BANGMOD INTERNATIONAL JOURNAL OF
MATHEMATICAL & COMPUTATIONAL SCIENCE
ISSN: 2408-154X (Print)
ISSN: 3057-0557 (Online)
https://bangmodjmcs.com
Volume 11 (2025)
Pages 157–183

Elegant Rational Contractive Conditions with Applications to Implicit Functional Integral Equations

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Received: 9 March 2025 / Accepted: 19 June 2025

Abstract This paper aims to introduce several new classes of contraction mappings inspired by convex and rational contraction mappings. We establish the existence and uniqueness of fixed points for each newly proposed contraction mapping in metric spaces. To validate our theoretical findings, we provide several illustrative examples that demonstrate cases where well-known fixed point results, such as the Banach contraction principle, the Kannan fixed point theorem, the Chatterjea fixed point theorem, the Jaggi fixed point theorem, and the Istrăţescu fixed point theorem, are not applicable. As an application, we employ our theoretical fixed point results to investigate the existence and uniqueness of solutions to nonlinear implicit integral equations.

MSC: 47H10, 54E50, 45G10

Keywords: Complete metric spaces; fixed points; nonlinear integral equations; rational type contractions

Published online: 26 June 2025

Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: S.I. Hassnain et al., Elegant Rational Contractive Conditions with Applications to Implicit Functional Integral Equations, Bangmod Int. J. Math. & Comp. Sci., Vol. 11 (2025), 157–183. https://doi.org/10.58715/bangmodjmcs.2025.11.8

1. Introduction and preliminaries

This section provides a brief historical overview of some evolution of the metric fixed point theory, which inspired us to write this article. The first famous metric fixed point result, named the Banach contraction principle, stated that if T is a self-mapping defined on a complete metric space (X, d) satisfying

$$d\left(Tx, Ty\right) \leq \delta d\left(x, y\right) \tag{1.1}$$

for all $x, y \in X$, where $0 \leq \delta < 1$, then T has a unique fixed point, that is, there exists a unique point $z \in X$ such that $Tz = z$. This result was established by Banach [1] in 1922 and is used to prove the existence and uniqueness of a solution for an integral equation under the appropriate conditions. Furthermore, it can be used to guarantee the existence of solutions for various mathematical equations, including ordinary differential equations, partial differential equations, fractional differential equations, matrix equations, and functional equations. Nowadays, the Banach contraction principle is the most classical metric fixed point result, which serves as a motivation for many famous fixed point results in this era.

Next, we provide a brief history of other famous metric fixed point results. For instance, Kannan [2] and Chatterjea [3] introduced the contractive conditions, which are separated from the Banach contractive condition as follows:

Definition 1.1 ([2, 3])**.** Let T be a self-mapping defined on a metric space (X, d) and

$$d(Tx, Ty) \leq kA(x, y) \tag{1.2}$$

for $x, y \in X$, where $k \in [0, 1/2)$ and $A : X \times X \to [0, \infty)$ is a given function.

(1) If A is defined by $A(x, y) = d(x, Tx) + d(y, Ty)$ for all $x, y \in X$, then T is called a Kannan contraction mapping (see [2]).
(2) If A is defined by $A(x, y) = d(x, Ty) + d(y, Tx)$ for all $x, y \in X$, then T is called a Chatterjea contraction mapping (see [3]).

Kannan [2], and Chatterjea [3] also claimed that each of the Kannan contraction mappings and Chatterjea contraction mappings has a unique fixed point if its domain is complete. These results are known as the Kannan fixed point theorem and the Chatterjea fixed point theorem, respectively. These results are not an enlargement of the Banach contraction principle. Afterward, Dass and Gupta [4] introduced a new contractive condition, named the rational contractive condition, which differs from the Banach, Kannan, and Chatterjea contractive conditions. Similarly, Jaggi [5] introduced the other rational contractive condition as follows:

Definition 1.2 ([5])**.** Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called a rational type contraction mapping if it is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx) d(y, Ty)}{d\left(x, y\right)} + \beta d\left(x, y\right) \tag{1.3}$$

for all $x, y \in X$ with $x \neq y$.

Following the introduction of the above definition, numerous researchers have investigated fixed point results under the contractive condition of rational type in metric spaces (see [6, 7] and the references therein).

On the other hand, Istrăţescu ([8–10]) introduced the convex contractive condition that does not imply the Banach contractive condition. Still, the existence and uniqueness of the fixed point for mappings satisfying such a condition are assured. The mentioned convex contractive condition is as follows:

Definition 1.3 ([8])**.** Let (X,d) be a metric space. A mapping $T:X\to X$ is called a convex contraction mapping if it is continuous and there are constants $a,b\in[0,1)$ such that $a+b<1$ and

$$d(T^2x,T^2y)\leq ad(Tx,Ty)+bd(x,y) \tag{1.4}$$

for all $x,y\in X$.

Our destinations in this paper are to introduce new contractive conditions inspired by the ideas of convex contraction mappings and rational contraction mappings and to provide the existence and uniqueness results of a fixed point for each proposed contraction mapping in complete metric space. Many illustrative examples are provided to validate the main results, while numerous famous fixed point results in the literature cannot be applied to these examples. All presented examples yield the effectiveness of our contractive condition. Moreover, the application of our theoretical fixed point results to establish the existence of a solution for a nonlinear implicit integral equation is demonstrated in the final section.

2. Fixed point results

In this section, we introduce the concept of a new generalization of Banach contraction mappings, inspired by the idea of convex contraction mappings in Definition 1.3, in terms of rational expressions, and prove fixed point results for such contractions.

Definition 2.1. Let (X,d) be a metric space. A mapping $T:X\to X$ is called a rational I_1-contraction mapping if there are constants $\alpha,\beta\in[0,1)$ such that $\alpha+\beta<1$ and

$$d(T^2x,T^2y)\leq \alpha d(Tx,Ty)+\frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)} \tag{2.1}$$

for all $x,y\in X$ with $x\neq y$.

It can be easily seen that each Banach contraction mapping with the Banach contractive constant $k\in[0,1)$ is a rational I_1-contraction mapping with constants $\alpha:=k$ and $\beta:=0$.

Theorem 2.2. *Let (X,d) be a complete metric space and $T:X\to X$ be a rational I_1-contraction mapping. Suppose that T is continuous. Then T has a unique fixed point. Moreover, for each $x_0\in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$, converges to a fixed point of T.*

Proof. Let $x_0\in X$. Define the Picard sequence $\{x_n\}$ in X by

$$x_n=Tx_{n-1}$$

for all $n\in\mathbb{N}$. If $x_n=x_{n-1}$ for some $n\in\mathbb{N}$, then x_{n-1} is a fixed point of T. Hence, we may assume that $x_n\neq x_{n-1}$ for all $n\in\mathbb{N}$. Let $k:=\alpha+\beta\in[0,1)$, where α,β are constants defined in Definition 2.1. From (2.1), for each $n \in \mathbb{N}$ with $n \geq 2$, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(T^2 x_{n-2}, T^2 x_{n-1}) \\
&\leq \alpha d(Tx_{n-2}, Tx_{n-1}) + \frac{\beta d(Tx_{n-2}, Tx_{n-1}) d(x_{n-2}, Tx_{n-2})}{d(x_{n-2}, x_{n-1})} \\
&= \alpha d(x_{n-1}, x_n) + \frac{\beta d(x_{n-1}, x_n) d(x_{n-2}, x_{n-1})}{d(x_{n-2}, x_{n-1})} \\
&= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) \\
&= (\alpha + \beta)\, d(x_{n-1}, x_n) \\
&= k d(x_{n-1}, x_n) \\
&\ \vdots \\
&\leq k^n d(x_0, x_1).
\end{aligned}
$$

For each $m, n \in \mathbb{N}$ such that $2 \leq n < m$, we have

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\
&\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + k^{n+2} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\
&= \left(k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}\right) d(x_0, x_1) \\
&\leq \left(k^n + k^{n+1} + k^{n+2} + \cdots\right) d(x_0, x_1) \\
&= \frac{k^n}{1-k} d(x_0, x_1). \qquad (2.2)
\end{aligned}
$$

By taking the limit as $m, n \to \infty$ in (2.2), we have $d(x_n, x_m) \to 0$. This is enough to conclude that $\{x_n\}$ is a Cauchy sequence in X. The completeness of X implies that $x_n \to z$ as $n \to \infty$ for some $z \in X$. Since T is the continuous, we have

$$
Tz = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (Tx_n) = \lim_{n\to n} x_{n+1} = z.
$$

Then z is a fixed point of T. Finally, we will prove that T has a unique fixed point. Suppose that w is another fixed point of T with $z \neq w$. Then we obtain

$$
\begin{aligned}
d(z, w) &= d(Tz, Tw) \\
&= d\left(T^2 z, T^2 w\right) \\
&\leq \alpha d(Tz, Tw) + \frac{\beta d(Tz, Tw)\, d(z, Tz)}{d(z, w)} \\
&\leq \alpha d(z, w) \\
&< d(z, w),
\end{aligned}
$$

which is a contradiction. Therefore, T has a unique fixed point. This completes the proof. ∎

The following example will describe the situation that Theorem 2.2 can guarantee the existence and uniqueness of a fixed point while the Banach contraction principle [1], Kannan fixed point theorem in [2], Chatterjea fixed point theorem in [3], the Jaggi fixed point theorem in [5], and Istrăţescu fixed point theorem in [8] cannot be applied.

Example 2.3. Let $X = \left[0, \frac{12}{11}\right]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \frac{x^6}{2}$$

for all $x \in X$. From the definition of T, we get

$$T^2x = \frac{x^{36}}{128}$$

for all $x \in X$.

First, we will show that the Banach contraction principle cannot be applied in this example. To show this, we will claim that T is not a Banach contraction mapping. Indeed, for $x = 0.509$ and $y = 1$, we get

$$d(Tx, Ty) = \left|\frac{(0.509)^6}{2} - \frac{1}{2}\right| > 0.491 > k\,(0.491) = kd(x, y)$$

for all $k \in [0, 1)$. It yields that T is not a Banach contraction mapping. Therefore, the Banach contraction principle is not applicable in this situation.

Second, we will show that the Kannan fixed point theorem cannot be applied in this example. To show this, we will claim that T is not a Kannan contraction mapping. Indeed, for $x = 0.4928$ and $y = 1$, we get

$$\begin{aligned} d\,(Tx, Ty) &= \left|\frac{(0.4928)^6}{2} - \frac{1}{2}\right| \\ &> 0.492838 \\ &= \frac{1}{2}(0.985676) \\ &> k(0.985676) \\ &> k\left[\left|0.4928 - \frac{(0.4928)^6}{2}\right| + \left|1 - \frac{1}{2}\right|\right] \\ &= k\left[\left|x - \frac{x^6}{2}\right| + \left|y - \frac{y^6}{2}\right|\right] \\ &= k[d(x, Tx) + d(y, Ty)] \end{aligned}$$

for all $k \in \left[0, \frac{1}{2}\right)$. It yields that T is not a Kannan contraction mapping. Therefore, the Kannan fixed point theorem is not helpful in this situation.

Third, we will show that the Chatterjea fixed point theorem cannot be applied in this example. To show this, we will claim that T is not a Chatterjea contraction mapping. Indeed, for $x = 0.252$ and $y = 1.09$, we get

$$\begin{aligned} d\,(Tx, Ty) &= \left|\frac{(0.252)^6}{2} - \frac{(1.09)^6}{2}\right| \\ &> 0.8384 \\ &= \frac{1}{2}(1.6768) \\ &> k(1.6768) \\ &> k\left[\left|0.252 - \frac{(1.09)^6}{2}\right| + \left|1.09 - \frac{(0.252)^6}{2}\right|\right] \end{aligned}$$

$$
\begin{aligned}
&= k\left[\left|x-\frac{y^6}{2}\right|+\left|y-\frac{x^6}{2}\right|\right] \\
&= k[d(x,Ty)+d(y,Tx)]
\end{aligned}
$$

for all $k \in \left[0, \frac{1}{2}\right)$, this implies that T does not satisfy the conditions of a Chatterjea contraction. Consequently, the classical Chatterjea fixed point theorem is not applicable in this setting.

Fourth, we will show that the Jaggi fixed point theorem cannot be applied in this example. To show this, we will claim that T is not a rational type contraction mapping. Indeed, for $x = 0.5$ and $y = 1.09$, we get

$$
\begin{aligned}
d\left(Tx,Ty\right) &= \left|\frac{(0.5)^6}{2}-\frac{(1.09)^6}{2}\right| \\
&> 0.83 \\
&> \frac{\left|0.5-\frac{(0.5)^6}{2}\right|\left|1.09-\frac{(1.09)^6}{2}\right|}{|0.5-1.09|}+|0.5-1.09| \\
&= \frac{\alpha\left|x-\frac{x^6}{2}\right|\left|y-\frac{y^6}{2}\right|}{|x-y|}+\beta\left|x-y\right| \\
&> \frac{\alpha d(x,Tx)d(y,Ty)}{d\left(x,y\right)}+\beta d\left(x,y\right)
\end{aligned}
$$

for all $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$. This implies that T is not a rational-type contraction mapping. Therefore, the Jaggi fixed point theorem is not applicable in this situation.

Fifth, we will show that the Istrăţescu fixed point theorem cannot be applied in this example. To show this, we will claim that T is not a convex contraction mapping. Indeed, for $x = 1.0885$ and $y = 1.09$, we get

$$
\begin{aligned}
d\left(T^2x,T^2y\right) &= \left|\frac{(1.0885)^{36}}{128}-\frac{(1.09)^{36}}{128}\right| \\
&> 0.008407 \\
&> \left|\frac{(1.0885)^6}{2}-\frac{(1.09)^6}{2}\right|+|1.0885-1.09| \\
&> a\left|\frac{x^6}{2}-\frac{y^6}{2}\right|+b\left|x-y\right| \\
&= ad(Tx,Ty)+bd(x,y)
\end{aligned}
$$

for all $a, b \in [0, 1)$ with $a + b < 1$. It yields that T is not a convex contraction mapping. Therefore, the Istrăţescu fixed point theorem is not applicable is this situation.

Finally, we will demonstrate that Theorem 2.2 guarantees the existence and uniqueness of a fixed point of T. It is easy to verify that T is continuous and that the space X is complete. Next, we will show that T is a rational I_1-contraction mapping with constants $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$. To this end, let $x, y \in X$ with $x \neq y$. We divide the proof into three cases.

Case 1 : If $x, y \in \left[0, \frac{12}{11}\right)$, then

$$\begin{aligned} d(T^2x, T^2y) &= \left|\frac{x^{36}}{128} - \frac{y^{36}}{128}\right| \\ &= \frac{1}{128}\left|x^{36} - y^{36}\right| \\ &\leq \frac{1}{2}\left|\frac{x^6}{2} - \frac{y^6}{2}\right| + \frac{\frac{1}{3}\left|\frac{x^6}{2} - \frac{y^6}{2}\right|\left|x - \frac{x^6}{2}\right|}{|x-y|} \\ &= \alpha d(Tx, Ty) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}. \end{aligned}$$

Figure 1 shows the validity of the above showing. In this figure, the blue surface is a function

$$\left[0, \frac{12}{11}\right)^2 \ni (x, y) \mapsto \left|\frac{x^{36}}{128} - \frac{y^{36}}{128}\right|$$

and the red surface is a function

$$\left[0, \frac{12}{11}\right)^2 \ni (x, y) \mapsto \frac{1}{2}\left|\frac{x^6}{2} - \frac{y^6}{2}\right| + \frac{\frac{1}{3}\left|\frac{x^6}{2} - \frac{y^6}{2}\right|\left|x - \frac{x^6}{2}\right|}{|x-y|}.$$

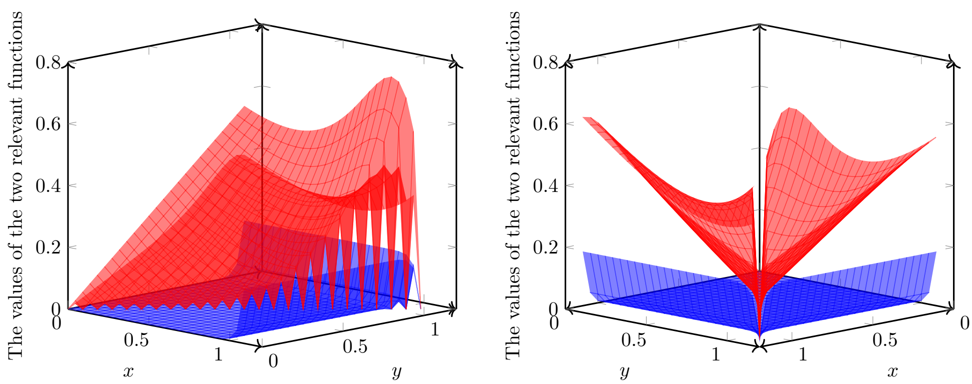

FIGURE 1. Showing the validity for Case 1 in Example 2.3.

Case 2: If $x \in \left[0, \frac{12}{11}\right)$ and $y = \frac{12}{11}$, then

$$\begin{aligned} d(T^2x, T^2y) &= \left| \frac{x^{36}}{128} - \frac{\left(\frac{12}{11}\right)^{36}}{128} \right| \\ &= \frac{1}{128} \left| x^{36} - \left(\frac{12}{11}\right)^{36} \right| \\ &\leq \frac{1}{2} \left| \frac{x^6}{2} - \frac{\left(\frac{12}{11}\right)^6}{2} \right| + \frac{\frac{1}{3} \left| \frac{x^6}{2} - \frac{\left(\frac{12}{11}\right)^6}{2} \right| \left| x - \frac{x^6}{2} \right|}{\left| x - \frac{12}{11} \right|} \\ &= \alpha d(Tx, Ty) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}. \end{aligned}$$

Figure 2 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$\left[0, \frac{12}{11}\right) \ni x \mapsto \left| \frac{x^{36}}{128} - \frac{\left(\frac{12}{11}\right)^{36}}{128} \right|$$

and the dash line is the graph of a function

$$\left[0, \frac{12}{11}\right) \ni x \mapsto \frac{1}{2} \left| \frac{x^6}{2} - \frac{\left(\frac{12}{11}\right)^6}{2} \right| + \frac{\frac{1}{3} \left| \frac{x^6}{2} - \frac{\left(\frac{12}{11}\right)^6}{2} \right| \left| x - \frac{x^6}{2} \right|}{\left| x - \frac{12}{11} \right|}.$$

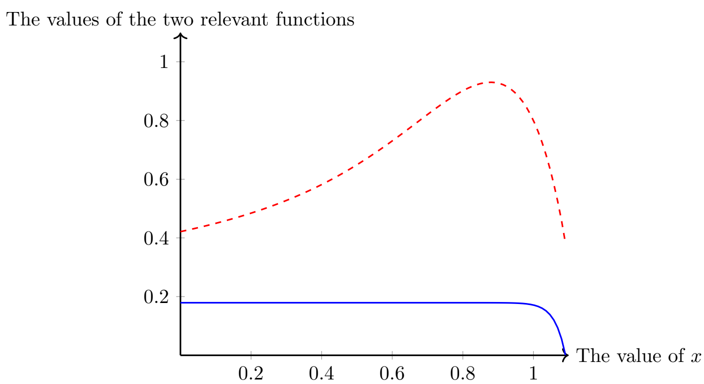

FIGURE 2. Showing the validity for Case 2 in Example 2.3.

Case 3: If $x = \frac{12}{11}$ and $y \in \left[0, \frac{12}{11}\right)$, then

$$\begin{aligned} d(T^2x, T^2y) &= \left| \frac{\left(\frac{12}{11}\right)^{36}}{128} - \frac{y^{36}}{128} \right| \\ &= \frac{1}{128} \left| \left(\frac{12}{11}\right)^{36} - y^{36} \right| \\ &\leq \frac{1}{2} \left| \frac{\left(\frac{12}{11}\right)^{6}}{2} - \frac{y^{6}}{2} \right| + \frac{\frac{1}{3} \left| \frac{\left(\frac{12}{11}\right)^{6}}{2} - \frac{y^{6}}{2} \right| \left| \frac{12}{11} - \frac{\left(\frac{12}{11}\right)^{6}}{2} \right|}{\left| \frac{12}{11} - y \right|} \\ &= \alpha d(Tx, Ty) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}. \end{aligned}$$

Figure 3 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$\left[0, \frac{12}{11}\right) \ni y \mapsto \left| \frac{\left(\frac{12}{11}\right)^{36}}{128} - \frac{y^{36}}{128} \right|$$

and the dash line is the graph of a function

$$\left[0, \frac{12}{11}\right) \ni y \mapsto \frac{1}{2} \left| \frac{\left(\frac{12}{11}\right)^{6}}{2} - \frac{y^{6}}{2} \right| + \frac{\frac{1}{3} \left| \frac{\left(\frac{12}{11}\right)^{6}}{2} - \frac{y^{6}}{2} \right| \left| \frac{12}{11} - \frac{\left(\frac{12}{11}\right)^{6}}{2} \right|}{\left| \frac{12}{11} - y \right|}.$$

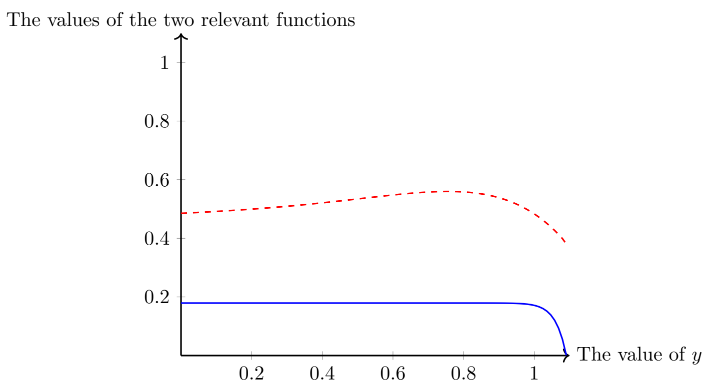

FIGURE 3. Showing the validity for Case 3 in Example 2.3.

From all the above cases, we conclude that inequality (2.1) holds for all $x, y \in X$. This implies that T is a rational I_1-contraction mapping. Therefore, all the conditions of Theorem 2.2 are satisfied, and the existence and uniqueness of a fixed point of T follows directly from Theorem 2.2.

Theorems 2.4, 2.5 and 2.6 can be proved by using a similar argumentation as in the proof of Theorem 2.2.

Theorem 2.4. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(Tx, Ty) + \frac{\beta d(y, Tx)d(y, Ty)}{d(Tx, Ty)} \tag{2.3}$$

for all $x, y \in X$ with $Tx \neq Ty$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.5. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(Tx, Ty) + \frac{\beta d(x, Tx)d(y, Ty)}{d(x, y)} \tag{2.4}$$

for all $x, y \in X$ with $Tx \neq Ty$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.6. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(Tx, Ty) + \frac{\beta d(Tx, Ty)d(x, y)}{d(x, Tx)} \tag{2.5}$$

for all $x, y \in X$ with $x \neq Tx$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Next, inspired by the concept of convex contraction mappings, we introduce a new rational type of convex contraction that does not satisfy the classical Banach contractive condition. The existence and uniqueness of fixed points for such mappings are established through alternative techniques, as demonstrated in Theorems 2.2, 2.4, 2.5, and 2.6.

Definition 2.7. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a rational I_2-contraction mapping if there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and

$$d(T^2x, T^2y) \leq \alpha d\,(x, y) + \frac{\beta d(Tx, Ty)d(x, Tx)}{d(x, y)} \tag{2.6}$$

for all $x, y \in X$ with $x \neq y$.

Example 2.8. Let $X = [0, 1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x^2}{10} & \text{if } x \in [0, 1) \\[2ex] \dfrac{1}{2} & \text{if } x = 1. \end{cases}$$

From the definition of T, we get

$$T^2x = \begin{cases} \dfrac{x^4}{1000} & \text{if } x \in [0, 1) \\[2ex] \dfrac{1}{40} & \text{if } x = 1. \end{cases}$$

We will show that T is a rational I_2-contraction mapping with constants $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Suppose that $x, y \in X$ with $x \neq y$. We will divide our showing into 3 cases.

Case 1: If $x, y \in [0,1)$, then

$$\begin{aligned}
d\left(T^2x, T^2y\right) &= \left|\frac{x^4}{1000} - \frac{y^4}{1000}\right| \\
&= \frac{1}{1000}\left|x^4 - y^4\right| \\
&= \frac{1}{500}\left|x^4 - y^4\right| - \frac{1}{1000}\left|x^4 - y^4\right| \\
&= \frac{1}{500}\left|x^4 - y^4\right| + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&= \frac{\left|x^2 - y^2\right|}{100}\frac{\left|x^2 + y^2\right|}{5} + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&\leq \frac{1}{100}\left|x^2 - y^2\right| + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&= \frac{|x-y|}{10}\frac{|x+y|}{10} + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&\leq \frac{1}{10}|x-y| + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&\leq \frac{1}{2}|x-y| + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|} \\
&= \alpha d(x,y) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}.
\end{aligned}$$

The validity of the above showing is demonstrated in Figure 4, which presents two different viewpoints of the surface. In this figure, the blue surface is a function

$$[0,1)^2 \ni (x,y) \mapsto \left|\frac{x^4}{1000} - \frac{y^4}{1000}\right|$$

and the red surface is a function

$$[0,1) \ni x, y \mapsto \frac{1}{2}|x-y| + \frac{\frac{1}{3}\left|\frac{x^2}{10} - \frac{y^2}{10}\right|\left|x - \frac{x^2}{10}\right|}{|x-y|}.$$

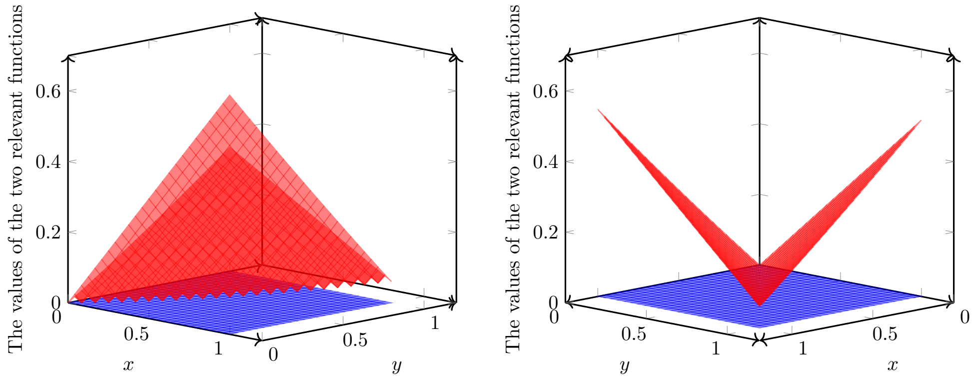

FIGURE 4. Showing the validity for Case 1 in Example 2.8.

Case 2: If $x = 1$ and $y \in [0, 1)$, then

$$\begin{aligned}
d\left(T^2x, T^2y\right) &= \left|\frac{1}{40} - \frac{y^4}{1000}\right| \\
&= \left|\frac{1}{1000} - \frac{y^4}{1000} + \frac{24}{1000}\right| \\
&\leq \left|\frac{1}{1000} - \frac{y^4}{1000}\right| + \frac{24}{1000} \\
&= \frac{1}{1000}\left|1 - y^4\right| + \frac{24}{1000} \\
&\leq \frac{1}{100}\left|1 - y^2\right| + \frac{24}{1000} \\
&\leq \frac{1}{10}\left|1 - y\right| + \frac{24}{1000} \\
&\leq \frac{1}{2}\left|1 - y\right| + \frac{1}{12} \\
&= \frac{1}{2}\left|1 - y\right| + \frac{\frac{1}{3}\left|\frac{1}{2} - 0\right|\left|1 - \frac{1}{2}\right|}{|1 - 0|} \\
&\leq \frac{1}{2}\left|1 - y\right| + \frac{\frac{1}{3}\left|\frac{1}{2} - \frac{y^2}{10}\right|\left|1 - \frac{1}{2}\right|}{|1 - y|} \\
&= \alpha d\left(x, y\right) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}.
\end{aligned}$$

Figure 5 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$[0, 1) \ni y \mapsto \left|\frac{1}{40} - \frac{y^4}{1000}\right|$$

and the dash line is the graph of a function

$$[0,1) \ni y \mapsto \frac{1}{2}|1-y| + \frac{\frac{1}{3}\left|\frac{1}{2}-\frac{y^2}{10}\right|\left|1-\frac{1}{2}\right|}{|1-y|}.$$

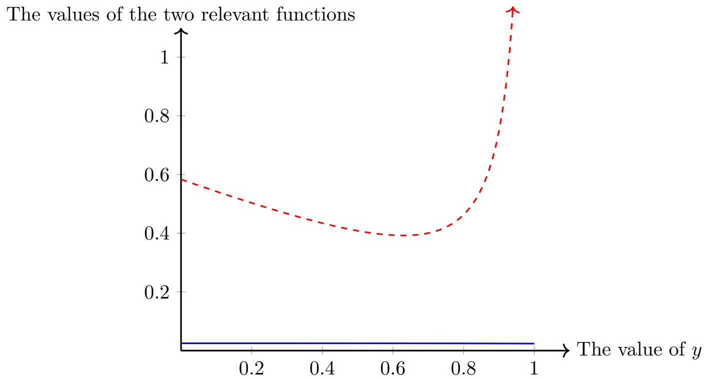

FIGURE 5. Showing the validity for Case 2 in Example 2.8.

Case 3: If $x \in [0,1)$ and $y = 1$, then

$$\begin{aligned}
d\left(T^2x, T^2y\right) &= \left|\frac{x^4}{1000} - \frac{1}{40}\right| \\
&= \left|\frac{x^4}{1000} - \frac{1}{1000} - \frac{24}{1000}\right| \\
&\leq \left|\frac{x^4}{1000} - \frac{1}{1000}\right| + \frac{24}{1000} \\
&= \frac{1}{1000}\left|x^4 - 1\right| + \frac{24}{1000} \\
&\leq \frac{1}{100}\left|x^2 - 1\right| + \frac{24}{1000} \\
&\leq \frac{1}{10}|x-1| + \frac{24}{1000} \\
&= \frac{1}{2}|x-1| + \frac{24}{1000} - \frac{2}{5}|x-1| \\
&\leq \frac{1}{2}|x-1| + \frac{\frac{1}{3}\left|\frac{x^2}{2}-\frac{1}{2}\right|\left|x-\frac{x^2}{10}\right|}{|x-1|} \\
&= \alpha d(x,y) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}.
\end{aligned}$$

Figure 6 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$[0,1) \ni x \mapsto \left|\frac{x^4}{1000} - \frac{1}{40}\right|$$

and the dash line is the graph of a function

$$[0,1) \ni x \mapsto \frac{1}{2}|x-1| + \frac{\frac{1}{3}\left|\frac{x^2}{2}-\frac{1}{2}\right|\left|x-\frac{x^2}{10}\right|}{|x-1|}.$$

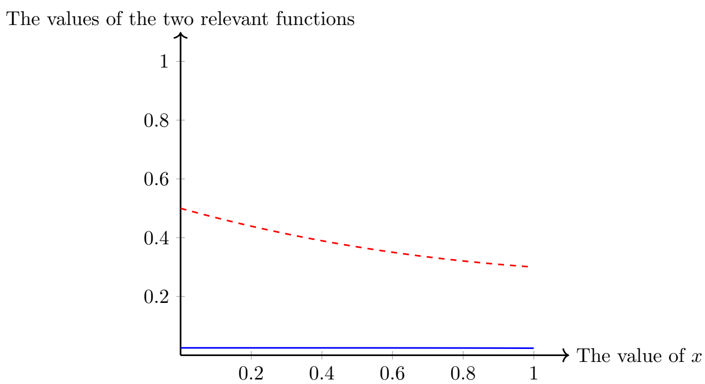

FIGURE 6. Showing the validity for Case 3 in Example 2.8.

From all cases, we obtain the inquality (2.6) holds for all $x, y \in X$. This means that T is a rational I_2-contraction mapping.

Example 2.9. Let $X = [0,1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{6} & \text{if } x \in [0,1) \\[2ex] \dfrac{1}{12} & \text{if } x = 1. \end{cases}$$

From the definition of T, we get

$$T^2x = \begin{cases} \dfrac{x}{36} & \text{if } x \in [0,1) \\[2ex] \dfrac{1}{72} & \text{if } x = 1. \end{cases}$$

We will show that T is a rational I_2-contraction mapping with constants $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Suppose that $x, y \in X$ with $x \neq y$. We will divide our showing into 3 cases.

Case 1: If $x, y \in [0, 1)$, then

$$\begin{aligned}
d\left(T^2x, T^2y\right) &= \left|\frac{x}{36} - \frac{y}{36}\right| \\
&= \frac{1}{36}\left|x - y\right| \\
&= \frac{1}{18}\left|x - y\right| - \frac{1}{36}\left|x - y\right| \\
&\leq \frac{1}{18}\left|x - y\right| + \frac{\frac{1}{3}\left|\frac{x}{6} - \frac{y}{6}\right|\left|x - \frac{x}{6}\right|}{\left|x - y\right|} \\
&\leq \frac{1}{2}\left|x - y\right| + \frac{\frac{1}{3}\left|\frac{x}{6} - \frac{y}{6}\right|\left|x - \frac{x}{6}\right|}{\left|x - y\right|} \\
&= \alpha d\left(x, y\right) + \frac{\beta d(Tx, Ty)d(x, Tx)}{d(x, y)}.
\end{aligned}$$

The validity of the above showing is demonstrated in Figure 7, which presents two different viewpoints of the surface. In this figure, the blue surface is a function

$$[0, 1)^2 \ni (x, y) \mapsto \left|\frac{x}{36} - \frac{y}{36}\right|$$

and the red surface is a function

$$[0, 1) \ni x, y \mapsto \frac{1}{2}\left|x - y\right| + \frac{\frac{1}{3}\left|\frac{x}{6} - \frac{y}{6}\right|\left|x - \frac{x}{6}\right|}{\left|x - y\right|}.$$

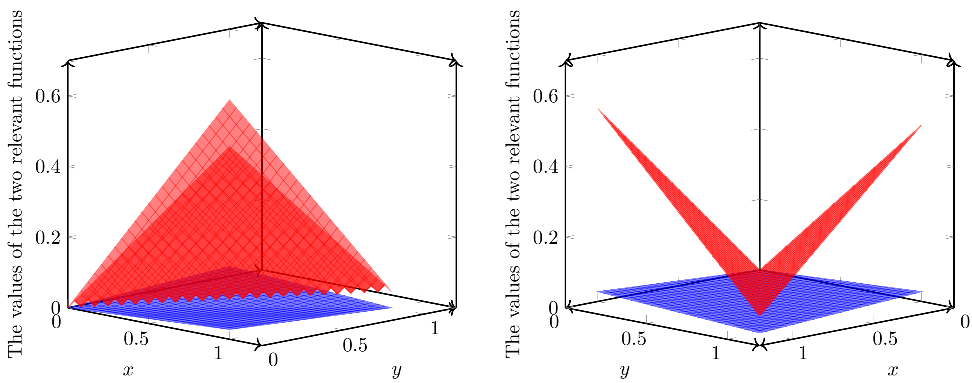

FIGURE 7. Showing the validity for Case 1 in Example 2.9.

Case 2: If $x = 1$ and $y \in [0, 1)$, then

$$\begin{aligned} d\left(T^2x, T^2y\right) &= \left|\frac{1}{72} - \frac{y}{36}\right| \\ &= \left|\frac{1}{36} - \frac{y}{36} - \frac{1}{72}\right| \\ &\leq \left|\frac{1}{36} - \frac{y}{36} - \frac{1}{72}\right| \\ &= \frac{1}{36}\left|1-y\right| + \frac{1}{72} \\ &\leq \frac{1}{3}\left|1-y\right| + \frac{1}{72} - \frac{11}{36}\left|1-y\right| \\ &\leq \frac{1}{3}\left|1-y\right| + \frac{\frac{1}{2}\left|\frac{1}{12} - \frac{y}{6}\right|\left|1 - \frac{1}{12}\right|}{\left|1-y\right|} \\ &= \alpha d\left(x,y\right) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}. \end{aligned}$$

Figure 8 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$[0,1) \ni y \mapsto \left|\frac{1}{72} - \frac{y}{36}\right|$$

and the dash line is the graph of a function

$$[0,1) \ni y \mapsto \frac{1}{3}\left|1-y\right| + \frac{\frac{1}{2}\left|\frac{1}{12} - \frac{y}{6}\right|\left|1 - \frac{1}{12}\right|}{\left|1-y\right|}.$$

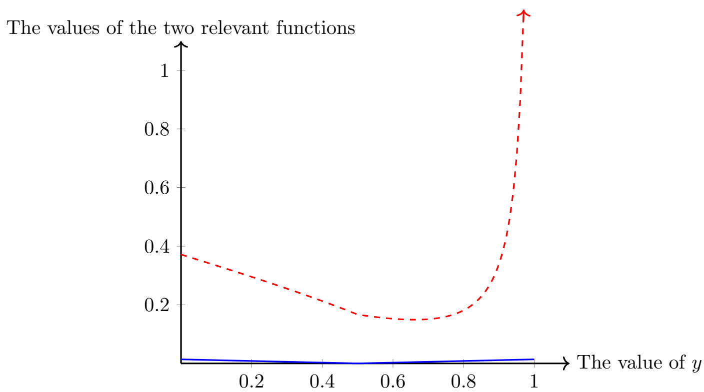

FIGURE 8. Showing the validity for Case 2 in Example 2.9.

Case 3: If $x \in [0,1)$ and $y = 1$, then

$$\begin{aligned}
d\left(T^2x, T^2y\right) &= \left|\frac{x}{36} - \frac{1}{72}\right| \\
&= \left|\frac{x}{36} - \frac{1}{36} + \frac{1}{72}\right| \\
&\leq \left|\frac{x}{36} - \frac{1}{36}\right| + \frac{1}{72} \\
&= \frac{1}{36}|x-1| + \frac{1}{72} \\
&= \frac{1}{3}|x-1| + \frac{1}{72} - \frac{11}{36}|x-1| \\
&\leq \frac{1}{3}|x-1| + \frac{\frac{1}{2}\left|\frac{x}{6} - \frac{1}{12}\right|\left|x - \frac{x}{6}\right|}{|x-1|} \\
&= \alpha d(x,y) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}.
\end{aligned}$$

Figure 9 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$[0,1) \ni x \mapsto \left|\frac{x}{36} - \frac{1}{72}\right|$$

and the dash line is the graph of a function

$$[0,1) \ni x \mapsto \frac{1}{3}|x-1| + \frac{\frac{1}{2}\left|\frac{x}{6} - \frac{1}{12}\right|\left|x - \frac{x}{6}\right|}{|x-1|}.$$

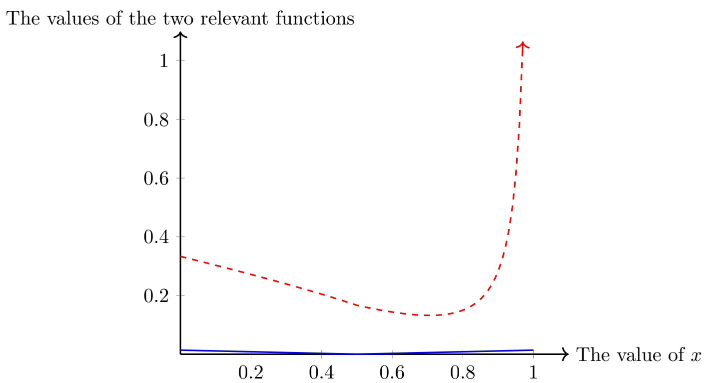

FIGURE 9. Showing the validity for Case 3 in Example 2.9.

From all cases, we obtain the inquality (2.6) holds for all $x, y \in X$. This means that T is a rational I_2-contraction mapping.

Theorem 2.10. *Let (X,d) be a complete metric space and $T : X \to X$ be a rational I_2-contraction mapping. Suppose that T is continuous. Then T has a unique fixed point.*

Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Proof. Let $x_0 \in X$. Define the Picard sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of T. Hence, we may assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. From (2.6), define $s = d(x_0, x_1) + d(x_1, x_2)$ and $\alpha + \beta =: k \in [0, 1)$. Then we obtain

$$\begin{aligned}
d(x_2, x_3) &= d(T^2x_0, T^2x_1) \\
&\leq \alpha d(x_0, x_1) + \frac{\beta d(Tx_0, Tx_1)d(x_0, Tx_0)}{d(x_0, x_1)} \\
&= \alpha d(x_0, x_1) + \frac{\beta d(x_1, x_2)d(x_0, x_1)}{d(x_0, x_1)} \\
&= \alpha d(x_0, x_1) + \beta d\,(x_1, x_2) \\
&\leq ks, \\
d(x_3, x_4) &= d(T^2x_1, T^2x_2) \\
&\leq \alpha d(x_1, x_2) + \frac{\beta d(Tx_1, Tx_2)d(x_1, Tx_1)}{d(x_1, x_2)} \\
&= \alpha d(x_1, x_2) + \frac{\beta d(x_2, x_3)d(x_1, x_2)}{d(x_1, x_2)} \\
&= \alpha d(x_1, x_2) + \beta d\,(x_2, x_3) \\
&\leq \alpha s + \beta\,(ks) \\
&\leq \alpha s + \beta s \\
&= ks, \\
d(x_4, x_5) &= d(T^2x_2, T^2x_3) \\
&\leq \alpha d(x_2, x_3) + \frac{\beta d(Tx_2, Tx_3)d(x_2, Tx_2)}{d(x_2, x_3)} \\
&= \alpha d(x_2, x_3) + \frac{\beta d(x_3, x_4)d(x_2, x_3)}{d(x_2, x_3)} \\
&= \alpha d(x_2, x_3) + \beta d\,(x_3, x_4) \\
&\leq \alpha\,(ks) + \beta\,(ks) \\
&= (\alpha + \beta)\,(ks) \\
&= k^2 s,
\end{aligned}$$

$$\begin{aligned}
d(x_5, x_6) &= d(T^2x_3, T^2x_4) \\
&\leq \alpha d(x_3, x_4) + \frac{\beta d(Tx_3, Tx_4)(x_3, Tx_3)}{d(x_3, x_4)} \\
&= \alpha d(x_3, x_4) + \frac{\beta d(x_4, x_5)d(x_3, x_4)}{d(x_3, x_4)}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha d(x_3, x_4) + \beta d\left(x_4, x_5\right) \\
&\leq \alpha\left(k^2 s\right) + \beta\left(k s\right) \\
&\leq \alpha\left(k s\right) + \beta\left(k s\right) \\
&= (\alpha+\beta)\left(k s\right) \\
&= k^2 s \\
&\vdots
\end{aligned}
$$

By the above relation, we obtain

$$
d(x_n, x_{n+1}) = \begin{cases} k^{\frac{n-1}{2}} s & \text{if } n \text{ is odd,} \\ k^{\frac{n}{2}} s & \text{if } n \text{ is even.} \end{cases}
$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ such that $n < m$. We will divide into two cases.

Case 1: If n is odd, we have

$$
\begin{aligned}
d(x_n, x_m) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + d\left(x_{n+2}, x_{n+3}\right) + \cdots + d\left(x_{m-1}, x_m\right) \\
&\leq k^{\frac{n-1}{2}} s + k^{\frac{n+1}{2}} s + k^{\frac{n+1}{2}} s + k^{\frac{n+3}{2}} s + k^{\frac{n+3}{2}} s + \cdots \\
&\leq 2\left(k^{\frac{n-1}{2}} s + k^{\frac{n+1}{2}} s + k^{\frac{n+3}{2}} s + \cdots\right) \\
&= 2\left(\frac{k^{\frac{n-1}{2}} s}{1-k}\right).
\end{aligned}
$$

Case 2: If n is even, we have

$$
\begin{aligned}
d(x_n, x_m) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + d\left(x_{n+2}, x_{n+3}\right) + \cdots + d\left(x_{m-1}, x_m\right) \\
&\leq k^{\frac{n}{2}} s + k^{\frac{n}{2}} s + k^{\frac{n+2}{2}} s + k^{\frac{n+2}{2}} s + \cdots \\
&\leq 2\left(k^{\frac{n}{2}} s + k^{\frac{n+2}{2}} s + k^{\frac{n+4}{2}} s + \cdots\right) \\
&= 2\left(\frac{k^{\frac{n}{2}} s}{1-k}\right).
\end{aligned}
$$

For all cases, we get

$$
d(x_n, x_m) = \begin{cases} 2\left(\dfrac{k^{\frac{n-1}{2}} s}{1-k}\right) & \text{if } n \text{ is odd,} \\ 2\left(\dfrac{k^{\frac{n}{2}} s}{1-k}\right) & \text{if } n \text{ is even.} \end{cases} \tag{2.7}
$$

This is sufficient to conclude that $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a point $x \in X$ such that $x_n \to z$ as $n \to \infty$ for some $z \in X$. Since T is continuous, it follows that

$$
Tz = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty}\left(T x_n\right) = \lim_{n\to n} x_{n+1} = z.
$$

Thus, z is a fixed point of T. We now proceed to prove the uniqueness of the fixed point. Suppose, for the sake of contradiction, that w is another fixed point of T with $w \neq z$.

Then we obtain

$$\begin{aligned} d(z,w) &= d\left(T^2 z, T^2 w\right) \\ &\leq \alpha d(z,w) + \frac{\beta d(Tz,Tw)d(z,Tz)}{d(z,w)} \\ &= \alpha d(z,w) \\ &< d(z,w), \end{aligned}$$

which is a contradiction. Therefore, T has a unique fixed point. This completes the proof. ∎

The following example illustrates a situation in which Theorem 2.10 guarantees the existence and uniqueness of a fixed point. In contrast, the Banach contraction principle in [1], the Kannan fixed point theorem in [2], the Chatterjea fixed point theorem in [3] are not applicable.

Example 2.11. Let $X = [0, 1.1]$, d be a usual metric on X and $T : X \to X$ be defined by

$$Tx = \frac{x^5}{2}$$

for all $x \in X$. From the definition of T, we get

$$T^2 x = \frac{x^{25}}{64}$$

for all $x \in X$. We begin by showing that the Banach contraction principle cannot be applied in this example. To do so, we claim that T is not a Banach contraction mapping. Indeed, for $x = 0.5188$ and $y = 1$, we obtain

$$d(Tx,Ty) = \left|\frac{(0.5188)^5}{2} - \frac{1}{2}\right| > 0.4812 > k\,(0.4812) = kd(x,y)$$

for all $k \in [0,1)$. Hence, T fails to satisfy the Banach contraction condition, and thus, the Banach contraction principle cannot be applied in this context.

Next, we show that the Kannan fixed point theorem cannot be applied in this example. To demonstrate this, we claim that T is not a Kannan contraction mapping. Indeed, for $x = 0.48$ and $y = 1$, we obtain

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{(0.48)^5}{2} - \frac{1}{2}\right| \\ &> 0.4872 \\ &= \frac{1}{2}(0.9744) \\ &> k(0.9744) \\ &> k\left[\left|0.48 - \frac{(0.48)^5}{2}\right| + \left|1 - \frac{1}{2}\right|\right] \\ &= k\left[\left|x - \frac{x^5}{2}\right| + \left|y - \frac{y^5}{2}\right|\right] \\ &= k[d(x,Tx) + d(y,Ty)] \end{aligned}$$

for all $k \in \left[0, \frac{1}{2}\right)$. It yields that T is not a Kannan contraction mapping. Therefore, the Kannan fixed point theorem is not useful is this situation.

Moreover, the Chatterjea fixed point theorem cannot be applied in this example. To demonstrate this, we claim that T is not a Chatterjea contraction mapping. Indeed, for $x = 0.3$ and $y = 1.1$, we obtain

$$\begin{aligned}
d\left(Tx, Ty\right) &= \left|\frac{(0.3)^5}{2} - \frac{(1.1)^5}{2}\right| \\
&= 0.80404 \\
&= \frac{1}{2}(1.60808) \\
&> k(1.60808) \\
&> k\left[\left|0.3 - \frac{(1.1)^5}{2}\right| + \left|1.1 - \frac{(0.3)^5}{2}\right|\right] \\
&= k\left[\left|x - \frac{y^5}{2}\right| + \left|y - \frac{x^5}{2}\right|\right] \\
&= k[d(x, Ty) + d(y, Tx)]
\end{aligned}$$

for all $k \in \left[0, \frac{1}{2}\right)$. This implies that T is not a Chatterjea contraction mapping. Therefore, the Chatterjea fixed point theorem is not applicable in this situation.

Finally, we will show that Theorem 2.10 guarantees the existence and uniqueness of a fixed point of T. It is easy to see that T is continuous and that the space X is complete. Furthermore, we will demonstrate that T is a rational I_2-contraction mapping with constants $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. Suppose that $x, y \in X$ with $x \neq y$. We divide the proof into three cases.

Case 1 : If $x, y \in [0, 1.1)$, then

$$\begin{aligned}
d(T^2x, T^2y) &= \left|\frac{x^{25}}{64} - \frac{y^{25}}{64}\right| \\
&= \frac{1}{64}\left|x^{25} - y^{25}\right| \\
&\leq \frac{1}{2}\left|x - y\right| + \frac{\frac{1}{3}\left|\frac{x^5}{2} - \frac{y^5}{2}\right|\left|x - \frac{x^5}{2}\right|}{|x - y|} \\
&= \alpha d\left(x, y\right) + \frac{\beta d(Tx, Ty)d(x, Tx)}{d(x, y)}.
\end{aligned}$$

The validity of the above showing is demonstrated in Figure 10, which presents two different viewpoints of the surface. In this figure, the blue surface is a function

$$[0, 1.1) \ni x, y \mapsto \left|\frac{x^{25}}{64} - \frac{y^{25}}{64}\right|$$

and the red surface is a function

$$[0, 1.1) \ni x, y \mapsto \frac{1}{2}\left|x - y\right| + \frac{\frac{1}{3}\left|\frac{x^5}{2} - \frac{y^5}{2}\right|\left|x - \frac{x^5}{2}\right|}{|x - y|}.$$

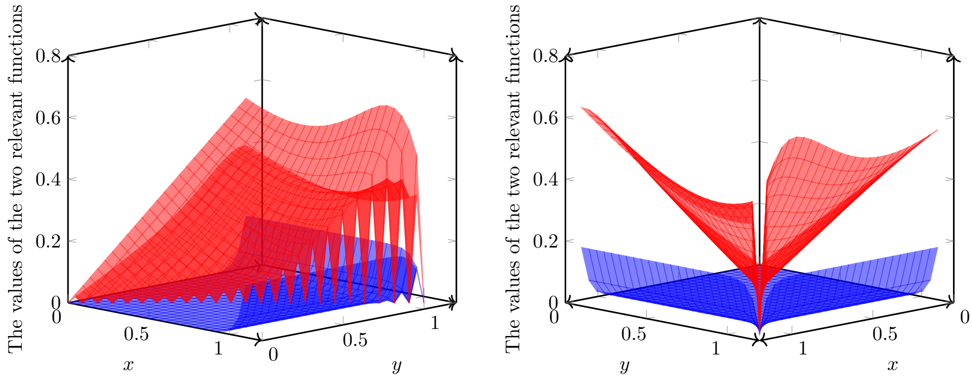

FIGURE 10. Showing the validity for Case 1 in Example 2.11.

Case 2: If $x \in [0, 1.1)$ and $y = 1.1$, then

$$\begin{aligned} d(T^2x, T^2y) &= \left| \frac{x^{25}}{64} - \frac{(1.1)^{25}}{64} \right| \\ &= \frac{1}{64} \left| x^{25} - (1.1)^{25} \right| \\ &\leq \frac{1}{2} |x - 1.1| + \frac{\frac{1}{3} \left| \frac{x^5}{2} - \frac{(1.1)^{25}}{2} \right| \left| x - \frac{x^5}{2} \right|}{|x - 1.1|} \\ &= \alpha d(x, y) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}. \end{aligned}$$

The validity of the above showing is demonstrated in Figure 11, which presents two different viewpoints of the surface. In this figure, the solid line is the graph of a function

$$[0, 1.1) \ni x \mapsto \left| \frac{x^{25}}{64} - \frac{(1.1)^{25}}{64} \right|$$

and the dash line is the graph of a function

$$[0, 1.1) \ni x \mapsto \frac{1}{2} |x - 1.1| + \frac{\frac{1}{3} \left| \frac{x^5}{2} - \frac{(1.1)^{25}}{2} \right| \left| x - \frac{x^5}{2} \right|}{|x - 1.1|}.$$

Case 3: If $x = 1.1$ and $y \in [0, 1.1)$, then

$$\begin{aligned} d(T^2x, T^2y) &= \left| \frac{(1.1)^{25}}{64} - \frac{y^{25}}{64} \right| \\ &= \frac{1}{64} \left| (1.1)^{25} - y^{25} \right| \\ &\leq \frac{1}{2} |1.1 - y| + \frac{\frac{1}{3} \left| \frac{(1.1)^5}{2} - \frac{y^5}{2} \right| \left| 1.1 - \frac{(1.1)^5}{2} \right|}{|1.1 - y|} \\ &= \alpha d(x, y) + \frac{\beta d(Tx, Ty) d(x, Tx)}{d(x, y)}. \end{aligned}$$

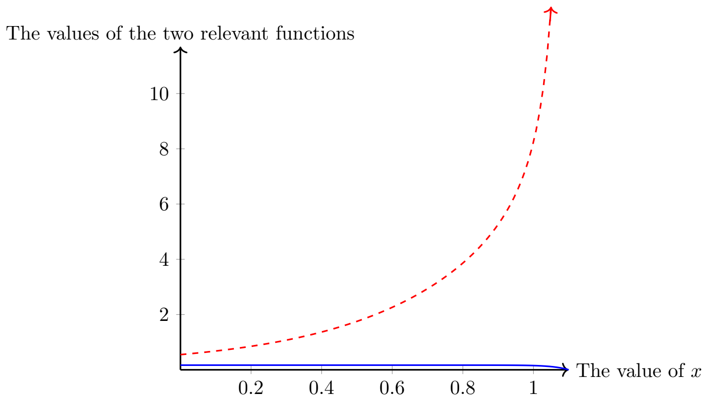

FIGURE 11. Showing the validity for Case 2 in Example 2.11.

Figure 12 shows the validity of the above showing. In this figure, the solid line is the graph of a function

$$[0,1.1) \ni y \mapsto \left| \frac{(1.1)^{25}}{64} - \frac{y^{25}}{64} \right|$$

and the dash line is the graph of a function

$$[0,1.1) \ni y \mapsto \frac{1}{2}\left|1.1-y\right| + \frac{\frac{1}{3}\left|\frac{(1.1)^5}{2} - \frac{y^5}{2}\right| \left|1.1 - \frac{(1.1)^5}{2}\right|}{|1.1-y|}.$$

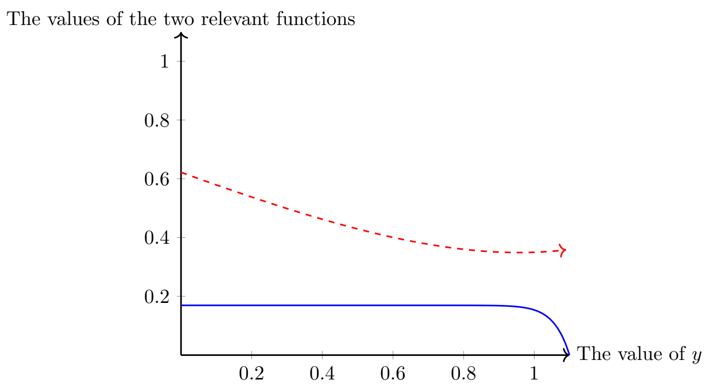

FIGURE 12. Showing the validity for Case 3 in Example 2.11.

From all cases, it follows that the inequality (2.6) holds for all $x, y \in X$, confirming that T is a rational I_2-contraction mapping. Consequently, all the hypotheses of Theorem 2.10 are satisfied, guaranteeing the existence and uniqueness of a fixed point for T as established in Theorem 2.10.

Theorems 2.12, 2.13, 2.14, 2.15, and 2.16 can be proved using a similar reasoning approach as in the proof of Theorem 2.10.

Theorem 2.12. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(x, y) + \frac{\beta d(x, Tx)d(y, Ty)}{d(x, y)} \tag{2.8}$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.13. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(y, Tx) + \frac{\beta d(x, Tx)d(y, Ty)}{d(x, y)} \tag{2.9}$$

for all $x, y \in X$ with $x \neq y$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.14. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \frac{\alpha d(x, y)d(y, Ty)}{d(Tx, Ty)} + \beta d(y, Ty) \tag{2.10}$$

for all $x, y \in X$ with $Tx \neq Ty$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.15. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d(y, Ty) + \frac{\beta d(y, Tx)d(y, Ty)}{d(Tx, Ty)} \tag{2.11}$$

for all $x, y \in X$ with $Tx \neq Ty$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Theorem 2.16. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is continuous and there are constants $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and*

$$d(T^2x, T^2y) \leq \alpha d\,(x, y) + \frac{\beta d(Tx, Ty)d(x, y)}{d\,(x, Tx)} \tag{2.12}$$

for all $x, y \in X$ with $x \neq Tx$. Then T has a fixed point. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

3. Applications on implicit functional integral equations

The theory of integral equations is a vast area of mathematics since it has various applications in physics, mechanics, engineering, bioengineering, control theory, and other disciplines related to real-world issues over the last three decades. This section aims to demonstrate the existence of integrable solutions for an implicit functional integral equation, utilizing the theoretical fixed point results established in the previous section.

Theorem 3.1. *Let $X = C[0,1]$ be the set of all continuous real-value functions on $[0,1]$ and d be a Chebyshev distance on X. Consider the mapping $T : X \to X$, formulating from the nonlinear implicit functional integral equation, which is defined for each $x \in X$ by*

$$(Tx)(t) = \alpha x(t) + \int_0^t k(t, \tau, x(\tau), (Tx)(\tau))\, d\tau \tag{3.1}$$

for all $t \in [0,1]$, where $\alpha \in [0,1)$ and $k : [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ is a given function. Suppose that the following conditions hold:

(1) T is continuous;

(2) there is $\beta \in [0, 1-\alpha)$ such that for each $x, y \in X$ with $x \neq y$ and for each $t \in [0,1]$, we have

$$|(fx)(t) - (fy)(t)| < (1-\alpha)|x(t) - y(t)|$$

and

$$|(f(Tx))(t) - (f(Ty))(t)| \leq \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)},$$

where fa is defined for each $a \in X$ by

$$(fa)(t) = \int_0^t k(t, \tau, a(\tau), (Ta)(\tau))\, d\tau.$$

Then T has a unique fixed point.

Proof. It is well-known that (X,d) is complete. Now, we will show that (2.6) holds. For each $x, y \in A$ with $x \neq y$ and $t \in [0,1]$, we have

$$\begin{aligned}
&\left|(T^2x)(t) - (T^2y)(t)\right| \\
= &\left|\alpha[(Tx)(t) - (Ty)(t)] + \int_0^t \left[k\left(t,\tau,(Tx)(\tau),(T^2x)(\tau)\right) - k\left(t,\tau,(Ty)(\tau),(T^2y)(\tau)\right)\right] d\tau\right| \\
\leq &\ \alpha|(Tx)(t) - (Ty)(t)| + \left|\int_0^t \left[k\left(t,\tau,(Tx)(\tau),(T^2x)(\tau)\right) - k\left(t,\tau,(Ty)(\tau),(T^2y)(\tau)\right)\right] d\tau\right| \\
\leq &\ \alpha|(Tx)(t) - (Ty)(t)| + \int_0^t \left|k\left(t,\tau,(Tx)(\tau),(T^2x)(\tau)\right) - k\left(t,\tau,(Ty)(\tau),(T^2y)(\tau)\right)\right| d\tau \\
\leq &\ \alpha|(Tx)(t) - (Ty)(t)| + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)} \\
\leq &\ \alpha\left[\alpha|(x(t) - y(t))| + |(fx)(t) - (fy)(t)|\right] + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)} \\
\leq &\ \alpha\left[\alpha|(x(t) - y(t))| + (1-\alpha)|x(t) - y(t)|\right] + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)} \\
\leq &\ \alpha|x(t) - y(t)| + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)} \\
\leq &\ \alpha d(x,y) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}
\end{aligned}$$

This implies that

$$d(T^2x, T^2y) \leq \alpha d(x,y) + \frac{\beta d(Tx,Ty)d(x,Tx)}{d(x,y)}$$

for all $x, y \in X$. Now, all conditions of Theorem 2.10 are satisfied. Therefore, T has a unique fixed point. ∎

Remark 3.2. The other theorems established in this paper provide various sufficient conditions under which the existence and uniqueness of solutions to the given implicit functional equations can be guaranteed by using a similar line of reasoning as in the proof of Theorem 3.1.

4. Conclusions and open problems

This paper introduced several novel classes of contraction mappings inspired by convex and rational contraction mappings. For each proposed class, we established fixed point theorems, ensuring the existence and uniqueness of fixed points in complete metric spaces. These results extended and generalized classical fixed point theorems, providing new tools for addressing situations where existing results were not applicable. To support our theoretical developments, we offered several illustrative examples. In particular, Example 2.3 served as a compelling case study that demonstrated the effectiveness of the newly introduced mapping. This example clearly showed that none of the five classical fixed point theorems, including the Banach contraction principle, the Kannan fixed point theorem, the Chatterjea fixed point theorem, the Jaggi fixed point theorem, and the Istrăţescu fixed point theorem, were applicable in the given situation. Nevertheless, our newly developed fixed point result successfully addressed the problem, clearly illustrating the strength and applicability of the proposed framework. As an application, we employed our fixed point results to investigate the existence and uniqueness of solutions to a class of nonlinear implicit integral equations, thereby demonstrating the practicality and potential for further applications of our findings.

Despite these contributions, several questions remain open. For instance:

- Can the assumption of continuity be omitted from the theoretical results established in this paper?
- In Example 2.11, it was observed that only the Banach contraction principle, the Kannan fixed point theorem, and the Chatterjea fixed point theorem were not applicable. It remains an open question whether one can construct an example where the Jaggi fixed point theorem and the Istrăţescu fixed point theorem are also inapplicable.
- Are there other rational-type contractions that can be formulated by combining classical or well-known fixed point theorems in a manner similar to the approach taken in this paper?

Acknowledgement

The authors are thankful to the referees and the editor for their comments and suggestions, which substantially improve the initial version of the paper. This work was supported by Thammasat University Research Unit in Fixed Points and Optimization.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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