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}M_{22i}\big)(2g_7-g_2-g_4)\right\}\xi(\tau)$$

$$+\eta_1(\tau)\sum_{i=1}^{N}\ \xi^T(\tau)\left[\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \left[\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\big(N_{11i}+\frac{N_{13i}}{3}\big)\right]\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}\right]\xi(\tau)$$

$$+\eta_1(\tau)\sum_{i=1}^{N}\ \xi^T(\tau)\left[\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \left[\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\big(N_{31i}+\frac{N_{33i}}{3}\big)\right]\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}\right]\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\ \xi^T(\tau)\left[\begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix}^T \left[\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}\big(N_{21i}+\frac{N_{23i}}{3}\big)\right]\begin{bmatrix} g_2 \\ g_4 \\ g_7 \end{bmatrix}\right]\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(e^{-2\zeta\eta_{1\mathrm{M}}}M_{31i}\big)(g_1-g_2)\right\}\xi(\tau)$$

$$+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\xi^T(\tau)\ \mathrm{sym}\left\{\begin{bmatrix} g_1 \\ g_2 \\ g_6 \end{bmatrix}^T \big(e^{-2\zeta\eta_{1\mathrm{M}}}M_{32i}\big)(2g_6-g_1-g_2)\right\}\xi(\tau)$$

$$+\eta_1(\tau)\sum_{i=1}^{N}\xi^T(\tau)\left[g_6^T\big[-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}T_{3\mathrm{a}_1 i}\big]g_6\right]\xi(\tau)$$

$$
\begin{aligned}
&-\eta_1(\tau)\sum_{i=1}^{N}\xi^T(\tau)\Big[e^{-2\zeta\eta_{1\mathrm{M}}}\big[g_6^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_1 i}\big)g_6\big]\Big]\xi(\tau)\\
&+(\eta_{1\mathrm{M}}-\eta_1(\tau))\sum_{i=1}^{N}\xi^T(\tau)\Big[g_7^T\big[-\eta_{1\mathrm{M}}e^{-2\zeta\eta_{1\mathrm{M}}}T_{3\mathrm{a}_1 i}\big]g_7\Big]\xi(\tau)\\
&-\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\sum_{i=1}^{N}\xi^T(\tau)\Big[\big[g_7^T\big(\eta_{1\mathrm{M}}T_{3\mathrm{b}_1 i}\big)g_7\big]\Big]\xi(\tau)\\
&-\sum_{i=1}^{N}\xi^T(\tau)\Bigg[e^{-2\zeta\eta_{1\mathrm{M}}}\begin{bmatrix} g_1-g_2\\ g_2-g_4\end{bmatrix}^T\\
&\times\begin{bmatrix} T_{3\mathrm{b}_2 i}+\frac{\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)}{\eta_{1\mathrm{M}}}X_{\mathrm{a}_1 i} & \frac{\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)}{\eta_{1\mathrm{M}}}X_{\mathrm{b}_1 i}+\frac{\eta_1(\tau)}{\eta_{1\mathrm{M}}}X_{\mathrm{b}_2 i}\\ * & T_{3\mathrm{b}_2 i}+\frac{\eta_1(\tau)}{\eta_{1\mathrm{M}}}X_{\mathrm{a}_2 i}\end{bmatrix}\\
&\times\begin{bmatrix} g_1-g_2\\ g_2-g_4\end{bmatrix}\Bigg]\xi^T(\tau),
\end{aligned}\tag{3.28}
$$

where

$$
\begin{aligned}
&\Omega_i(\eta(\tau))-L_i(\tau,\varepsilon_i(\tau),\varepsilon_i(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma,\mu_i(\tau)\\
&\quad=\Omega_{0i}+\eta_1(\tau)\Omega_{11i}+\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)\Omega_{12i}\\
&\qquad+\frac{e^{-2\zeta\eta_{1\mathrm{M}}}}{\eta_{1\mathrm{M}}}\big(\eta_{1\mathrm{M}}-\eta_1(\tau)\big)(g_1-g_2)^TX_{\mathrm{b}_2 i}T_{3\mathrm{b}_2 i}^{-1}X_{\mathrm{b}_2 i}^T(g_1-g_2)\\
&\qquad+\frac{e^{-2\zeta\eta_{1\mathrm{M}}}}{\eta_{1\mathrm{M}}}\big(\eta_1(\tau)\big)(g_2-g_4)^TX_{\mathrm{b}_1 i}T_{3\mathrm{b}_2 i}^{-1}X_{\mathrm{b}_1 i}^T(g_2-g_4),
\end{aligned}\tag{3.29}
$$

$$
\begin{aligned}
X_{\mathrm{a}_1 i}&=T_{3\mathrm{b}_2 i}-X_{\mathrm{b}_2 i}T_{3\mathrm{b}_2 i}^{-1}X_{\mathrm{b}_2 i}^T,\\
X_{\mathrm{a}_2 i}&=T_{3\mathrm{b}_2 i}-X_{\mathrm{b}_1 i}^TT_{3\mathrm{b}_2 i}X_{\mathrm{b}_1 i},\quad i=1,2,\dots,N.
\end{aligned}
$$

It is easy to check that

$$
\sum_{i=1}^{N}h_i\|\varepsilon_i(\tau)\|^2\le V(\tau,\varepsilon_i(\tau)),\qquad\forall\tau\ge 0.\tag{3.30}
$$

By Schur compiment, for $i=1,2,\dots,N$, $\Omega_i(\eta(\tau))<0$ is eguivalent to $\Pi_{1i}<0$, $\Pi_{2i}<0$, respectively. Then it can be derived that

$$
\begin{aligned}
\dot V(\tau,\varepsilon_i(\tau))\quad\le\quad&-2\zeta V(\tau,\varepsilon_i(\tau))-\sum_{i=1}^{N}L_i(\tau,\varepsilon_i(\tau),\varepsilon_i(\tau-\eta_1(\tau)),\int_{\tau-\delta_1(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma,\mu_i(\tau))\\
&+\sum_{i=1}^{N}\xi^T(\tau)\Omega_i(\eta(\tau))\xi(\tau),\quad i=1,2,\dots,N,\quad\forall\tau\ge 0.
\end{aligned}\tag{3.31}
$$

Since $\sum_{i=1}^{N}L_i\big(\tau,\varepsilon_i(\tau),\varepsilon_i(\tau-\eta_2(\tau)),\int_{\tau-\delta_2(\tau)}^{\tau}\varepsilon_i(\varsigma)d\varsigma,\mu_i(\tau)\big)>0$ and $\Omega_i(\eta(\tau))<0$, we have

$$\dot{V}(\tau, \varepsilon_i(\tau)) \leq -2\zeta V(\tau, \varepsilon_i(\tau)), \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0. \tag{3.32}$$

Combining the two sides of (3.32) from 0 to τ yields

$$V(\tau, \varepsilon_i(\tau)) \leq V(0, \varepsilon_i(0))e^{-2\zeta\tau}, \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0. \tag{3.33}$$

Additionally, including condition (3.3) in consideration (3.33), We now posses

$$\sum_{i=1}^{N} h_i \|\varepsilon_i(\tau)\|^2 \leq V(\tau, \varepsilon_i(\tau)) \leq V(0, \varepsilon_i(0))e^{-2\zeta\tau}, \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0.$$

Using the Lyapunov-Krasovskii theorem and the close-loop error dynamical network (2.5), solution $\|\varepsilon_i(\tau, \upsilon)\|$ fulfilling the radial unboundedness of $\dot{V}(\tau, \varepsilon_i(\tau))$,

$$\|\varepsilon_i(\tau, \upsilon)\| \leq \sqrt{\frac{\Upsilon_i}{h_i}} e^{-\zeta\tau}, \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0,$$

According to this, the hybrid controller (2.2) makes the exponential stability of the error complex network (2.5).

As a result, the drive system (2.3) and the controlled react CDNs (2.1) are EFPS. Additionally, we obtain the following from (3.31) and $V(\tau, \varepsilon_i(\tau)) \geq 0$,

$$\begin{aligned}\dot{V}(\tau, \varepsilon_i(\tau)) \leq & -\sum_{i=1}^{N} L_i\Big(\tau, \varepsilon_i(\tau), \varepsilon_i(\tau - \eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma, \mu_i(\tau)\Big), \\ & i = 1, 2, \ldots, N, \quad \forall \tau \geq 0.\end{aligned} \tag{3.34}$$

Combining the two sides of (3.34) from 0 to τ yields,

$$\begin{aligned}&\sum_{i=1}^{N} \int_0^{\tau} L_i\Big(\tau, \varepsilon_i(\tau), \varepsilon_i(\tau - \eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma, \mu_i(\tau)\Big)d\tau \\ &\qquad \leq V(0, \varepsilon_i(0)) - V(\tau, \varepsilon_i(\tau)) \leq V(0, \varepsilon_i(0)), \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0,\end{aligned}$$

because of $V(\tau, \varepsilon_i(\tau)) \geq 0$. Hence, letting $\tau \to \infty$, we finally obtain that,

$$\begin{aligned}\mathbb{J} &= \sum_{i=1}^{N} \int_0^{\infty} L_i\Big(\tau, \varepsilon_i(\tau), \varepsilon_i(\tau - \eta_1(\tau)), \int_{\tau-\delta_1(\tau)}^{\tau} \varepsilon_i(\varsigma)d\varsigma, \mu_i(\tau)\Big)d\tau \\ &\leq V(0, \varepsilon_i(0)) = \mathbb{J}^*, \quad i = 1, 2, \ldots, N, \quad \forall \tau \geq 0.\end{aligned}$$

This completes the proof of the theorem.

4. A numerical Example

In this section, we present an example In order to demonstrate the efficacy of the outcome and the suggested management plan by the disturbed, the driving mechanism of Chua's circuit system with time-varying delays [8], which employs CDNs with uncoupled nodes (2.1).

$$\begin{aligned}
\dot{s}_1(t) &= g_1\left(s_2(t-h_1(t)) - \frac{1}{7}\left(2s_1^3(t) - s_1(t)\right)\right), \\
\dot{s}_2(t) &= s_1(t) - g_4 s_2(t) + s_3(t-h_1(t)), \\
\dot{s}_3(t) &= -g_2 s_2(t) + g_3 \int_{t-d_1(t)}^{t} s_1^2(\theta)d\theta,
\end{aligned} \tag{4.1}$$

and we take the system (2.1) as identical nodes of network (response networks), which is given by

$$\begin{aligned}
\begin{pmatrix} \dot{\chi}_{i1}(\tau) \\ \dot{\chi}_{i2}(\tau) \\ \dot{\chi}_{i3}(\tau) \end{pmatrix} &= \begin{pmatrix} \lambda_1\left(\chi_{i2}(\tau-\eta_1(\tau)) - \frac{1}{7}\left(2\chi_{i1}^3(\tau) - \chi_{i1}(\tau)\right)\right) \\ \chi_{i1}(\tau) - \lambda_4\chi_{i2}(\tau) + \chi_{i3}(\tau-\eta_1(\tau)) \\ -\lambda_2\chi_{i2}(\tau) + \lambda_3 \int_{\tau-\delta_1(\tau)}^{\tau} \chi_{i1}^2(\varsigma)\, d\varsigma \end{pmatrix} \\
&+\omega_1 \sum_{j=1}^{N} \alpha_{ij} A\chi_j(\tau) + \omega_2 \sum_{j=1}^{N} \beta_{ij} B\chi_j(\tau-\eta_1(\tau)) \\
&+\omega_3 \sum_{j=1}^{N} \gamma_{ij} C \int_{\tau-\delta_1(\tau)}^{\tau} \chi_j(\varsigma)d\varsigma + \mathbb{U}_i(\tau), \quad i=1,2,\ldots,N, \quad \tau \geq 0,
\end{aligned} \tag{4.2}$$

where $\lambda_1 = 7$, $\lambda_2 = \frac{100}{7}$, $\lambda_3 = 0.07$, and $\lambda_4 = 1.5$ are real positive constants, respectively. The initial condition function $\Theta(\tau) = [0.65\cos\tau, 0.3\cos\tau, -0.2\cos\tau]^T$, the time-varying delay functions $\eta_1(\eta) = 0.4\sin 2\tau$, $\eta_2(\eta) = 0.3\sin^2\tau$, $\delta_1(\tau) = 0.2\cos^2\tau$, and $\delta_2(\tau) = 0.1\cos^2\tau$, the coupling strength $\omega_1 = 0.2$, $\omega_2 = 0.3$, $\omega_3 = 0.2$, given positive constants $\zeta = 0.1$, $\eta_1 = 0.4$, $\eta_2 = 0.3$, $\delta_1 = 0.2$, and $\delta_2 = 0.1$. It is stable at the equilibrium point $\varsigma(\tau) = 0$, $\varsigma(\tau-\eta_1(\tau)) = 0$, $\int_{\tau-\delta_1(\tau)}^{\tau} \varsigma(\vartheta)d\vartheta = 0$.
The Jacobian matrices are

$$\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.5 & 0 \\ 0 & -\frac{100}{7} & 0 \end{bmatrix}, \quad \Psi_{\eta_1} = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Psi_{\delta_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The innercoupling matrices are

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Given positive definite matrices

$$\Phi_{11i} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \Phi_{22i} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix},$$
$$\Phi_{33i} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}, \quad \Phi_{44i} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix},$$

$$D_{1i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{2i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{3i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ i = 1, 2, \dots, 5,$$

and the following matrices for the coupling configuration:

$$A_1 = \begin{bmatrix} -2 & 0 & 1 & 0 & 1 \\ 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -2 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix},$$

Solution: From the conditions (3.1) and (3.2) of Theorem 3.1 utilizing MATLAB's LMI Toolbox, we determine that the cost function value's upper bound is

$$J^* = 4.71866.$$

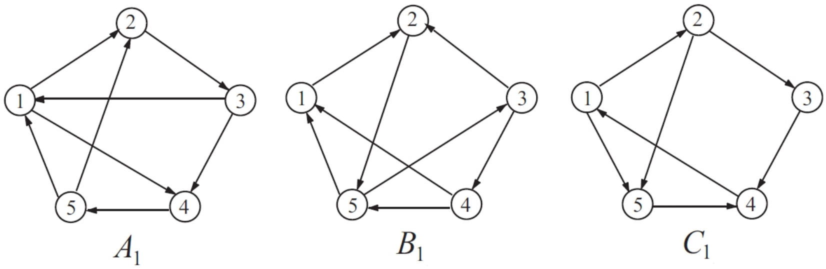

FIGURE 1. The complicated network's topological structure with $N = 5$.

5. Conclusions

In this work introduces a novel development criterion for exponential function projective synchronization and guaranteed cost control in complex dynamical networks with asymmetric, multi-connection topologies and mixed time-varying delays. By designing a new Lyapunov-Krasovskii functional and utilizing advanced matrix inequalities, we derived sufficient LMI conditions ensuring synchronization and cost performance. Numerical simulations confirm the practicality and robustness of the proposed approach. Further research may explore extensions to fractional-order CDNs, incorporation of stochastic disturbances, or real-time adaptive implementations in cyber-physical systems. These would broaden the applicability of the framework to more diverse nonlinear and networked phenomena.

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