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Existence Theorems of Fixed Points for $(\mathcal{Z}, \lambda)$-Enriched Contraction in Convex Metric Spaces

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Abstract In this paper, we introduce the simulation function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and define a mapping $T : X \to X$ as a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$, which generalizes the Banach contraction principle. To indicate the relevance of our new results, we present some important particular cases of the fixed point theorem along with supportive examples.

MSC: 54H25, 47H10, 54C30

Keywords: Simulation Function; $(\mathcal{Z}, \lambda)$-Enriched Contraction; Convex Metric Spaces; Convex Structure

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1. Introduction

The concept of a metric space was first introduced by Frchet in 1906 [1]. Since then, mathematicians have explored the existence and uniqueness of fixed points using the Banach contraction principle, which has been extended to various generalized metric spaces [2].

Fixed point theory plays a crucial role in nonlinear analysis, particularly in studying the existence and approximation of solutions for nonlinear functional equations, differential equations, integral equations, and integro-differential equations. One of the most widely used metrical fixed point theorems in nonlinear analysis is undoubtedly Banachs contraction mapping principle, which is based on a symmetric contraction. Given a mapping $T : X \to X$, there exists a constant $c \in [0, 1)$ satisfying the condition

$$d(Tx, Ty) \leq cd(x, y), \text{ for all } a, y \in X. \tag{1.1}$$

A point $x \in X$ is called a fixed point of T if it satisfies $T(x) = x$.

Following this fundamental principle, numerous researchers have generalized it by introducing various types of contractions in metric spaces [3–8]. In particular, Rhoades [9], compared several contraction conditions defined on metric spaces.

In 2015, Khojasteh, Shukla and Radenovic [10] introduced the concept of a simulation function$\zeta \in \mathcal{Z}$ and defined a mapping $T : X \to X$ as a $\mathcal{Z}$-contraction, generalizing Banachs contraction principle. This concept unifies several known types of contractions by incorporating conditions involving both $d(Tx, Ty)$ and $d(x, y)$ in complete metric spaces, ultimately proving the existence of a fixed point for T.

Later, in 2021, Berinde and Păcurar [11], established existence and uniqueness results for fixed points under symmetric contractive-type conditions in convex metric spaces. Their findings also provided approximation results for certain classes of such mappings. Further, research on the topic, we refer [12–14].

In this study, we introduce the simulation function $\zeta \in \mathcal{Z}$ and define a mapping $T : X \to X$ as a $(\mathcal{Z}, \lambda)$-enriched contraction, where $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$. We then prove a fixed point existence theorem for T in a convex metric space.

2. Preliminaries

In this section, we present some definitions and lemmas that are needed for our main results presented in section 3.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a metric space, and d is called a metric on X.

Definition 2.2. [15] Let (X, d) be a metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for all } u \in X. \tag{2.1}$$

A metric space (X, d) endowed with a convex structure W is called a Takahashi convex metric space and is usually denoted by (X, d, W).

Definition 2.3. [11] Let (X, d, W) be a convex metric space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist $c \in [0, 1)$ and $\lambda \in [0, 1)$ such that

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \leq cd(x, y), \forall x, y \in X. \tag{2.2}$$

To specify the parameters c and λ involved in (2.2), we called T a (λ, c)-enriched contraction.

Definition 2.4. [11] Let (X, d, W) be a complex valued metric space. A continuous function $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if, for all $x, y \in X$ and any $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \precsim \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for all } u \in X. \tag{2.3}$$

A metric space (X, d) endowed with a convex structure W is called a complex valued convex metric space and is usually denoted by (X, d, W).

Lemma 2.5. [11] *Let (X, d, W) be a complex valued convex metric space. For all $x, y \in X$ and for all $\lambda \in [0, 1]$, such that*

$$d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y). \tag{2.4}$$

Lemma 2.6. [11] *Let (X, d, W) be a complex valued convex metric space. For all $x, y \in X$ and for all $\lambda \in [0, 1]$, such that*

$$d(x, W(x, y; \lambda)) = (1 - \lambda)d(x, y) \text{ and } d(W(x, y; \lambda), y) = \lambda d(x, y) \tag{2.5}$$

Definition 2.7. [10] Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions;

(ζ1) $\zeta(0, 0) = 0$;
(ζ2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ3) if $\{t_n\}, \{s_n\}$ are sequence in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 2.8. [10] Let (X, d) be a metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied

$$\zeta\left(d(Tx, Ty), d(x, y)\right) \geq 0, \quad \text{for all } x, y \in X. \tag{2.6}$$

If T is $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then $d(Tx, Ty) < d(x, y)$ for all distinct $x, y \in X$.

From Definition 2.8 we defined some contraction in convex metric space as follows;

Definition 2.9. Let (X, d, W) be a convex metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$, $\lambda \in [0, 1)$. Then T is called a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$, if the following condition is satisfied

$$\zeta\Big(d(W(x, Tx; \lambda), W(y, Ty; \lambda)), d(x, y)\Big) \geq 0, \quad \text{for all } x, y \in X. \tag{2.7}$$

3. Main Results

In this section, we introduce the simulation function $\zeta \in \mathcal{Z}$ and a mapping $T : X \to X$ is a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$, and prove the existance theorem of fixed point of T in convex metric space.

Our notion of (ζ, λ)-enriched contraction simultaneously exploits the flexibility of simulation functions (cf. [10]) and the relaxation mechanism of enriched/Krasnosel'skii-type iterations (cf. [11]).

Compared with classical $\mathcal{Z}$-contractions [2, 10, 11], the parameter $\lambda \in [0, 1)$ interpolates between the Picard step ($\lambda = 0$) and a relaxed step ($\lambda > 0$), which can improve stability on nonexpansive or weakly contractive operators. When $\zeta(t, s) = cs - t$ with $c \in [0, 1)$ and $W(x, y; \lambda) = \lambda x + (1-\lambda)y$, our results recover Banach-type theorems as special cases, while allowing a broader range of admissible contractive behaviours encoded by ζ.

Moreover, the proofs quantify the asymptotic regularity of the relaxed sequence and clarify how λ influences the convergence speed via the one-step decrease inequality.

Theorem 3.1. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$. Let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = W(x_n, Tx_n; \lambda), \;\; n \geq 0, x_0 \in X. \tag{3.1}$$

Then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ defined in (3.1). Suppose that $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. Since

$$d(x_{n+1}, x_n) = d\left(W(x_n, Tx_n; \lambda), W(x_{n-1}, Tx_{n-1}; \lambda)\right). \tag{3.2}$$

From (3.1), (3.2) and Definition 2.7 ($\zeta 2$), we have

$$\zeta\Big(d\Big(W(x_n, Tx_n; \lambda), W(x_{n-1}, Tx_{n-1}; \lambda)\Big), d(x_n, x_{n-1})\Big) = \zeta\Big(d(x_{n+1}, x_n), d(x_n, x_{n-1})\Big) \tag{3.3}$$

$$< d(x_n, x_{n-1}) - d(x_{n+1}, x_n). \tag{3.4}$$

Since, ζ is a $(\mathcal{Z}, \lambda)$-enriched contraction, from Definition 2.9 and (3.3), we have

$$\zeta\Big(d(x_{n+1}, x_n), d(x_n, x_{n-1})\Big) \geq 0, \tag{3.5}$$

and from (3.4), we have

$$d(x_n, x_{n-1}) - d(x_{n+1}, x_n) > 0, \tag{3.6}$$

it follows that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

So, the sequence $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers. Then, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Assume that $r > 0$. From Definition 2.7 ($\zeta 3$), we replace $t_n := d(x_{n+1}, x_n)$ and $s_n := d(x_n, x_{n-1})$, we see that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = \lim_{n\to\infty} d(x_n, x_{n-1}), \tag{3.8}$$

it follows that

$$\limsup_{n\to\infty} \zeta\Big(d(x_{n+1}, x_n), d(x_n, x_{n-1})\Big) < 0. \tag{3.9}$$

From (3.5) and (3.9), a contradiction, then $r = 0$. Therefore, $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Lemma 3.2. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Then sequence $\{x_n\}$ defined in (3.1) is a bounded sequence.*

Proof. Assume that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1, \tag{3.10}$$

and

$$d(x_m, x_{n_k}) \leq 1, \text{ for } n_k \leq m \leq x_{n_{k+1}-1}. \tag{3.11}$$

Therefore,

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Taking $k \to \infty$, and Theorem 3.1, we obtain that

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1. \tag{3.12}$$

From Definition 2.9, we have

$$\begin{aligned} 0 &< \zeta\Big(d(W(x_{n_{k+1}-1}, Tx_{n_{k+1}-1}; \lambda), W(x_{n_k-1}, Tx_{n_k-1}; \lambda)), d(x_{n_{k+1}-1}, x_{n_k-1})\Big) \\ &= \zeta\Big(d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})\Big) \\ &< d(x_{n_{k+1}-1}, x_{n_k-1}) - d(x_{n_{k+1}}, x_{n_k}), \end{aligned}$$

it follows that

$$d(x_{n_{k+1}}, x_{n_k}) < d(x_{n_{k+1}-1}, x_{n_k-1}).$$

We obtain that

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &< d(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) \\ &\leq 1 + d(x_{n_k}, x_{n_k-1}). \end{aligned}$$

Taking $k \to \infty$ and using Theorem 3.1, we have

$$\lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = 1. \tag{3.13}$$

Since T is a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$, (3.12), (3.13) and Definition 2.7 $(\zeta 2), (\zeta 3)$, we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta\Big(d(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})\Big) \\ &< \limsup_{k\to\infty} \Big[d(x_{n_{k+1}-1}, x_{n_k-1}) - d(x_{n_{k+1}}, x_{n_k})\Big] \\ &\leq 0. \end{aligned}$$

A contradiction. Therefore, $\{x_n\}$ is a bounded sequence.

Theorem 3.3. *Let (X,d,W) be a complete convex metric space and let $T:X\to X$ be a $(\mathcal{Z},\lambda)$-enriched contraction with respect to $\zeta\in\mathcal{Z}$ and $\lambda\in[0,1)$. Then T has a fixed point.*

Proof. Let $x_0\in X$, if there exists $m\in\mathbb{N}$ such that $x_m=x_{m+1}$, then

$$\begin{aligned} 0 &= d(x_m,x_{m+1}) \\ &= d(x_m,W(x_m,Tx_m;\lambda)) \\ &= \lambda d(x_m,x_m)+(1-\lambda)d(x_m,Tx_m) \\ &= (1-\lambda)d(x_m,Tx_m) \\ \therefore d(x_m,Tx_m) &= 0, \end{aligned}$$

it implies that x_m is a fixed point of T. So, we can suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}$. Let

$$D_n=\sup\{d(x_i,x_j):i,j\geq n,n\in\mathbb{N}\}. \tag{3.14}$$

By Lemma 3.2, we have $\{x_n\}$ is bounded sequence, it follows that $D_n<\infty$ for any $n\in\mathbb{N}$ and $\{D_n\}$ is a positive decreasing sequence, there exists $d\geq 0$ such that

$$\lim_{n\to\infty} D_n=d. \tag{3.15}$$

Assume that $d>0$. From definition of D_n, for each $k\in\mathbb{N}$ there exists $n_k,m_k\in\mathbb{N}$ such that $m_k>n_k\geq k$ and

$$D_k-\frac{1}{k}<d(x_{m_k},x_{n_k})\geq D_k. \tag{3.16}$$

Hence,

$$\lim_{n\to\infty} d(x_{m_k},x_{n_k})=d. \tag{3.17}$$

From Definition 2.9 and Definition 2.7 ($\zeta 2$), we have

$$\begin{aligned} 0 &\leq \zeta\Big(d(W(x_{m_k-1},Tx_{m_k-1};\lambda),W(x_{n_k-1},Tx_{n_k-1};\lambda)),d(x_{m_k-1},x_{n_k-1})\Big) \\ &= \zeta\Big(d(x_{m_k},x_{n_k}),d(x_{m_k-1},x_{n_k-1})\Big) \\ &< d(x_{m_k-1},x_{n_k-1})-d(x_{m_k},x_{n_k}), \end{aligned}$$

it follows that

$$\begin{aligned} d(x_{m_k},x_{n_k}) &< d(x_{m_k-1},x_{n_k-1}) \\ &\leq d(x_{m_k-1},x_{m_k})+d(x_{m_k},x_{n_k})+d(x_{n_k},x_{n_k-1}). \end{aligned}$$

Taking $k\to\infty$ and Theorem 3.1, we have

$$\begin{aligned} d=\lim_{k\to\infty} d(x_{m_k},x_{n_k}) &\leq \lim_{k\to\infty} d(x_{m_k-1},x_{n_k-1}) \\ &\leq \lim_{k\to\infty} d(x_{m_k-1},x_{m_k})+\lim_{k\to\infty} d(x_{m_k},x_{n_k})+\lim_{k\to\infty} d(x_{n_k},x_{n_k-1}) \\ &= \lim_{k\to\infty} d(x_{m_k},x_{n_k})=d, \end{aligned}$$

it implies that

$$\lim_{k\to\infty} d(x_{m_k-1},x_{n_k-1})=d. \tag{3.18}$$

Since T isa $(\mathcal{Z},\lambda)$-enriched contraction with respect to $\zeta\in\mathcal{Z}$ and $\lambda\in[0,1)$, (3.17), (3.18) and Definition 2.7 ($\zeta 3$), we have

$$0\leq \limsup_{k\to\infty}\zeta\Big(d(x_{m_k-1},x_{n_k-1}),d(x_{m_k},x_{n_k})\Big)<0.$$

A contradiction, it follows that $d=0$. Then $\lim\limits_{n\to\infty}D_n=0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X,d,W) is a complete, there exists $p\in X$ such that $\lim\limits_{n\to\infty}x_n=p$.

Now, we shall prove that $p\in Fix(T)$. Assume that $p\neq Tp$, then $d(p,Tp)>0$. Using, Definition 2.9 and Definition 2.7 ($\zeta 2$), we have

$$\begin{aligned}
0 &\leq \zeta\Big(d(W(x_n,Tx_n;\lambda),W(p,Tp;\lambda)),d(x_n,p)\Big)\\
&< d(x_n,p)-d(W(x_n,Tx_n;\lambda),W(p,Tp;\lambda))\\
&= d(x_n,p)-\Big[d(x_{n+1},W(p,Tp;\lambda))\Big],
\end{aligned}$$

taking $n\to\infty$, it follows that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty}\zeta\Big(d(W(x_n,Tx_n;\lambda),W(p,Tp;\lambda)),d(x_n,p)\Big)\\
&\leq \limsup_{n\to\infty}\Big(d(x_n,p)-\Big[d(x_{n+1},W(p,Tp;\lambda))\Big]\Big)\\
&= -d(p,W(p,Tp;\lambda))\\
&= -(1-\lambda)d(p,Tp).
\end{aligned}$$

A contradiction. Therefore, p is a fixed point of T.

Finally, we define a mapping $\zeta_c:[0,\infty)\times[0,\infty)\to\mathbb{R}$ by

$$\zeta_c(t,s)=cs-t,\quad \text{for all } s,t\in[0,\infty). \tag{3.19}$$

We see that

(1) $\zeta_c(0,0)=c(0)-0=0$,
(2) $\zeta_c(t,s)=cs-t<s-t,\ c\in[0,1)$,
(3) if $\{t_n\},\{t_n\}\subseteq[0,\infty)$ with $\lim\limits_{n\to\infty}t_n=r=\lim\limits_{n\to\infty}s_n, r>0$ we have

$$\limsup_{n\to\infty}\zeta_c(t_n,s_n)=\limsup_{n\to\infty}(cs_n-t_n), \tag{3.20}$$

we see that, if $c=0$ and (3.20), we have

$$\limsup_{n\to\infty}\zeta_c(t_n,s_n)=\limsup_{n\to\infty}(-t_n)<0,$$

if $0<c<1$ and (3.20), we have

$$\limsup_{n\to\infty}\zeta_c(t_n,s_n)<\limsup_{n\to\infty}(s_n-t_n)=0.$$

Hence, $\zeta_c\in\mathcal{Z}$.

Corollary 3.4. *Under the assumptions of Theorem 3.3, the fixed point of T is unique.*

Proof. Suppose $p,q\in X$ are fixed points of T. Using the (ζ,λ)-enriched contraction inequality with $x=p$ and $y=q$, and the fact that $Tp=p$ and $Tq=q$, we obtain

$$\zeta(d(W(p,Tp;\lambda),\,W(q,Tq;\lambda))\,,\,d(p,q))\geq 0.$$

Because $W(p, Tp; \lambda) = W(p, p; \lambda) = p$ and $W(q, Tq; \lambda) = q$, this becomes $\zeta(d(p, q), d(p, q)) \geq 0$. By the defining properties of a simulation function, this is possible only when $d(p, q) = 0$; hence $p = q$.

Corollary 3.5. *Let (X, d, W) be a complete convex metric space and let $T : X \to X$ be a $(\mathcal{Z}, c)$-enriched contraction. Then T has a fixed point in X.*

Proof. Taking $\zeta = \zeta_c$ in Theorem 3.3, we have T is a $(\mathcal{Z}, \lambda)$-enriched contraction with respect to $\zeta_c \in \mathcal{Z}$ and $\lambda \in [0, 1)$. Therefore, T has a fixed point in X.

The following example support our theorem 3.3.

Example 3.6. Let $X = [0, 1]$ and d be a Euclidean metric on X. Set $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for any $\lambda \in [0, 1)$. Then (X, d, W) is a complete convex metric space.

Let us define $T : X \to X$ by $Tx = 1 - x$ and $\zeta(t, s) = cs - t$.

Then, the mapping T satisfies all the conditions of theorem 3.3, with $\lambda = \frac{1}{2}$ and for $c \in (0, 1)$ (if the case $c = 0$ is admitted, then take $c \in [0, 1)$) and hence admit a fixed point $p = \frac{1}{2}$.

Remark 3.7. The above toy example illustrates the mechanism; in Section 4 we include more substantive applications where the enriched step is indispensable.

Theorem 3.8. *Let (X, d, W) be a convex metric space endowed with a directed graph $G = (X, \mathcal{E})$, and let $T : X \to X$ be a self-mapping such that:*

(1) T is a (ζ, λ)-enriched contraction with respect to some simulation function $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$;

(2) there exists $x_0 \in X$ such that the iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = W(x_n, Tx_n; \lambda),$$

satisfies $(x_n, x_{n+1}) \in \mathcal{E}$ for all $n \in \mathbb{N}$.

Then $\{x_n\}$ converges to a fixed point $p \in X$, and this fixed point is the limit of a walk in the graph G.

Proof. Let $x_0 \in X$ be given, and define the sequence $\{x_n\}$ by

$$x_{n+1} = W(x_n, Tx_n; \lambda), \quad \text{for all } n \in \mathbb{N}.$$

Since T is a (ζ, λ)-enriched contraction, we can apply Theorem 3.1 to conclude that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0,$$

and that the sequence $\{x_n\}$ is Cauchy in X. As (X, d) is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. We next show that p is a fixed point of T. From the recursive definition and the continuity of W and T, we have

$$x_{n+1} = W(x_n, Tx_n; \lambda) \longrightarrow W(p, Tp; \lambda).$$

On the other hand, $x_{n+1} \to p$ implies that

$$W(p, Tp; \lambda) = p.$$

Since W is the convex combination map with parameter $\lambda \in (0, 1)$, it follows that $p = Tp$. Thus, p is a fixed point of T.

Finally, by assumption, we have $(x_n, x_{n+1}) \in \mathcal{E}$ for all $n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ defines a walk in the graph G converging to the fixed point p.

This completes the proof.

4. Applications and Numerical Examples

We illustrate the applicability of the (ζ, λ)-enriched scheme via two simple models. Throughout we take $W(x, y; \lambda) = \lambda x + (1 - \lambda)y$ and $\zeta(t, s) = cs - t$ with $c \in [0, 1)$.

Example 4.1. (Fixed point form of an ODE steady state) Consider the scalar ODE $\dot{x} = -x + \cos x$. A steady state satisfies $x = \cos x$, i.e., the fixed point equation $x = T(x)$ with $T(x) = \cos x$ on $X = [0, 1]$. Starting from $x_0 = 0$, define $x_{n+1} = W(x_n, Tx_n; \lambda)$ with $\lambda = 0.3$. The generated errors $e_n = |x_n - x_*|$ (with $x_* \approx 0.739085$) decay monotonically; see Figure 1.

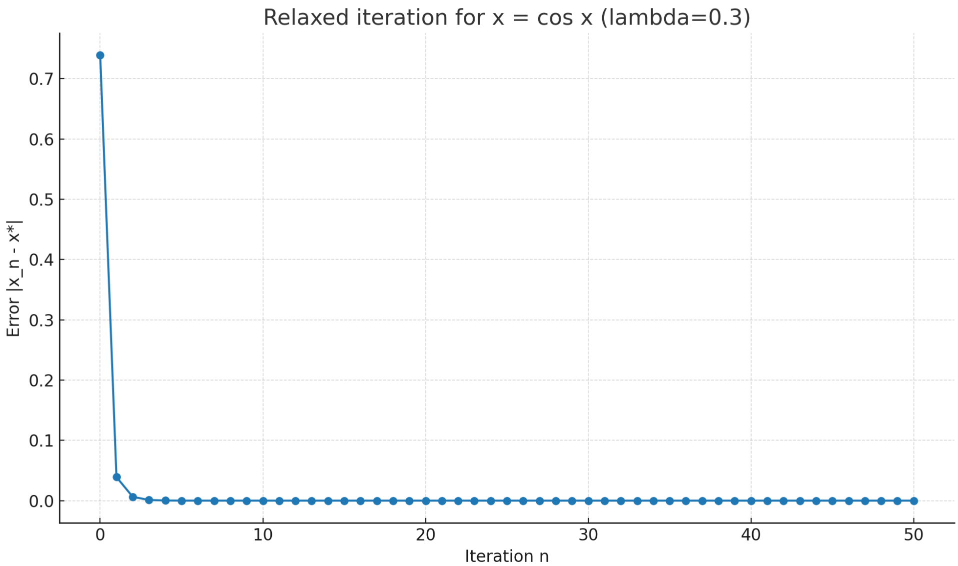

Figure 1. Error decay per iteration for the relaxed iteration mapping $T(x) = \cos x$ with $\lambda = 0.3$. The x-axis represents iteration count and the y-axis represents error magnitude.

Example 4.2. (Convex feasibility via alternating relaxed steps) Let $C_1 = [0, 0.8]$ and $C_2 = [0.3, 1] \subset \mathbb{R}$. Consider the composition $T = P_{C_2} \circ P_{C_1}$ of metric projections. With $x_0 = 1$, the relaxed iteration $x_{n+1} = W(x_n, Tx_n; \lambda)$ with $\lambda = 0.5$ converges to a point in $C_1 \cap C_2 = [0.3, 0.8]$; the error trajectory is shown in Figure 2.

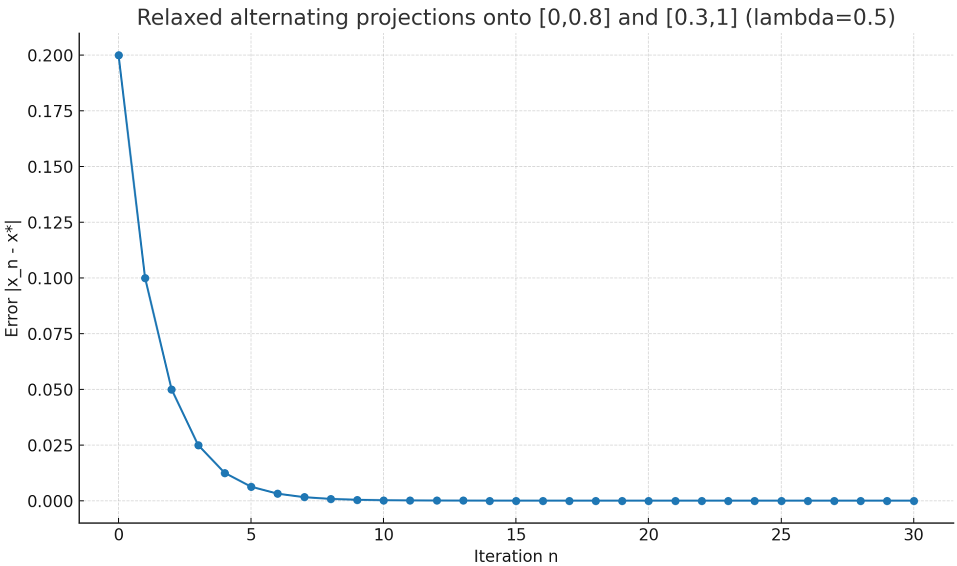

FIGURE 2. Convergence trajectory of the relaxed projection scheme onto $C_1 \cap C_2$. The x-axis and y-axis represent coordinate values in $\mathbb{R}^2$.

5. Conclusion

In this paper, we introduce a novel concept of $(\mathcal{Z}, \lambda)$-Enriched Contraction with respect to $\zeta \in \mathcal{Z}$ and $\lambda \in [0, 1)$ which generalize the Banach contraction principle.To illustrate our results, we present specific cases with supportive examples. it will be interesting to extend the obtain results in other generalized spaces such as: probabilistic, fuzzy, or cone metric spaces.

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