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AN ALGORITHM FOR APPROXIMATING SOLUTIONS OF SPLIT HAMMERSTEIN INTEGRAL EQUATIONS

Abdulmalik Usman Bello[1,2*], **Monday Ogudu Nnakwe**[3], **Aisha Aminu Adam** [1*]
[1] *Charles Chidume Mathematics Institute, Mathematics Institute, African University of Science and Technology, PMB 681, Abuja 900107, Nigeria*
E-mail: uabdulmalik@aust.edu.ng
[2] *Federal University Dutsinma, Katsina, Nigeria*
E-mail: mon0002@auburn.edu
[3] *Department of Mathematics and Statistics, Auburn University, Alabama, 36849, USA*
E-mail: aaminu@aust.edu.ng

**Corresponding author.*

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Abstract In this paper, we study split Hammerstein integral equations of the form: $u \in H_1$ such that $u + K_1F_1u = 0$ and $A(u) + K_2F_2(Au) = 0$, where K_1, F_1 are maximal monotone maps defined on a real Hilbert space H_1, with $D(K_1) = D(F_1) = H_1$; K_2, F_2 are maximal monotone maps defined on a real Hilbert space H_2, with $D(K_2) = D(F_2) = H_2$ and A, a bounded linear map from H_1 to H_2. The sequence of the algorithm is proved to converge strongly to a solution of the split Hammerstein integral equation. The theorem proved, improves, unifies and complements some important related recent results in the literature.

MSC: 47H09; 47H05; 47J25; 47J05

Keywords: Nonlinear maps; Nonlinear Hammerstein equations; Strong convergence

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1. Introduction

Let Ω be a measurable and bounded subset of $\mathbb{R}^n$ and dy be a σ-finite measure on Ω. A nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_{\Omega} k(x,y)f(y,u(y))dy = w(x), \tag{1.1}$$

where the unknown function u and the function w lie in a suitable Banach space of functions, say, $\mathcal{F}(\Omega,\mathbb{R})$. The function $k : \Omega \times \Omega \to \mathbb{R}$ is the kernel of the equation while $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a real-valued measurable function. Equation (1.1) can be put in an abstract form as

$$u + KFu = 0, \tag{1.2}$$

where the operators $F, K : \mathcal{F}(\Omega,\mathbb{R}) \to \mathcal{F}(\Omega,\mathbb{R})$ are defined by

$$Fu(x) = f(x,u(x)) \qquad \text{and} \qquad Kv(x) = \int_{\Omega} k(x,y)v(y)dy,\ x \in \Omega, \tag{1.3}$$

respectively, where, without loss of generality, we have taken the function $w \equiv 0$. Hammerstein integral equations have applications in different areas of science and engineering. For instance, it can be used to describe the final state of a spatially distributed population (see, e.g., [34] and [41]). Hammerstein equations are also intimately connected with nonsmooth calculus of variation. Consider the following energy functional defined by

$$Ju = \int_{\Omega} (h(u(t)) - f(s,u(s)))ds, \tag{1.4}$$

where h denotes the kinetic energy of the system and f is the potential energy generator of the superposition operator. The functional J, in general, is not differentiable. However, it admits generalized gradient or subgradient in the sense, for instance, of Clarke's generalized gradient (see, e.g., [23]). Thus the problem of minimizing the functional J can now be seen as the Euler Lagrange inclusion

$$Lu \in \partial Fu, \tag{1.5}$$

where L is a linear operator and ∂F is the generalized Clarke's gradient. Equation (1.5), in turn, is equivalent to the following Hammerstein inclusion problem

$$u + KFu \ni 0. \tag{1.6}$$

Consider also the following nonlinear boundary value problem

$$\begin{cases} -\Delta u = f(x,u(x)),\ x \in \Omega, \\ u(x) = 0,\ x \in \partial\Omega. \end{cases} \tag{1.7}$$

where Ω is a smooth subset of $\mathbb{R}^n$. Define the operator $K : \mathcal{F}(\Omega,\mathbb{R}) \to \mathcal{F}(\Omega,\mathbb{R})$ by $Kg = u$, where u is the unique solution of the corresponding linear boundary value problem:

$$\begin{cases} -\Delta u = g, \\ u(x) = 0,\ x \in \partial\Omega, \end{cases} \tag{1.8}$$

and $Fu(x) = f(x,u(x))$. Then, (1.7) can be put in the form of (1.2).

Let H be a real Hilbert space. A map $A : D(A) \subset H \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0,\ \forall\ x,y \in D(A), \tag{1.9}$$

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where $D(A)$ is the domain of A. Several existence results have been proved for approximatuing solutions of (1.2) when the operators F and K are monotone (See, for instance, Browder [4, 29, 30], Browder and De Figueiredo [5, 6], Brézis and Browder [26, 28], Appel *et al.* [25], and Cardinali and Papageorgiou [9]).

In general, Hammerstein integral equations do not have closed form solutions. Thus, developing algorithms for approximating their solutions is of great interest. Let $A : H \to H$ be a nonlinear operator, A is said to be angle bounded with angle $\beta > 0$ if and only if

$$\langle Ax - Ay, z - y\rangle \leq \beta\langle Ax - Ay, x - y\rangle, \tag{1.10}$$

for any $x, y, z \in H$. For $y = z$, inequality (1.10) implies the monotonicity of A. A monotone *linear* operator $A : H \to H$ is said to be *angle bounded* with angle $\alpha > 0$, if

$$|\langle Ax, y\rangle - \langle Ay, x\rangle| \leq 2\alpha\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}}, \tag{1.11}$$

for all $x, y \in H$. Brézis and Browder, in [27], proved the following approximation result for angle bounded operators defined by suitable Galerkin method.

Theorem 1.1 (Brézis and Browder, [27]). *Let H be a separable Hilbert space and C be a closed subspace of H. Let $K : H \to C$ be a bounded continuous monotone operator and $F : C \to H$ be an angle-bounded and weakly compact mapping. For a given $f \in C$, consider the Hammerstein equation*

$$(I + KF)u = f \tag{1.12}$$

and its nth Galerkin approximation given by

$$(I + K_nF_n)u_n = P^*f, \tag{1.13}$$

where $K_n = P_n^*KP_n : H \to C$ and $F_n = P_nFP_n^* : C_n \to H$, with the symbols having their usual meanings (see, e.g., Pascali [38]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.13) *admits a unique solution u_n in C_n and $\{u_n\}$ converges strongly in H to the unique solution $u \in C$ of the equation* (1.12).

Attempts have been made to develop iterative algorithms for approximating solutions of (1.2) (see, e.g., Mann [35]). However, most of these results require the inverse of the operator K not only to exist but also be strongly monotone. These requirements do not only limit the class of operators involved, but is also not convenient for implementation. The first satisfactory result for approximating solution of Hammerstein equation was given by Chidume and Zegeye (see [19–21]). They considered the product space $\mathcal{Q} = H \times H$ and defined the auxillary operator $T : \mathcal{Q} \to \mathcal{Q}$ by

$$T[u, v] = [Fu - v, Kv + u], \quad u, v \in \mathcal{Q}. \tag{1.14}$$

It can be easily seen that u^* solves (1.2) if and only if $T[u^*, v^*] = 0$ with $v^* = Fu^*$. The auxiliary operator T gave an insight on how to develop a coupled algorithm for computing solutions of (1.2). The same authors (see Chidume and Zegeye [21]) defined the following coupled algorithm: for $u_0, v_0 \in H$, define the sequences $\{u_n\}$ and $\{v_n\}$ recursively by

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n),\ n \geq 0, \tag{1.15}$$

$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n),\ n \geq 0, \tag{1.16}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying appropriate conditions. For more recent results on the approximation of solutions of Hammerstein equations (see, e.g., Chidume and Djitte [14–16], Chidume and Ofeodu [31], Chidume *et al.* [17, 18, 32, 33], Chidume and Bello [13]), and Minjibir and Muhammad [36]).

Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is given by the following:

$$\text{find } x^* \in C \text{ such that } \; Ax^* \in Q, \tag{1.17}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear map.

This problem was introduced by Censor and Elfving [11] for the modeling of inverse problems stemming from phase retrievals, image processing and intensity modulated radiation therapy (IMRT) (see, e.g., [3, 7, 8]). The SFP has also been successfully applied in other areas such as immaterial science, computerized tomography, antenna design, sensor networks, data denoising and data compression (see, e.g., [1, 2, 10, 12, 24, 40]).

Combining the split feasibility problem and the coupling technique introduced by Chidume and Zegeye in [21], we introduce *split Hammerstein integral equation*. Let H_1 and H_2 be real Hilbert spaces. Let $F_1, K_1 : H_1 \longrightarrow H_1$ and $F_2, K_2 : H_2 \longrightarrow H_2$ be monotone operators. Define the sets Ω_1 and Ω_2 by

$$\Omega_1 = \{u \in H_1 : u + K_1F_1u = 0, \text{ with } v = F_1u\},$$

and

$$\Omega_2 = \{u \in H_2 : u + K_2F_2u = 0, \text{ with } v = F_2u\}.$$

The *split Hammerstein integral equation* is to find

$$u^* \in \Omega_1 \qquad \text{such that} \qquad Au^* \in \Omega_2, \tag{1.18}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear map.

We prove a strong convergence result for approximating solution of (1.18). This problem improves, unifies and complements many existing results on Hammerstein integral equation in the literature.

2. PRELIMINARIES

Definition 2.1. Let $A : H \to 2^H$ be a map.

The map A is called *maximal monotone* if A is monotone and the graph $G(A)$ of A,

$$G(A) := \{(u,v) \in H \times H : v \in A(u)\},$$

is not properly contained in the graph of any other monotone map.

$$dom(A) := \{u \in H : A(u) \neq \emptyset\}.$$

The resolvent of A with a parameter $\lambda > 0$ is defined by

$$J_\lambda^A (I + \lambda A)^{-1},$$

where I is the identity map.

Remark 2.2. (i) The map A is monotone if and only if the resolvent J_λ^A of A is single-valved and firmly nonexpansive.

(ii) The map A is maximal monotone if and only if the resolvent J_λ^A of A is single-valved, firmly nonexpansive and $dom(J_\lambda^A) = H$.

(iii) Moreover, $0 \in A(u^*) \iff u^* \in F(J_\lambda^A)$, where $F(J_\lambda^A)$ is the fixed point set of J_λ^A.

(3) The map A is called *β-inverse strongly monotone* (ism) with constant $\beta > 0$ if,

$$\langle u - v, \eta_u - \eta_v \rangle \geq \beta ||\eta_u - \eta_v||^2, \ \forall \ u, v \in H, \ \eta_u \in Au, \ \eta_v \in Av.$$

Definition 2.3. Let $T : H \to H$ be map.

(1) The map T is called *firmly nonexpansive* if, $\langle u-v, Tu-Tv \rangle \geq ||Tu-Tv||^2, \ \forall \, u, v \in H$.

(2) The map T is called *α-averaged* if $T = (1 - \alpha)I + \alpha S$,
where $\alpha \in (0, 1)$, $S : H \to H$ is a *nonexpansive map* and I is the identity map.

Thus firmly nonexpansive maps (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators) are averaged.

Lemma 2.4. [42] (1) *T is nonexpansive if and only if $(I - T)$ is $\frac{1}{2}$-ism.*

(2) *If T is v-ism and $\gamma > 0$, then, γT is $\frac{v}{\gamma}$-ism.*

(3) *T is averaged if and only if $(I - T)$ is v-ism for some $v > \frac{1}{2}$. Indeed, for $\eta \in (0, 1)$, T is η-averaged if and only if $(I - T)$ is $\frac{1}{2\eta}$-ism.*

(4) *If T_1 is η_1-averaged and T_2 is η_2-averaged, where η_1, $\eta_2 \in (0, 1)$, then, $T_1 o T_2$ is η-averaged, where $\eta = \eta_1 + \eta_2 - \eta_1\eta_2$.*

(5) *If T_1 and T_2 are averaged and have a common fixed point, then, $F(T_1 o T_2) = F(T_1) \cap F(T_2)$.*

Lemma 2.5. [22] *Let H be a real Hilbert space and $F, K : H \to H$ be maps with $D(F) = D(K) = H$. Let $\mathcal{Q} = H \times H$ and $A : \mathcal{Q} \to \mathcal{Q}$ be the map defined by*

$$Aw := (Fu - v, Kv + u), \ \forall \, w = (u, v) \in \mathcal{Q}. \tag{2.1}$$

Assume that F and K are maximal monotone and satisfy the range condition. Then, A is maximal monotone and also satisfies the range condition.

Lemma 2.6. [37, 39] *Let E be nonempty, closed and convex subset of a real Hilbert space H. Let $h : E \to E$ be an averaged map. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) h x_n, \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in F(h)$.

3. Main Result

Lemma 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $A_1 : H_1 \to 2^{H_1}$ and $A_2 : H_2 \to 2^{H_2}$ be two maximal monotone maps and $B : H_1 \to H_2$ be a bounded linear map with $B \neq 0$ and $\Omega =: \{u \in H_1 : 0 \in A_1(u) \ \text{ and } \ 0 \in A_2(Bu)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) J_\lambda^{A_1} \left[I - \gamma B^*(I - J_\lambda^{A_2})B \right] x_n, \end{cases} \tag{3.1}$$

where $\gamma \in \left(0, \frac{2}{L}\right)$, *with* $L = ||B^*B||$, *and* $\{\alpha_n\} \subset [0,1]$, *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then, the sequence* $\{x_n\}$ *converges strongly to a point,* $x^* \in \Omega$.

Proof. First, we show that $I - \gamma B^*(I - J_\lambda^{A_2})B$ is averaged.

Let $p \in \mathcal{F}$. Then, $J_\lambda^{A_1}(p) = p$ and $J_\lambda^{A_2}(Bp) = Bp$. Define $V := I - \gamma B^*(I - J_\lambda^{A_2})B$. Then, $V(p) = p$. By Remark (2.2) and Definition (2.3), we have that $J_\lambda^{A_1}$ and $J_\lambda^{A_2}$ are averaged maps. Since, $J_\lambda^{A_2}$ is averaged, then, by Lemma 2.4, we have that $I - J_\lambda^{A_2}$ is v-ism with $v > \frac{1}{2}$. So,

$$\begin{aligned}
\langle B^*(I - J_\lambda^{A_2})Bx - B^*(I - J_\lambda^{A_2})By, x - y\rangle &= \langle (I - J_\lambda^{A_2})Bx - (I - J_\lambda^{A_2})By, Bx - By\rangle \\
&\geq v||(I - J_\lambda^{A_2})Bx - (I - J_\lambda^{A_2})By||^2 \\
&\geq \frac{v}{L}||B^*(I - J_\lambda^{A_2})Bx - B^*(I - J_\lambda^{A_2})By||^2
\end{aligned}$$

Hence, $\gamma B^*(I - J_\lambda^{A_2})B$ is $\frac{v}{\gamma L}$-ism. This implies that $I - \gamma B^*(I - J_\lambda^{A_2})B$ is averaged.

Now, Setting $h := I - \gamma B^*(I - J_\lambda^{A_2})B$, Algorithm (3.5) reduces to the following algorithm:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)(J_\lambda^{A_1} oh)x_n, \end{cases} \tag{3.2}$$

By lemma 2.4, we have that $J_\lambda^{A_1} oh$ is averaged and $F(J_\lambda^{A_1} oh) = F(J_\lambda^{A_1}) \cap F(h)$. Also, by Lemma 2.6 and the fact that $J_\lambda^{A_1} oh$ is averaged, we have that $\{x_n\}$ converges strongly to a point, $x^* \in F(J_\lambda^{A_1} oh)$, i.e., $x^* \in F(J_\lambda^{A_1})$ and $x^* \in F(h)$. By Remark 2.2(iii), we have that

$$x^* \in F(J_\lambda^{A_1}) \iff 0 \in A_1(x^*). \tag{3.3}$$

Also, $x^* \in F(h) \implies \gamma B^*(I - J_\lambda^{A_2})Bx^* = 0$.

Now, for $J_\lambda^{A_2}(Bx^*) = Bx^* + \mathcal{D}$, with $B^*(\mathcal{D}) = 0$, combined with the fact that $J_\lambda^{A_2}(Bp) = Bp$, we get that

$$||J_\lambda^{A_2}(Bx^*) - J_\lambda^{A_2}(Bp)||^2 = ||Bx^* - Bp||^2 + ||\mathcal{D}||^2.$$

Since $J_\lambda^{A_2}$ is nonexpansive, we have that $\mathcal{D} = 0$. Thus, $J_\lambda^{A_2}(Bx^*) = Bx^*$, and by Remark 2.2(iii), we have that

$$Bx^* \in F(J_\lambda^{A_2}(Bx^*)) \iff 0 \in A_2(Bx^*). \tag{3.4}$$

Applying inclusions (3.3) and (3.4), we conclude that $x^* \in \mathcal{F}$. ∎

Now, we prove our main theorem.

Theorem 3.2. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $\mathcal{Q}_1 := H_1 \times H_1$ *and* $\mathcal{Q}_2 := H_2 \times H_2$. *Let* $K_1, F_1 : H_1 \to H_1$ *and* $K_2, F_2 : H_2 \to H_2$ *be two maximal monotone maps and* $B : H_1 \to H_2$ *be a bounded linear map with* $B \neq 0$ *and* $\Omega := \{u \in H_1 : u + K_1F_1u = 0,\ Bu + K_2F_2(Bu) = 0,\ \text{with}\ v = F_1u,\ y = F_2(Bu)\} \neq \emptyset$. *Let* $G_1 : \mathcal{Q}_1 \to \mathcal{Q}_1$ *and* $G_2 : \mathcal{Q}_2 \to \mathcal{Q}_2$ *be two maps defined by* $G_1(u, v) := (F_1u - v, K_1v + u)$

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and $G_2(t,y) := (F_2(t)-y, K_2y+t)$*, respectively. Let* $\{x_n\}$ *be a sequence generated iteratively by*

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) J_\lambda^{G_1} \left[I - \gamma B^*(I - J_\lambda^{G_2})B \right] x_n, \end{cases} \tag{3.5}$$

where $\gamma \in \left(0, \frac{2}{L}\right)$*, with* $L = ||B^*B||$*, and* $\{\alpha_n\} \subset [0,1]$*, with* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*. Then, the sequence* $\{x_n\}$ *converges strongly to a point,* $x^* \in \Omega$*.*

Proof. By Lemma 2.5, G_1 and G_2 are maximal monotone maps. Setting $A_1 = G_1$ and $A_2 = G_2$ in Lemma 3.1, the result of Theorem 3.2 is immediate. ∎

Now in Theorem 3.2, setting $H_1 = H_2 = H$ and $B = I$, the identity on H, we obtain the following corollary for approximating a common solution of two Hammerstein integral equations.

Corollary 3.3. *Let* H *be a real Hilbert spaces and* $\mathcal{Q} := H \times H$*. Let* $K_1, F_1 : H \to H$ *and* $K_2, F_2 : H \to H$ *be two maximal monotone maps and* $\Omega := \{u \in H : u + K_1F_1u = 0, \text{ and } u + K_2F_2u = 0\} \neq \emptyset$*. Let* $G_1 : \mathcal{Q} \to \mathcal{Q}$ *and* $G_2 : \mathcal{Q} \to \mathcal{Q}$ *be two maps defined by* $G_1(u,v) := (F_1u - v, K_1v + u)$ *and* $G_2(t,y) := (F_2(t) - y, K_2y + t)$*, respectively. Then, under the conditions of Theorem 3.2, the sequence generated by (3.5) converges strongly to a point in* Ω*.*

Example 3.4. Let $H_i = \mathcal{L}_2([0,1]),\ i = 1, 2$, and $||u||_2 = \left(\int_0^1 |u(t)|^2 dt\right)^{\frac{1}{2}}$.
Let F_i, $K_i : H_i \to H_i$ be defined by $(F_iu)(t) = (t+1)u_i(t)$ and $(K_iv)(t) = tv_i(t)\ \forall\ t \in [0,1]$.
Let $B : H_1 \to H_2$ be defined by $(Bu)(t) = \frac{t^2}{2}u(t)\ \forall\ t \in [0,1]$.
Clearly, F_i and K_i are maximal monotone for each i. B is a bounded linear map with $(B^*u)(t) = \frac{t^2}{2}u(t)\ \forall\ t \in [0,1]$ and $||B^*B||_2 = \frac{1}{2}$. Observe that $u^*(t) = 0\ \forall\ t \in [0,1]$ is the solution of the equations $u + K_1F_1u = 0$ and $Bu + K_2F_2Bu = 0$. Also, by Lemma 2.5, G_1 and G_2 are maximal monotone.
Algorithm 3.2 can be written as follows:
Choose $x_0 \in H_1$ with $\gamma = 2$, and $\alpha_n = \frac{1}{n+1}$. Then, compute the $(n+1)$th iteration as follows:

$$\left\{ x_{n+1}(t) = \alpha_n x_0(t) + (1-\alpha_n) J_\lambda^{G_1} \left[I - \gamma B^*(I - J_\lambda^{G_2})B \right] x_n(t). \right.$$

Conclusion

In this paper, we have studied *split Hammerstein integral equation problem* and proved a strong convergence theorem for approximating solution of the problem in real Hilbert spaces. The problem studied improves, unifies and complements several other related results in the literature.

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