BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
Volume 9 (2023)
Pages 72–91
MATHEMATICAL & COMPUTATIONAL SCIENCE ISSN: 2408-154X
https://bangmodjmcs.com/

DESCENT MODIFIED CONJUGATE GRADIENT METHODS FOR VECTOR OPTIMIZATION PROBLEMS

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Received: 23 October 2023 / Accepted: 25 December 2023

Abstract Scalarization approaches transform vector optimization problems (VOPs) into single-objective optimization. These approaches are quite elegant; however, they suffer from the drawback of necessitating the assignment of weights to prioritize specific objective functions. In contrast, the conjugate gradient (CG) algorithm provides an attractive alternative that does not require the conversion of any objective function or assignment of weights. Nevertheless, the set of Pareto-optimal solutions is obtainable. We introduce three CG techniques for solving VOPs by modifying their search directions. We consider modifying the search directions of the Fletcher-Reeves (FR), Conjugate Descent (CD), and Dai-Yuan (DY) CG techniques to obtain their descent property without the use of any line search, as well as to achieve good convergence properties. The sufficient descent property of these techniques are established without any line search and achieve global convergence using Wolfe line search. Numerical experiments are conducted to demonstrate the implementation and efficiency of the proposed techniques.

MSC: 90C29; 90C52; 90C30; 90C26; 49M37
Keywords: Conjugate gradient method; sufficient descent condition; vector optimization

Published online: 31 December 2023

1. Introduction

Conjugate gradient (CG) techniques for solving vector optimization problems (VOPs) have begun to gain substantial attention from researchers since their introduction to the vector setting by Lucambio Pérez and Prudente [37]. Their simplicity and minimal memory requirements are the factors in developing interest in the vector setting, just as they were in classical optimization [2].

In the following, we consider an unconstrained vector optimization problem of the form

$$Minimize_Q\ F(z),\ z \in \mathbb{R}^n, \tag{1.1}$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is in C^1 (continuously differentiable functions) and $Q \subset \mathbb{R}^m$ is closed, convex and pointed cone with nonempty interior. The partial order defined in $\mathbb{R}^m$, $\preccurlyeq_Q$, generated by Q is

$$a \preccurlyeq_Q b \Longleftrightarrow b - a \in Q,$$

and $\prec_Q$, generated by $int(Q)$ is

$$a \prec_Q b \iff b - a \in int(Q).$$

If $Q = \mathbb{R}^m_+$, then problem (1.1) is considered to be multiobjective optimization. If $Q = \mathbb{R}_+$ and $\mathbb{R}^m = \mathbb{R}$, then problem (1.1) reduces to single-objective optimization.

The VOPs have diverse applications in bi-level programming, cancer treatment planning, engineering, environmental analysis, location science, management science, and statistics. See for instance the references, [13, 17, 18, 25, 28, 30, 33, 45].

Furthermore, the VOPs are known to be solved by scalarization approaches. It involves transforming a vector optimization problem into an appropriate scalar optimization problem with a real-valued objective function, [31, 36]. However, the choice of weights can significantly impact the results, and finding a satisfactory set of weights that represents your preferences can sometimes be challenging. Additionally, this technique does not guarantee the discovery of all Pareto-optimal solutions. Hence, it is crucial to thoroughly contemplate alternative techniques for solving VOPs. Therefore, CG algorithms provide attractive alternatives that do not have this restriction and improve vector optimization problem-solving strategies.

Over the last twenty years, the interest in studying descent-based algorithms to solve VOPs, initially designed for single-objective optimization, has been increasing. We can trace the use of descent-based algorithms for VOPs to at least 2005, with Drummond and Svaiter [14] article on the steepest descent technique and that of Bonnel et al. [6] article on proximal technique. Since then, several other works in this direction have followed suit, [3, 4, 7–10, 15, 19, 21, 23, 24, 26, 38, 42].

In 2018, Lucambio Pérez and Prudente [37] studied some CG techniques by extending them to vector optimization. Before delving into their contribution to this study, let us look at some properties of Q. The positive polar-cone of Q is given by

$$Q^* := \{p \in \mathbb{R}^m \mid \langle p, z\rangle \geq 0,\ \forall\ z \in Q\}. \tag{1.2}$$

Please cite this article as: J. Yahaya et al., Descent modified conjugate gradient methods for vector optimization problems, Bangmod J-MCS., Vol. 9 (2023) 72–91.

Note that since Q is closed and convex. Then, $Q = Q^{**}$. Suppose $C \subseteq Q^* \setminus \{0\}$ is compact, and Q^* is defined to be the conic hull of a convex hull of C as follows

$$Q^* = cone(conv(C)). \tag{1.3}$$

Now, for a given Q (closed, convex and pointed cone with nonempty interior), the set

$$C = \{p \in Q^* \mid ||p|| = 1\}, \tag{1.4}$$

satisfies (1.3). Throughout this paper, we consider C to be as defined in (1.4).

Define $\theta : \mathbb{R}^m \to \mathbb{R}$ as

$$\theta(z) := \sup\{\langle z, p\rangle \mid p \in C\}. \tag{1.5}$$

By the compactness of C, we have that θ is well-defined.

Next, define $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(z, d) := \theta(JF(z)d) = \sup\{\langle JF(z)d,\ p\rangle \mid p \in C\}. \tag{1.6}$$

For a given point z we represent the Jacobian of F by $JF(z)$.

Now, we can describe a CG technique as

$$z_{k+1} = z_k + \alpha_k d_k,\ \ k \geq 1, \tag{1.7}$$

where $\alpha_k > 0$ is the step size or step length which is obtainable through a line search technique, and d_k is the search direction defined by

$$d_k := \begin{cases} u(z_k), & k = 1, \\ u(z_k) + \beta_k d_{k-1}, & k \geq 2. \end{cases} \tag{1.8}$$

Here, the algorithmic parameter β_k can be chosen from the following options

$$\beta_k^{FR} := \frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}, \tag{1.9}$$

$$\beta_k^{CD} := \frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, d_{k-1})}, \tag{1.10}$$

$$\beta_k^{DY} := \frac{-\psi(z_k, u(z_k))}{\psi(z_k, d_{k-1}) - \psi(z_{k-1}, d_{k-1})}, \tag{1.11}$$

$$\beta_k^{PRP} := \frac{-\psi(z_k, u(z_k)) + \psi(z_{k-1}, u(z_k))}{-\psi(z_{k-1}, u(z_{k-1}))}, \tag{1.12}$$

$$\beta_k^{HS} := \frac{-\psi(z_k, u(z_k)) + \psi(z_{k-1}, u(z_k))}{\psi(z_k, d_{k-1}) - \psi(z_{k-1}, d_{k-1})}, \tag{1.13}$$

$$\beta_k^{LS} := \frac{-\psi(z_k, u(z_k)) + \psi(z_{k-1}, u(z_k))}{-\psi(z_{k-1}, d_{k-1})}, \tag{1.14}$$

they are called the Fletcher-Reeves (FR), Conjugate Descent (CD), Dai-Yuan (DY), Polak-Ribiére-Polyak (PRP), Hestenes-Stiefel (HS), and Liu-Storey (LS), respectively. Notice that if $\mathbb{R}^m = \mathbb{R}$ and $Q = \mathbb{R}_+$, then $\psi(\cdot, \cdot) = \langle \cdot, \cdot \rangle$ and $JF(x) = \nabla f(x)$. Hence, the above listed CG parameters ((1.9)- (1.14)) become the classical FR, CD, DY, PRP, HS, and LS, respectively.

Definition 1.1. A direction d is Q-descent direction (Q-DD) for F at z if

$$\psi(z,d) < 0. \tag{1.15}$$

A z is Q-critical point for F if

$$\psi(z,d) \geq 0, \tag{1.16}$$

for all d.

Definition 1.2. A direction d is said to fulfil *sufficient descent condition* (SDC) at z if

$$\psi(z,d) \leq c\psi(z,u(z)), \tag{1.17}$$

for some $c > 0$.

Lucambio Pérez and Prudente [37] introduced the β_k parameters (1.9)-(1.13). This study specifically investigated the CG techniques (1.9)-(1.13), that only CD and DY satisfy the sufficient descent conditions (SDC). The study extended upon some concepts found in [1, 11, 12, 22]. Additionally, it introduced the concepts of Zoutendijk and Wolfe conditions in this context. The study established the global convergence of these techniques and conducted a numerical experiment to demonstrate their implementation. Notably, the nonnegative PRP and HS outperformed the others, while DY and CD showed better performance than FR. Subsequently, many other works in this direction followed suit. Readers can refer to [24, 26, 38, 47] for further developments in this area.

In 2022, Goncalves et al. in [23] proposed the β_k parameter (1.14) and two of its modifications, and their global convergence was established using Wolfe and Armijo line searches. It is worth noting that the search direction LS (1.14) could not achieve SDC in this setting. The authors established the global convergence of this technique by assuming the descent condition. However, their modified LS achieve this property without any line search.

Recently, He et al. [26] proposed some spectral conjugate gradient techniques for VOPs. While Yahaya and Kumam [47] proposed the first hybrid CG techniques for VOPs and established their global convergence using the strong Wolfe conditions (SWC). The presented numerical experiments show that these hybrid CG techniques appear to be promising.

Motivated by the significant impact [48, 49], we present the generalization of [49] to the vector setting and two of its variations for the CD and DY CG techniques. We propose three CG techniques for solving VOPs by modifying their search directions. In particular, we consider modifying the search directions of the Fletcher-Reeves (FR), Conjugate Descent (CD), and Dai-Yuan (DY) CG techniques to obtain their descent property without the use of any line search and to achieve good convergence properties. We establish their descent property without line search and obtain their global convergence using Wolfe line search. Additionally, we present numerical experiments to demonstrate the implementation and efficiency of the proposed techniques.

In Section 2, we present some basic and preliminary results related to vector optimization. In Section 3, we investigate the sufficient descent property and the global convergence of the proposed techniques. In Section 4, we present and discuss numerical results. Finally, in Section 5, we provide some closing remarks.

2. Preliminaries

In this section, we present some basic notions and preliminary results in VOP used in this paper. For some notable preliminaries in VOP, see the references [14, 35, 37].

The aim, in vector optimization is to minimize a finite set of objective functions simultaneously. Rarely does a single point minimize all objective functions at once. In this setting, an alternative notion of optimality is needed. The concepts of *Pareto-optimality* and *weak Pareto optimality* are utilized.

Definition 2.1. [20] A point $\bar{z} \in \mathbb{R}^n$ is Pareto optimal or efficient if and only if $\nexists\ z \in \mathbb{R}^n$ such that $F(z) \preccurlyeq_Q F(\bar{z})$ and $F(z) \neq F(\bar{z})$.

Definition 2.2. [20] A point $\bar{z} \in \mathbb{R}^n$ is weak Pareto optimal or weak efficient if and only if $\nexists\ z \in \mathbb{R}^n$ such that $F(z) \prec_Q F(\bar{z})$.

Notice that when $\bar{z} \in \mathbb{R}^n$ represents a Pareto optimal point, it also qualifies as a weak Pareto point. However, the reverse statement is often not true.

Other properties of Q are

$$-Q = \{z \in \mathbb{R}^m \mid \langle z, p \rangle \leq 0,\ \forall\ p \in Q^*\},$$

and

$$-int(Q) = \{z \in \mathbb{R}^m \mid \langle z, p \rangle < 0,\ \forall\ p \in Q^* \setminus \{0\}\}.$$

In multiobjective optimization setting, $Q = \mathbb{R}^m_+$, implies $Q^* = Q$ and C is taken to be the canonical basis in $\mathbb{R}^m$. Assuming Q is a polyhedral cone, Q^* also possesses polyhedral characteristics. Furthermore, C can be regarded as a finite set of extremal rays belonging to the polyhedral cone Q^*.

For a given point z, the term $\text{Im}(JF(z))$ represents the image on $\mathbb{R}^m$ generated by $JF(z)$. A necessary condition for $Q-optimality$ of $\bar{z} \in \mathbb{R}^n$ is given as

$$-int(Q) \cap Im(JF(\bar{z})) = \emptyset, \tag{2.1}$$

when the condition (2.1) is fulfilled, we classify the point $\bar{z} \in \mathbb{R}^n$ as *stationary or Q-critical*. On the contrary, if $\bar{z} \in \mathbb{R}^n$ does not meet the criteria for Q-critical, then there is a h belonging to $\mathbb{R}^n$ for which $JF(\bar{z})h$ falls within $-int(Q)$. This signifies that h serves as a Q-DD (1.15) for F at the point $\bar{z}$. In other words, we have a positive s for which $F(\bar{z} + \bar{r}h) \prec_Q F(\bar{z})$, for all $0 < \bar{r} < s$. See, for instance, [35] for a full discussion on this.

Lemma 2.3. [14] *Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is in C^1. Then, the statements below hold:*
(a) $\psi(z, z' + \alpha d) \leq \psi(z, z') + \alpha\psi(z, d)$, for $z, z', d \in \mathbb{R}^n$ and $\alpha \geq 0$;

(b) The mapping $(z, d) \longmapsto \psi(z, d)$ is continuous;

(c) $|\psi(z, d) - \psi(z', d)| \leq \|JF(z) - JF(z')\|\|d\|$, for $z, z', d \in \mathbb{R}^n$;

(d) Let $\|JF(z) - JF(z')\| \leq L\|z - z'\|$, then $|\psi(z, d) - \psi(z', d)| \leq L\|d\|\|z - z'\|$.

We define $u : \mathbb{R}^n \to \mathbb{R}^n$ and $v : \mathbb{R}^n \to \mathbb{R}$, respectively by

$$u(z) := argmin\left\{\psi(z, d) + \frac{\|d\|^2}{2} \mid d \in \mathbb{R}^n\right\} \tag{2.2}$$

and

$$v(z) := \psi(z, u(z)) + \frac{\|u(z)\|^2}{2}. \tag{2.3}$$

Given that the real-valued function $\psi(z,\cdot)$ is closed and convex, and $d \longmapsto \frac{\|d\|^2}{2}$ is strictly convex, then $u(z)$ exists and is unique. The function $u(z)$ allows us to develop the concept of the steepest descent direction in the vector minimization setting. It is worth noting that in scalar optimization, we have $\psi(z,d) = \langle \nabla F(z), d\rangle$, $u(z) = -\nabla F(z)$, and $v(z) = -\frac{\|\nabla F(z)\|^2}{2}$, respectively.

Consider the convex quadratic problem

$$\begin{cases} Minimize\ \alpha + \frac{1}{2}\|d\|^2, \\ subject\ to\ [JF(z)d]_i \le \alpha,\ i = 1,2,\dots,m, \end{cases} \tag{2.4}$$

with linear inequality constraints, see for instance, [16]. We say that the step size, $\alpha > 0$ can be acquired through an exact line search if

$$\psi(z+\alpha d, d) = 0. \tag{2.5}$$

We now give the vector Wolfe conditions that was introduced by Lucambio Pérez and Prudente [38].

Definition 2.4. [37] Suppose $d \in \mathbb{R}^n$ is a Q-descent and $e \in Q$, we have

$$0 < \langle p,\ e\rangle \le 1, \tag{2.6}$$

for all $p \in C$.

Now, $\alpha > 0$ fulfils the *standard Wolfe condition* (WWC) if

$$F(z+\alpha d) \preccurlyeq_Q F(z) + \rho\alpha\psi(z,d)e \tag{2.7}$$

$$\psi(z+\alpha d, d) \ge \sigma\psi(z,d), \tag{2.8}$$

where $0 < \rho < \sigma < 1$. Furthermore, $\alpha > 0$ fulfils the *strong Wolfe condition* (SWC) if

$$F(z+\alpha d) \preccurlyeq_Q F(z) + \rho\alpha\psi(z,d)e \tag{2.9}$$

$$|\psi(z+\alpha d, d)| \le \sigma|\psi(z,d)|. \tag{2.10}$$

It is interesting to know that the vector $e \in Q$ given in (2.6), always exists. Specifically, for multiobjective optimization, we take e to be $[1,\cdots,1]^T \in \mathbb{R}^m$, Q and C are considered as $\mathbb{R}^m_+$, and canonical basis of $\mathbb{R}^m$, respectively.

Let us now conclude this section with the following important Lemmas.

Lemma 2.5. [14] *(a) let z be a Q-critical for F, then $u(z) = 0$ and $v(z) = 0$. (b) suppose z is not Q-critical for F, then $u(z) \ne 0$, $v(z) < 0$, $\psi(z,u(z)) < -\frac{\|u(z)\|^2}{2} < 0$ and $u(z)$ Q-DD for F at z. (c) The u and v are continuous maps.*

Lemma 2.6. [37] *Let p and q be any scalars and $t \ne 0$, then following hold:*

(a) $pq \le \frac{p^2}{2} + \frac{q^2}{2}$,

(b) $2pq \le 2t^2p^2 + \frac{q^2}{2t^2}$,

(c) $(p+q)^2 \le 2p^2 + 2q^2$,

(d) $(p+q)^2 \le (1+2t^2)p^2 + [1 + \frac{1}{2t^2}]q^2$.

https://doi.org/10.58715/bangmodjmcs.2023.9.6

3. Convergence analysis

In this section, we propose the modified CG techniques, prove their SDC and investigate their convergence properties.

Assumption 1. Suppose that the cone Q is finitely generated and there exists an open set Δ for which the $\mathcal{L} := \{z \mid F(z) \preccurlyeq_Q F(z_1)\} \subset \Delta$, where $z_1 \in \mathbb{R}^n$ and there exists $L > 0$ such that JF satisfies $\|JF(z) - JF(z^{'})\| \leq L\|z - z^{'}\|$ for all $z, z^{'} \in \Delta$.

Assumption 2. Suppose a sequence $\{D_k\}_{k\in\mathbb{N}} \subset F(\mathcal{L})$ and $D_{k+1} \preccurlyeq_Q D_k$, for all k, then there exists $\mathcal{D} \in \mathbb{R}^m$ for which $\mathcal{D} \preccurlyeq_Q D_k$. That is, all monotone nonincreasing sequences in $F(\mathcal{L})$ are bounded below.

We emphasize that, these assumptions are natural extension of those considered in the classical optimization.

The following Lemma is called the Zoutendijk condition

Lemma 3.1. [37] *Suppose Assumptions 1 and 2 hold. Consider the iteration (1.7), with d_k being Q-DD for F and α_k fulfils the WWC (2.7). Then,*

$$\sum_{k=1}^{\infty} \frac{\psi^2(z_k, d_k)}{||d_k||^2} < +\infty. \tag{3.1}$$

We give the outline of the propose CG algorithm.

Algorithm 1: Descent Conjugate Gradient Scheme (DCGS)

Step 0: Let $0 < \rho < \sigma < 1$, $e \in Q$ as in (2.6), $z_1 \in \mathbb{R}^n$ be given and initialize $k \longleftarrow 1$.

Step 1: Compute $u(z_k)$ and $v(z_k)$ as in (2.2) and (2.3), respectively. If $v(z_k) = 0$, then STOP.

Step 2: Compute

$$d_k = \begin{cases} u(z_k), & if\ k = 1, \\ \ell_k u(z_k) + \beta_k d_{k-1}, & if\ k \geq 2, \end{cases} \tag{3.2}$$

where β_k is an algorithmic parameter.

Step 3: Compute $\alpha_k > 0$ by using the conditions (2.9).

Step 4: Set z_{k+1} as in (1.7), for $k \longleftarrow k + 1$ and move to **Step 1.**

Remark 3.2.
- In Step 1, the algorithm compute the steepest descent direction, if the optimal value $v(z_k) = 0$, we have Q-critical, so it stop. Otherwise, move to Step 2 by computing d_k with an appropriate ℓ_k and β_k;
- In Step 3, it is required that the step size α_k is greater than zero and satisfies the strong Wolfe conditions (2.9);
- If d_k is a Q-descent direction of F at z_k, under Assumptions 1 and 2, it is possible to show that there exist intervals of positive step sizes that satisfy such conditions; see [[37], Proposition 3.2];
- In Step 4, we keep updating the iterates; this process is repeated until a solution is obtained or the maximum number of iterations is reached.

The main goal in this paper is to modify the search direction as indicated in (3.2), where $\ell_k := \left(1 - \beta_k \frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\right)$. Notice that if $\beta_k := \beta_k^{FR}$ in (3.2), we deduce that

$$d_k := \left(1 - \beta_k^{FR} \frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\right) u(z_k) + \beta_k^{FR} d_{k-1}, \tag{3.3}$$

using (1.9), we have

$$\begin{aligned} d_k &:= \left(1 - \frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))} \frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\right) u(z_k) + \left(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}\right) d_{k-1}, \\ &= \left(1 - \frac{\psi(z_k, d_{k-1})}{\psi(z_{k-1}, u(z_{k-1}))}\right) u(z_k) + \left(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}\right) d_{k-1}. \end{aligned}$$

Thus,

$$d_k := u(z_k) + \left(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}\right) d_{k-1} - \frac{\psi(z_k, d_{k-1})}{\psi(z_{k-1}, u(z_{k-1}))} u(z_k). \tag{3.4}$$

From (3.4), we can easily deduce that

$$\psi(z_k, d_k) \leq \psi(z_k, u(z_k)), \ \forall \ k \geq 1. \tag{3.5}$$

If we apply exact line search in equation (3.4), it reduces to

$$d_k := u(z_k) + \left(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}\right) d_{k-1}.$$

This is the same as the case when $\ell_k = 1$, in (3.2).

Again, if $\beta_k := \beta_k^{CD}$ in (3.2) and $\ell_k := \left(1 - \beta_k^{CD} \frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\right)$. We can easily deduce that

$$\psi(z_k, d_k) \leq \psi(z_k, u(z_k)), \ \forall \ k \geq 1. \tag{3.6}$$

Additionally, if $\beta_k := \beta_k^{DY}$ in (3.2) and $\ell_k := \left(1 - \beta_k^{DY} \frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\right)$. We can easily deduce that

$$\psi(z_k, d_k) \leq \psi(z_k, u(z_k)), \ \forall \ k \geq 1. \tag{3.7}$$

Remark 3.3. Observe that we established sufficient descent conditions for the search directions of the FR, CD, and DY CG techniques without any line search. In contrast to the existing results in [37], where the FR could not achieve SDC, while CD and DY achieved it but with the condition of a strong Wolfe line search.

Theorem 3.4. *Let $\{z_k\}$ be generated by Algorithm 1 with $\beta_k^{FR} \geq 0$ as defined in (1.9) and suppose Assumptions 1 and 2 hold. If the Zoutendijk condition (3.1) holds, then*

$$\liminf_{k \to \infty} \|u(z_k)\| = 0. \tag{3.8}$$

Proof. Suppose we have a constant δ for which

$$\|u(z_k)\| > \delta, \ for \ all \ k \geq 1. \tag{3.9}$$

Now, from (3.2), we have

$$d_k = \ell_k u(z_k) + \beta_k^{FR} d_{k-1}, \ k \geq 2. \tag{3.10}$$

https://doi.org/10.58715/bangmodjmcs.2023.9.6

Squaring (3.10) and applying Lemma 2.6(d) with $p = \ell_k||u(z_k)||$, $q = \beta_k^{FR}||d_{k-1}||$, and $t = 1$, we get

$$||d_k||^2 \leq \Big(\ell_k||u(z_k)|| + \beta_k^{FR}||d_{k-1}||\Big)^2 \leq 3\ell_k^2||u(z_k)||^2 + \frac{3}{2}(\beta_k^{FR})^2||d_{k-1}||^2. \quad (3.11)$$

$$\leq 3\Big(1 - \beta_k^{FR}\frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\Big)^2||u(z_k)||^2 + \frac{3}{2}\Big(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, u(z_{k-1}))}\Big)^2||d_{k-1}||^2$$

$$= 3\Big(1 - \frac{\psi(z_k, d_{k-1})}{\psi(z_{k-1}, u(z_{k-1}))}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, u(z_{k-1}))}||d_{k-1}||^2$$

$$\leq 3\Big(1 + \frac{2|\psi(z_k, d_{k-1})|}{-\psi(z_{k-1}, u(z_{k-1}))} + \frac{\psi^2(z_k, d_{k-1})}{\psi^2(z_{k-1}, u(z_{k-1}))}\Big)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, u(z_{k-1}))}||d_{k-1}||^2.$$

By the strong Wolfe condition (2.9), we have

$$\leq 3\Big(1 + \frac{2\sigma|\psi(z_{k-1}, d_{k-1})|}{\psi(z_{k-1}, u(z_{k-1}))} + \frac{\sigma^2\psi^2(z_{k-1}, d_{k-1})}{\psi^2(z_{k-1}, u(z_{k-1}))}\Big)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, u(z_{k-1}))}||d_{k-1}||^2.$$

By SDC (3.5), we have

$$\leq 3(1 + 2\sigma + \sigma^2)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, u(z_{k-1}))}||d_{k-1}||^2.$$

Divide through by $\psi^2(z_k, d_k)$, we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \leq 3(1 + 2\sigma + \sigma^2)\frac{\|u(z_k)\|^2}{\psi^2(z_k, d_k)} + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, u(z_{k-1}))}\frac{\|d_{k-1}\|^2}{\psi^2(z_k, d_k)}.$$

Again, by the SDC (3.5), we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \leq 3(1 + 2\sigma + \sigma^2)\frac{\|u(z_k)\|^2}{\psi^2(z_k, u(z_k))} + \frac{3}{2}\frac{\|d_{k-1}\|^2}{\psi^2(z_{k-1}, u(z_{k-1}))}. \quad (3.12)$$

Now, using Lemma 2.5 (b) and (3.9), we get

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \leq (1 + 2\sigma + \sigma^2)\frac{12}{\delta^2} + \frac{3}{2}\frac{\|d_{k-1}\|^2}{\psi^2(z_{k-1}, u(z_{k-1}))}.$$

Repeating this continuously, we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \leq (1 + 2\sigma + \sigma^2)\frac{12k}{\delta^2} + \frac{3}{2}\frac{\|d_1\|^2}{\psi^2(z_1, u(z_1))}.$$

Applying Lemma 2.5 (b) and (3.9), we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \leq (1 + 2\sigma + \sigma^2)\frac{12k}{\delta^2} + \frac{6}{\delta^2} \leq \frac{6}{\delta^2}((2 + 4\sigma + 2\sigma^2)k + 1).$$

Thus,

$$\sum_{k=1}^{\infty} \frac{\psi^2(z_k, d_k)}{\|d_k\|^2} \geq \frac{\delta^2}{6} \sum_{k=1}^{\infty} \frac{1}{((2+4\sigma+2\sigma^2)k+1)} = \infty. \tag{3.13}$$

This is a contradiction. Hence, we have (3.8) which complete the proof. ∎

Theorem 3.5. *Let $\{z_k\}$ be generated by Algorithm 1 with $\beta_k^{CD} \geq 0$ as defined in (1.10) and suppose Assumptions 1 and 2 hold. If the Zoutendijk condition (3.1) holds, then*

$$\liminf_{k\to\infty} \|u(z_k)\| = 0. \tag{3.14}$$

Proof. Suppose we have a constant δ for which

$$\|u(z_k)\| > \delta, \;\; for\ all\ k \geq 1. \tag{3.15}$$

Now, from (3.2), we have

$$d_k = \ell_k u(z_k) + \beta_k^{CD} d_{k-1}, \;\; k \geq 2. \tag{3.16}$$

Squaring (3.16) and applying Lemma 2.6(d) by using $p = \ell_k||u(z_k)||$, $q = \beta_k^{CD}||d_{k-1}||$, and $t = 1$, we get

$$||d_k||^2 \leq \Big(\ell_k||u(z_k)|| + \beta_k^{CD}||d_{k-1}||\Big)^2 \leq 3\ell_k^2||u(z_k)||^2 + \frac{3}{2}(\beta_k^{CD})^2||d_{k-1}||^2. \tag{3.17}$$

$$\leq 3\Big(1 - \beta_k^{CD}\frac{\psi(z_k, d_{k-1})}{\psi(z_k, u(z_k))}\Big)^2||u(z_k)||^2 + \frac{3}{2}\Big(\frac{\psi(z_k, u(z_k))}{\psi(z_{k-1}, d_{k-1})}\Big)^2||d_{k-1}||^2$$

$$= 3\Big(1 - \frac{\psi(z_k, d_{k-1})}{\psi(z_{k-1}, d_{k-1})}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, d_{k-1})}||d_{k-1}||^2$$

$$\leq 3\Big(1 + \frac{2|\psi(z_k, d_{k-1})|}{-\psi(z_{k-1}, d_{k-1})} + \frac{\psi^2(z_k, d_{k-1})}{\psi^2(z_{k-1}, d_{k-1})}\Big)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, d_{k-1})}||d_{k-1}||^2.$$

By the SWC (2.9), we have

$$\leq 3(1+2\sigma+\sigma^2)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, d_{k-1})}||d_{k-1}||^2.$$

Applying the SDC (3.6), we get

$$\leq 3(1+2\sigma+\sigma^2)||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, d_{k-1})}||d_{k-1}||^2.$$

Divide through by $\psi^2(z_k, d_k)$, we have

$$\frac{||d_k||^2}{\psi^2(z_k, d_k)} \leq 3(1+2\sigma+\sigma^2)\frac{\|u(z_k)\|^2}{\psi^2(z_k, d_k)} + \frac{3}{2}\frac{\psi^2(z_k, u(z_k))}{\psi^2(z_{k-1}, d_{k-1})}\frac{\|d_{k-1}\|^2}{\psi^2(z_k, d_k)}.$$

Applying SDC (3.6), we have

$$\frac{||d_k||^2}{\psi^2(z_k, d_k)} \leq 3(1+2\sigma+\sigma^2)\frac{\|u(z_k)\|^2}{\psi^2(z_k, u(z_k))} + \frac{3}{2}\frac{\|d_{k-1}\|^2}{\psi^2(z_{k-1}, u(z_{k-1}))}.$$

https://doi.org/10.58715/bangmodjmcs.2023.9.6

From (3.12), the results follows and therefore we have

$$\sum_{k=1}^{\infty} \frac{\psi^2(z_k, d_k)}{\|d_k\|^2} \geq \frac{\delta^2}{6} \sum_{k=1}^{\infty} \frac{1}{((2+4\sigma+2\sigma^2)k+1)} = \infty. \tag{3.18}$$

This is a contradiction. Hence, we have (3.14) which complete the proof. ∎

Theorem 3.6. *Let $\{z_k\}$ be generated by Algorithm 1 with $\beta_k^{DY} \geq 0$ as defined in (1.11) and suppose Assumptions 1 and 2 hold. If the Zoutendijk condition (3.1) holds, then*

$$\liminf_{k\to\infty} \|u(z_k)\| = 0. \tag{3.19}$$

Proof. Suppose we have a constant δ for which

$$\|u(z_k)\| > \delta, \; for \; all \; k \geq 1. \tag{3.20}$$

Now, from (3.2), we have

$$d_k = \ell_k u(z_k) + \beta_k^{DY} d_{k-1}, \; k \geq 2. \tag{3.21}$$

Squaring (3.21) and applying Lemma 2.6(d) by using $p = \ell_k||u(z_k)||$, $q = \beta_k^{DY}||d_{k-1}||$, and $t = 1$, we get

$$||d_k||^2 \leq \Big(\ell_k||u(z_k)|| + \beta_k^{DY}||d_{k-1}||\Big)^2 \leq 3\ell_k^2||u(z_k)||^2 + \frac{3}{2}(\beta_k^{DY})^2||d_{k-1}||^2. \tag{3.22}$$

$$\leq 3\Big(1-\beta_k^{DY}\frac{\psi(z_k,d_{k-1})}{\psi(z_k,u(z_k))}\Big)^2||u(z_k)||^2+\frac{3}{2}\Big(\frac{\psi(z_k,u(z_k))}{\psi(z_k,d_{k-1})-\psi(z_{k-1},d_{k-1})}\Big)^2||d_{k-1}||^2$$

$$= 3\Big(1-\frac{\psi(z_k,d_{k-1})}{\psi(z_k,d_{k-1})-\psi(z_{k-1},d_{k-1})}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k,u(z_k))}{(\psi(z_k,d_{k-1})-\psi(z_{k-1},d_{k-1}))^2}||d_{k-1}||^2$$

$$\leq 3\Big(\frac{1}{1-\frac{\psi(z_k,d_{k-1})}{\psi(z_{k-1},d_{k-1})}}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k,u(z_k))}{(\psi(z_k,d_{k-1})-\psi(z_{k-1},d_{k-1}))^2}||d_{k-1}||^2$$

$$\leq 3\Big(\frac{1}{1-q_k}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k,u(z_k))}{(\psi(z_k,d_{k-1})-\psi(z_{k-1},d_{k-1}))^2}||d_{k-1}||^2.$$

where $q_k = \frac{\psi(z_k,d_{k-1})}{\psi(z_{k-1},d_{k-1})}$. By the SWC (2.9) we have $q_k \in [-\sigma, \sigma]$ with $0 < \sigma < 1$. Thus,

$$||d_k||^2 \leq 3\Big(\frac{1}{1+\sigma}\Big)^2||u(z_k)||^2 + \frac{3}{2}\frac{\psi^2(z_k,u(z_k))}{(\sigma-1)^2\psi^2(z_{k-1},d_{k-1})}||d_{k-1}||^2.$$

Divide through by $\psi^2(z_k, d_k)$, we have

$$\frac{||d_k||^2}{\psi^2(z_k,d_k)} \leq \frac{3}{(1+\sigma)^2}\frac{\|u(z_k)\|^2}{\psi^2(z_k,d_k)} + \frac{3}{2}\frac{\psi^2(z_k,u(z_k))}{\psi^2(z_{k-1},d_{k-1})}\frac{\|d_{k-1}\|^2}{\psi^2(z_k,d_k)}.$$

Applying SDC (3.7), we have

$$\frac{||d_k||^2}{\psi^2(z_k,d_k)} \leq \frac{3}{(1+\sigma)^2}\frac{\|u(z_k)\|^2}{\psi^2(z_k,u(z_k))} + \frac{3}{2}\frac{\|d_{k-1}\|^2}{\psi^2(z_{k-1},u(z_{k-1}))}.$$

Now, using Lemma 2.5 (b) and (3.15), we get

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \le \frac{1}{(1+\sigma)^2}\frac{12}{\delta^2} + \frac{3}{2}\frac{\|d_{k-1}\|^2}{\psi^2(z_{k-1}, u(z_{k-1}))}.$$

Repeating this continuously, we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \le \frac{1}{(1+\sigma)^2}\frac{12k}{\delta^2} + \frac{3}{2}\frac{\|d_1\|^2}{\psi^2(z_1, u(z_1))}.$$

Applying Lemma 2.5 (b) and (3.15), we have

$$\frac{\|d_k\|^2}{\psi^2(z_k, d_k)} \le \frac{1}{(1+\sigma)^2}\frac{12k}{\delta^2} + \frac{6}{\delta^2} \le \frac{6}{\delta^2}\left(\frac{2k+(1+\sigma)^2}{(1+\sigma)^2}\right).$$

Thus,

$$\sum_{k=1}^{\infty}\frac{\psi^2(z_k, d_k)}{\|d_k\|^2} \ge \frac{\delta^2}{6}\sum_{k=1}^{\infty}\left(\frac{(1+\sigma)^2}{2k+(1+\sigma)^2}\right) = \infty. \tag{3.23}$$

This is a contradiction. Hence, we have (3.19) which complete the proof. ∎

4. Numerical experiments

In this section, we report the performance of the proposed techniques. The purpose is to assess their implementation and robustness in solving benchmark test problems derived from a wide range of multiobjective optimization research articles in the literature. The Algorithms were implemented using double precision Fortran 90, and the experiments were conducted on a PC with the following specifications: Intel Core i5-1135G7 CPU running at 2.4GHz, and 16 GB of RAM.

Notice that the vector $e \in Q$ given in (2.6), always exists. Specifically, for multiobjective optimization, we take e to be $[1, \cdots, 1]^T \in \mathbb{R}^m$, Q and C are considered as $\mathbb{R}^m_+$, and canonical basis of $\mathbb{R}^m$, respectively.

Below, we present a summary of the techniques under consideration, including their initial parameter values. This encompasses both our proposed techniques and those employed for comparison purposes:

- DFR: Descent Fletcher-Reeves CG technique;
- DCD: Descent conjugate descent CG technique;
- DDY: Descent Dai-Yuan CG technique.

An essential part of these techniques is the computation of the steepest descent direction, denoted as $u(z)$. To achieve this, we made use of Algencan, a versatile augmented Lagrangian code designed for solving nonlinear problems [5]. In addition, the selection of the step size was performed using a line search strategy that satisfies condition (2.9). Below are the initial parameters utilized in the implementation of our proposed techniques for these line searches:

- $\rho = 10^{-4}, \quad c = 0.7, \quad \sigma = 10^{-1}$.

Furthermore, Lemma 2.5 establishes that $z \in \mathbb{R}^n$ represents a Q-critical point of F only if $v(z) = 0$. Based on this findings, the experimentation process involved executing all the implemented techniques until the point of convergence, defined as $v(z) \ge -5 \times eps^{\frac{1}{2}}$. Here, $v(z)$ is defined as $\psi(z, u(z)) + \frac{\|u(z)\|^2}{2}$, and eps corresponds to the machine precision, approximately 2.22×10^{-16}. Alternatively, the process terminates if the maximum number of iterations, $max.It = 5000$, is exceeded.

Let us now discuss on the provided tables. Table 1 presents essential information regarding the selected test problems. In the first column, we have the names of the problems, such as "MGH" corresponding to the problem introduced by Mor'e, Garbow, and Hillstrom in [40], and "SLC2" aligning with the second problem proposed by Sch"utze, Lara, and Coello in [43]. The second column indicates the type of problem (convex or otherwise), the third and Fourth columns, labeled as "n" and "m," respectively, indicate the variables under consideration and the objectives of the problems. To generate the starting points, a box constraint was utilized, defined as z_1 with the lower and upper bounds denoted in the fifth column, while the last column shows the corresponding references. These results are established by solving each of the test problems 200 times, using a starting point from uniform random distribution within a defined box defined in the fifth column of Table 1. The process involved starting from various initial points to explore the solution space.

Tables 2, 3 and 4 present the results of Algorithm 1 with the modified search direction of FR, CD, and DY β_k parameters, the results are compared with the FR, CD, and DY CG techniques considered in [37]: "%", "It", "Fe", and "Ge". In this case, "%" denotes the percentage of execution that has attained a critical point and for the successful execution, while "It", "Fe", and "Ge" indicate the median number of the 200 runs for the iterations, functions, and gradients evaluations, respectively. It is important to emphasize that each evaluation of objective functions or gradient in the corresponding computation is accounted for in their respective columns. We emphasize that, despite obtaining sufficient descent conditions without using any line search, our results are not better than those given in [37] with the usual search directions. This shows that modification of the search direction in the vector setting may yield good convergence properties but not necessarily good numerical results.

TABLE 1. List of Test Problems

problems	convex	n	m	z_1	refs
JOS1	✓	1000	2	$[-10000, 10000]^n$	[29]
SLC2	✓	10	2	$[-100, 100]^n$	[43]
	✓	500	2	$[-100, 100]^n$	[43]
SLCDT1	×	2	2	$[-5, 5]^n$	[44]
AP1	✓	2	3	$[-100, 100]^n$	[3]
AP2	✓	2	2	$[-100, 100]^n$	[3]
AP3	×	2	2	$[-100, 100]^n$	[3]
Lov1	✓	2	2	$[-100, 100]^n$	[34]
Lov3	×	2	2	$[-100, 100]^n$	[34]
Lov4	×	2	2	$[-100, 100]^n$	[34]
FDS	✓	2	3	$[-2, 2]^n$	[15]
MMR1	×	2	2	$[0, 1]^n$	[39]
MOP1	✓	2	2	$[-100000, 100000]^n$	[29]
MOP2	×	2	2	$[-1, 1]^n$	[29]
MOP3	×	2	2	$[-\pi, \pi]$	[29]
MOP5	×	2	3	$[-1, 1]^n$	[29]
MOP7	✓	2	2	$[-400, 400]^n$	[29]
DGO1	✓	2	2	$[-10, 13]^n$	[29]
MLF1	×	2	2	$[-100, 100]^n$	[29]
SP1	✓	2	2	$[-100, 100]^n$	[29]
SSFYY2	×	2	2	$[-100, 100]^n$	[29]
SK1	×	2	2	$[-100, 100]^n$	[29]
Hil1	×	2	2	$[0, 1]^n$	[27]
KW2	×	2	2	$[-3, 3]^n$	[32]
Toi4	✓	4	2	$[-100, 100]^n$	[46]
Toi8	×	2	2	$[-1, 1]^n$	[46]
Toi9	×	4	4	$[-100, 100]^n$	[46]
MGH26	×	4	4	$[-1, 1]^n$	[40]
MGH33	✓	10	10	$[-1, 1]^n$	[40]
PNR	✓	2	2	$[-1, 1]^n$	[41]
SLCDT2	✓	10	3	$[-100, 100]^n$	[44]

TABLE 2. DFR and FR techniques results

	DFR				FR			
Problem	%	It	Fe	Ge	%	It	Fe	Ge
JOS1	100	1	2	4	100	1	2	4
SLC2 (n=10)	100	76	515	459.5	100	21.5	163.5	133
SLC2 (n=500)	100	1195	8419.5	6908	100	140	950	765.5
SLCDT1	100	2	18.5	18.5	100	2	18.5	18.5
AP1	100	851	7686.5	6000.5	100	150.5	1385.5	1104
AP2	100	1	2	4	100	1	2	4
AP3	100	196.5	1379	1020.5	100	58.5	446	355.5
Lov1	100	3	6	8	100	3	6	8
Lov3	100	3	9	11	100	3	9	11
Lov4	100	1	5	7	100	1	5	7
FDS	100	210.5	1933	18.5	100	87	851	737.5
MMR1	100	60.5	426.5	312.5	100	42.5	300.5	224.5
MOP1	100	1	2	4	100	1	2	4
MOP2	100	73.5	499.5	431	100	48	335	305
MOP3	100	14	70	62	100	13.5	64	56
MOP5	100	1	14	16	100	1	14	16
MOP7	100	7	21	24	100	7	21	24
DGO1	100	2	12.5	12.5	100	2	12.5	12.5
MLF1	100	1	7	7.5	100	1	7	7.5
SP1	100	5	10	12	100	6	12	14
SSFYY2	100	1	9	10	100	1	9	10
SK1	100	2	20	20	100	2	20	20
Hil1	100	874.5	6973.5	5471	100	150.5	1011.5	765.5
KW2	100	730.5	7753	5881	100	172	1200	1003.5
Toi4	100	4	37	33	100	4	37	33
Toi8	100	1	7	9	100	1	7	9
Toi9	100	1354	20336	15950.5	100	130.5	1714.5	1290.5
MGH26	100	3	46	41.5	100	3	46.5	41.5
MGH33	100	1	82	71	100	1	82	71
PNR	100	1	3	5	100	1	3	5
SLCDT2	100	495	5056	3983	100	63.5	591.5	514.5

TABLE 3. DCD and CD techniques results

	DCD				CD			
Problem	%	It	Fe	Ge	%	It	Fe	Ge
JOS1	100	1	2	4	100	1	2	4
SLC2 (n=10)	100	20.5	166.5	142.5	100	10	98	85.5
SLC2 (n=500)	100	1202.5	11910	9639	100	49.5	362.5	310
SLCDT1	100	2	16	16	100	2	18.5	18.5
AP1	100	836	7551	5894.5	100	47.5	482.5	432.5
AP2	100	1	2	4	100	1	2	4
AP3	100	22	194	163.5	100	25	217	186.5
Lov1	100	3	6	8	100	3	6	8
Lov3	100	3	9	11	100	3	9	11
Lov4	100	1	5	7	100	1	5	7
FDS	100	207.5	1906	1530.5	100	33.5	332.5	297.5
MMR1	100	58.5	412.5	302.5	100	22.5	160	139
MOP1	100	1	2	4	100	1	2	4
MOP2	100	70.5	480	418	100	23	159.5	146.5
MOP3	100	14	68	60	100	12	56	49.5
MOP5	100	1	14	16	100	1	14	16
MOP7	100	7	21	24	100	7	21	24
DGO1	100	2	12.5	12.5	100	2	12.5	12.5
MLF1	100	1	7	7.5	100	1	7	7.5
SP1	100	5	10	12	100	7.5	15	17
SSFYY2	100	1	9	10	100	1	9	10
SK1	100	2	20	20	100	2	20	20
Hil1	100	737.5	6623.5	5189	100	47.5	326.5	280.5
KW2	100	451	2955	2289	100	52.5	369	323
Toi4	100	4	37	33	100	4	31	28
Toi8	100	1	7	9	100	1	7	9
Toi9	100	1497.5	22652.5	16726.5	100	57.5	751.5	620.5
MGH26	100	3	46	41.5	100	3	44	42
MGH33	100	1	82	71	100	1	82	71
PNR	100	1	3	5	100	1	3	5
SLCDT2	100	135	1198.5	987	100	39.5	413.5	376

Table 4. DDY and DY techniques results

	DDY				DY			
Problem	%	It	Fe	Ge	%	It	Fe	Ge
JOS1	100	1	2	4	100	1	2	4
SLC2 (n=10)	100	20.5	166.5	142.5	100	12	123.5	113.5
SLC2 (n=500)	100	477.5	4073	3290	100	35	273	239
SLCDT1	100	2	18	18	100	2	18.5	18.5
AP1	100	836	7551	5894.5	100	32.5	321.5	289.5
AP2	100	1	2	4	100	1	2	4
AP3	100	22.5	195.5	165	100	26.5	210.5	184.5
Lov1	100	3	6	8	100	3	6	8
Lov3	100	3	9	11	100	3	9	11
Lov4	100	1	5	7	100	1	5	7
FDS	100	207.5	1906	1530.5	100	23.5	232.5	207.5
MMR1	100	58.5	412.5	302.5	100	17	121.5	102
MOP1	100	1	2	4	100	1	2	4
MOP2	100	70.5	480	418	100	17.5	115.5	94
MOP3	100	14	68	60	100	11	51.5	47
MOP5	100	1	14	16	100	1	14	16
MOP7	100	7	21	24	100	7	21	24
DGO1	100	2	12.5	12.5	100	2	12.5	12.5
MLF1	100	1	7	7.5	100	1	7	7.5
SP1	100	5	10	12	100	8	16	18
SSFYY2	100	1	9	10	100	1	9	10
SK1	100	2	20	20	100	2	20	20
Hil1	100	737	6623.5	5189	100	32	225.5	203
KW2	100	448.5	2940	2276.5	100	35	242.5	230.5
Toi4	100	4	37	33	100	4	31	28
Toi8	100	1	7	9	100	1	7	9
Toi9	100	1611	23112	16487	100	37	489	424
MGH26	100	3	45.5	40	100	3	45.5	40.5
MGH33	100	1	82	71	100	1	82	71
PNR	100	1	3	5	100	1	3	5
SLCDT2	100	182	1965.5	1630.5	100	30.5	326	283.5

5. Closing Remarks

We have proposed three modified conjugate gradient (CG) techniques for solving vector optimization problems. Specifically, we have modified the search directions of the Fletcher-Reeves (FR), conjugate descent (CD), and Dai-Yuan (DY) CG techniques to obtain their descent property without the use of any line search and to achieve good convergence properties. We have established the descent property without line search and achieved global convergence using the Wolfe line search. Additionally, we present numerical experiments to demonstrate the implementation and efficiency of the proposed techniques.

Statements and Declarations

Conflict of interest The authors declare that they have no competing interest.

Data availability Not applicable.

Acknowledgments The authors express their gratitude for the financial assistance received from the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), King Mongkut's University of Technology Thonburi (KMUTT). Additionally, the authors acknowledge the financial support provided by the Mid-Career Research Grant (N41A640089). The first author acknowledges the support provided by the Petchra Pra Jom Klao PhD scholarship of KMUTT (Contract No. 23/2565).

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https://doi.org/10.58715/bangmodjmcs.2023.9.6
Bangmod J-MCS 2023