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GUARANTEED PURSUIT TIME OF A LINEAR DIFFERENTIAL GAME WITH GENERALIZED GEOMETRIC CONSTRAINTS ON PLAYERS CONTROL FUNCTIONS

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Abstract In this study, simple motion differential game of one pursuer and one evader is considered in $\mathbb{R}^n$. Control functions of the players are subject to more generalized geometric constraints. Pursuit is said to be completed at a finite time Γ whenever the distance between both players at time Γ is zero. We obtain an estimate for the guaranteed pursuit time Γ and construct pursuer's strategy that ensure completion of pursuit at the time Γ.

MSC: 47H09
Keywords: Grönwall, Pursuit; Control functions; Guaranteed pursuit time

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1. Introduction

In a pursuit differential game, guaranteed pursuit time is a finite time, say T, for which pursuit is completed on or before the time T. Finding or estimating this time requires construction of strategy for the pursuers such that for any control of the evader, the strategy ensures that the pursuers wins the game. There is substantial literature on this class of problem (details can be found in the following references: [1], [2], [5], [7], [8], [9], [10], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [24].

In some of the problems considered, players are not allowed to move freely in the space considered. That is, players are only allowed to move within a confined area. Alternatively, we say that state variables are constrained. Research problem of this kind has been studied and fundamental results were obtained in [3], [21], [23] ,[25], and some references therein.

The works that are more relevant to this paper are pursuit differential games (such as [1], [3], [4], [5], [6], [8], [10], [20], [21], [23] and [25]). In particular, the papers ([6], [9], [10]) and [23],) which deals with problems involving guaranteed pursuit time, are even the most relevant.

Ibragimov [20] obtained optimal pursuit time and constructed optimal strategies of the players for a pursuit problem whose motion of the payers is described by $\dot{x}(t) = a(t)u(t),\ \dot{y}(t) = a(t)v(t)$. Constraints are considered on both control of the players and on the state variables in the space $\mathbb{R}^n$.

Ibragimov [22] considered a two-person zero sum pursuit-evasion differential game in the Hilbert space l_2 with integral constraints on both players control parameters and obtained an explicit form of the optimal pursuit time by constructing the players optimal strategies.

The Grönwall-type constraint is a generalization of the geometric constraint recently introduced in the work of [12]. The authors considered a simple pursuit-evasion differential game and obtained the optimal pursuit time with the Grönwall-type constraint imposed on players' controls. Also, in [11], a fixed duration pursuit-evasion differential game was studied with Grönwall-type constraints imposed on the players controls and sufficient conditions for the completion of pursuit as well as evasion were obtained.

Simple motion differential game problem of many pursuers and one evader were investigated in ([3],[20], [21], [23]) under phase constraints. Pshenichnyi [23] studied this class of problem where maximum speed of all players equals 1 (that is, $\rho_i = \sigma = 1$) and proved that pursuit is completed if the initial positions of the evader lies in the convex hull of that of the m pursuers, and if otherwise, evasion is guaranteed.

In the space R^3, Kuchkarov et. al [25] considered the problem with $\rho_i = \sigma = 1$ on the surface of a given cylinder M under some certain conditions, pursuit was shown to be completed at some finite time T and if these conditions fails, evasion is shown to be possible.

In [3], Ivanov investigated a differential game problem in a compact subset A of $\mathbb{R}^n$ having nonempty interior, where all players (with maximum speed 1) are confined within the set A. Under the assumption that if the dimension of the space exceeds the number of the pursuers (that is, $m < n$), evasion is guaranteed and if otherwise, pursuit is completed. In addition, the guaranteed pursuit time T is estimated to be $(n^3 - 2n^2 + n + 1)d$, where d is the diameter of A.

Alias et. al [21] studied and improved the results in [3] by estimating the guaranteed pursuit time T by a second degree polynomial in n (that is, $T \leq \frac{\sqrt{n}+1}{2}a + \frac{n(n-1)a}{2}$) where a is the length of the $n-$dimensional cube A (a convex compact subset of $\mathbb{R}^n$). Throughout the game, all players must not leave the cube A. Also assuming that the dimension of the space does not exceed the number of the pursuers (that is, $m \geq n$), pursuit is shown to be completed at the estimated guaranteed pursuit time T.

It is worth noting that in most of the aforementioned pursuit differential game problems, integral constraints are imposed on control functions of the players. In view of this fact and motivated by the work in [5], we consider a linear differential game problem in the space $\mathbb{R}^n$ with generalized geometric constraints on control functions of the players. To this end, we find an estimate for the guaranteed pursuit time and prove completion of pursuit at the estimated time.

2. Problem formulation

Consider the pursuit problem of one pursuer and one evader with dynamics governed by

$$\dot{z} + \lambda z = v - u, \tag{2.1}$$

where $z, u, v \in \mathbb{R}^n$ and λ is a given positive number, $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are control parameters of the pursuer and evader respectively, which are denoted by $u(t)$ and $v(t)$, $0 \leq t < \infty$.

Pursuit starts from initial positions

$$z(0) = z_0, \tag{2.2}$$

at time $t = 0$, where $z_0 \in \mathbb{R}^n$. Let Γ be a given number.

Definition 2.1. A measurable function $u(t)$, $0 \leq t \leq \Gamma$, is called an admissible control of the pursuer, if it satisfies the following

$$\|u(t)\| \leq \rho e^{k \int_0^t e^{\lambda s}\, ds}, \tag{2.3}$$

where ρ and k are given positive number and non-negative number respectively.

Definition 2.2. A measurable function $v(t), 0 \leq t \leq \Gamma$, is called an admissible control of the evader if it satisfies the following

$$\|v(t)\| \leq \sigma e^{k \int_0^t e^{\lambda s}\, ds}, \tag{2.4}$$

where σ and k are given positive number and non-negative number respectively.

If we replace the parameters u, v, in equation(2.1) by some admissible controls $u(t), v(t), 0 \leq t \leq \Gamma$, then it follows from the theory of differential equations that the initial value problem (2.1), (2.2) has a unique solution on the time interval $[0, \Gamma]$. The solution

$$z(t) = (z_1(t), z_2(t), \dots, z_n(t)),\ 0 \leq t \leq \Gamma,$$

of the differential equations (2.1) is considered in the space of functions $f(t) = (f_1(t), f_2(t), \dots, f_n(t))$ with absolutely continuous coordinates $f_i(t)$, defined on the interval $0 \leq t \leq \Gamma$.

Lemma 2.3 (Grönwall-type constraint). [11] *Let $\eta(t),\ t \geq 0$ be a measurable function and β and k be non-negative real numbers. Then*

$$\|\eta(t)\| \leq \beta e^{k \int_0^t a(s)\, ds}, \tag{2.5}$$

whenever

$$\|\eta(t)\|^2 \leq \beta^2 + 2k \int_0^t a(s) \|\eta(s)\|^2\, ds. \tag{2.6}$$

As a consequence of Lemma 2.3, we consider the constraint (2.5) which also generalizes the geometric constraint in this paper. That is, instead of the inequality (2.6), which is the Grönwall-type constraint.

Definition 2.4. A function of the form

$$U(t, v) = v + \xi(t) = (v_1 + \xi_1(t), v_2 + \xi_2(t), \cdots, v_n + \xi_n(t)),\ 0 \leq t \leq \Gamma,$$

where $U(t, v) = v + \xi(t) \in \mathbb{R}^n$, is called strategy of the pursuer, if for any admissible control of the evader $v = v(t),\ 0 \leq t \leq \Gamma$, the following inequality

$$\|U(t, v(t))\| \leq \rho e^{k \int_0^t e^{\lambda s}\, ds},$$

holds.

Definition 2.5. Pursuit is said to be completed at time $\Gamma > 0$, in the differential game (2.1) - (2.4), from the initial positions $z_0 = (z_{10}, z_{20}, \dots, z_{n0})$, if there exists a strategy of the pursuer $U(t, v(t))$, such that for any admissible control of the evader $v(t),\ 0 \leq t \leq \Gamma$, the solution $z(t),\ 0 \leq t \leq \Gamma$, equals zero at some time $\tau,\ 0 \leq \tau \leq \Gamma$, i.e., $z(\tau) = 0$ at some $\tau,\ 0 \leq \tau \leq \Gamma$.
In the sequel, the time Γ is called guaranteed pursuit time.

Once the admissible controls of the players are chosen, the corresponding motions of the players takes the form

$$z(t) = e^{-\lambda t} z_0 + \int_0^t e^{-\lambda(t-s)}(v(s) - u(s, v(s)))\, ds.$$

The pursuers' goal is to minimize the guaranteed pursuit time, Γ, through out the game while the evaders' is contrary.

Research Problem:
Find the guaranteed pursuit time with respect to the game (2.1)-(2.4).

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Theorem 2.6. *Let $\rho > \sigma$. The time $\Gamma \geq \Gamma' > 0$,*

$$\Gamma' := \frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{k} ln\left(\frac{\|z_0\|\, k e^{\frac{k}{\lambda}}}{\rho - \sigma}\right)\right), \tag{2.7}$$

is a guaranteed pursuit time in the game (2.1)-(2.4).

Proof. Let $v(t), 0 \leq t \leq \Gamma$, be an arbitrary control of the evader. Construct the pursuers' strategy as follows

$$U(t, v(t)) = \begin{cases} v(t) + \frac{z_0(\rho-\sigma)}{\|z_0\|} e^{k\int_0^t e^{\lambda s}\, ds}, & 0 \leq t \leq \Gamma', \\ v(t), & \Gamma' < t \leq \Gamma. \end{cases} \tag{2.8}$$

Clearly, the strategy is admissible for the time interval $\Gamma' < t \leq \Gamma$, since $\rho > \sigma$ and for $0 \leq t \leq \Gamma'$, we have

$$\begin{aligned} \|U(t, v(t))\| &= \left\| v(t) + \frac{z_0(\rho - \sigma)}{\|z_0\|} e^{k\int_0^t e^{\lambda s}\, ds} \right\| \\ &\leq \|v(t)\| + \left\| \frac{z_0(\rho - \sigma)}{\|z_0\|} e^{k\int_0^t e^{\lambda s}\, ds} \right\| \\ &= \|v(t)\| + (\rho - \sigma) e^{k\int_0^t e^{\lambda s}\, ds} \\ &\leq \sigma e^{k\int_0^t e^{\lambda s}\, ds} + \rho e^{k\int_0^t e^{\lambda s}\, ds} - \sigma e^{k\int_0^t e^{\lambda s}\, ds} \\ &= \rho e^{k\int_0^t e^{\lambda s}\, ds}. \end{aligned}$$

Thus, the strategy (2.8) is indeed admissible.

Next, we show that $z(\Gamma) = 0$. Let $v(t), 0 \leq t \leq \Gamma$, be an admissible control of the evader. Replacing $u(t), 0 \leq t \leq \Gamma$, in the equation

$$\begin{aligned} &\dot{z}(t) + \lambda z(t) = v(t) - u(t),\ 0 \leq t \leq \Gamma, \\ &z(0) = z_0, \end{aligned}$$

by the strategy $u(t, v(t))$ constructed in (2.8), we obtain

$$
\begin{aligned}
z(\Gamma) =& e^{-\lambda\Gamma} z_0 + \int_0^{\Gamma} e^{-\lambda(\Gamma-t)}(v(t) - u(t, v(t)))\, dt \\
=& e^{-\lambda\Gamma}\left[z_0 + \int_0^{\Gamma} e^{\lambda t}(v(t) - u(t, v(t)))\, dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 + \int_0^{\Gamma} e^{\lambda t} v(t)\, ds - \int_0^{\Gamma} e^{\lambda t} u(t, v(t))\, dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 + \int_0^{\Gamma} e^{\lambda t} v(t)\, dt - \left(\int_0^{\Gamma'} e^{\lambda t} v(t)\, dt + \int_{\Gamma'}^{\Gamma} e^{\lambda t} v(t)\, dt \right) \right] \\
& - e^{-\lambda\Gamma}\left[\int_0^{\Gamma'} e^{\lambda t}\left(\frac{z_0(\rho-\sigma) e^{k\int_0^t e^{\lambda s}\, ds}}{\|z_0\|} \right) dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 - \frac{z_0(\rho-\sigma)}{\|z_0\|} \int_0^{\Gamma'} e^{\lambda t} e^{k\int_0^t e^{\lambda s}\, ds}\, dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 - \frac{z_0(\rho-\sigma)}{\|z_0\|} \int_0^{\Gamma'} e^{\lambda t} e^{\frac{k}{\lambda}[e^{\lambda t}-1]}\, dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 - \frac{z_0(\rho-\sigma) e^{-\frac{k}{\lambda}}}{\|z_0\|} \int_0^{\Gamma'} e^{\lambda t} e^{\frac{k}{\lambda}[e^{\lambda t}]}\, dt \right] \\
=& e^{-\lambda\Gamma}\left[z_0 - \frac{z_0(\rho-\sigma) e^{-\frac{k}{\lambda}}}{\|z_0\|} \left(\frac{1}{k} e^{\frac{k}{\lambda}\left[e^{\lambda\Gamma'}-1\right]} \right) \right] \\
=& e^{-\lambda\Gamma}\frac{z_0}{\|z_0\|}\left[\|z_0\| - \frac{(\rho-\sigma)}{k e^{\frac{k}{\lambda}}} \left(e^{\frac{k}{\lambda}\left[e^{\lambda\Gamma'}-1\right]} \right) \right]
\end{aligned} \tag{2.9}
$$

having Γ' defined by (2.7), we therefore obtain

$$
\begin{aligned}
z(\Gamma) =& e^{-\lambda\Gamma}\frac{z_0}{\|z_0\|}\left[\|z_0\| - \frac{(\rho-\sigma)}{k e^{\frac{k}{\lambda}}} \left(\frac{\|z_0\|\, k e^{\frac{k}{\lambda}}}{\rho-\sigma} \right) \right] \\
=& e^{-\lambda\Gamma}\frac{z_0}{\|z_0\|}\left(\|z_0\| - \|z_0\| \right) \\
=& e^{-\lambda\Gamma}\frac{z_0}{\|z_0\|}(0) \\
=& 0.
\end{aligned}
$$

Thus, we conclude that $z(t) = 0, 0 \leq t \leq \Gamma$ at the time Γ. Hence Γ is a guaranteed pursuit time.
This completes the proof of the theorem.

∎

3. Illustrative Example

Consider the game problem (2.1)-(2.4) in the space $\mathbb{R}^3$. Let the maximum speed of the pursuer $\rho = 5$, with that of that of the evader $\sigma = 3$. Also, let $k = 2$, $\lambda = \frac{1}{2}$ and $\Gamma = 3$.

The dynamic equations of the players now becomes

$$\dot{z} + \frac{1}{2}z = v - u,$$

Pursuit starts from initial positions $x_0 = (0,0,1)$ and $y_0 = (1,0,0)$ of the pursuer and evader respectively, we denote $z_0 = y_0 - x_0 = (1,0,-1)$ and $v(t) = (1,t,t^2)$. The control functions satisfying the constraints

$$\|u(t)\| \leq 5e^{2\int_0^t e^{\frac{1}{2}s}\,ds} \text{ and } \|v(t)\| \leq 3e^{2\int_0^t e^{\frac{1}{2}s}\,ds}.$$

From the formula (2.7), we obtain that pursuit is completed at the time

$$\Gamma' := 2\ln\left(1 + \frac{1}{4}ln\left(\frac{2\sqrt{2}e^4}{5-3}\right)\right) = 1.47.$$

The constructed strategy (2.8) now takes form

$$U(t, v(t)) = \begin{cases} v(t) + (\sqrt{2}, 0, -\sqrt{2})e^{2\int_0^t e^{\frac{1}{2}s}\,ds}, & 0 \leq t \leq 1.47, \\ v(t), & 1.47 < t \leq 3. \end{cases} \tag{3.1}$$

Indeed, the strategy is admissible, since, for $0 \leq t \leq 1.47$, we have

$$\begin{aligned} \|U(t, v(t))\| &= \left\| \left(1 + \sqrt{2}e^{2\int_0^t e^{\frac{1}{2}s}\,ds},\ t,\ t^2 - \sqrt{2}e^{2\int_0^t e^{\frac{1}{2}s}\,ds}\right) \right\| \\ &= \left\| \left(1 + \sqrt{2}e^{4\left(e^{\frac{t}{2}}-1\right)},\ t,\ t^2 - \sqrt{2}e^{4\left(e^{\frac{t}{2}}-1\right)}\right) \right\| \\ &\leq 385 \end{aligned}$$

and it follows similarly for $1.47 < t \leq 3$, since $\rho = 5 > 3 = \sigma$.

Finally, we show that the solution $z(3) = \mathbf{0}$. That is

$$\begin{aligned} z(3) =& e^{-\frac{3}{2}} \left[z_0 + \int_0^3 e^{\frac{1}{2}t}(v(t) - U(t, v(t)))\,dt \right] \\ =& e^{-\frac{3}{2}} \left(z_0 + \int_0^3 v(t)e^{\frac{1}{2}t}\,dt - \int_0^3 v(t)e^{\frac{1}{2}t}\,dt - \int_0^{1.47} U(t, v(t))e^{\frac{1}{2}t}\,dt \right) \\ =& \mathbf{0}. \end{aligned}$$

In view of this, the time $\Gamma = 3$ is indeed a guaranteed pursuit time in the game (2.1)-(2.4).

4. Conclusion

We study a linear pursuit differential game with generalized geometric constraint imposed on players controls. Sufficient conditions of completion of the game at a finite time Γ (defined as (2.7)) were obtained and pursuers' optimal strategy was constructed in explicit form (see equation(2.8)).

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Bangmod J-MCS 2023

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