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Semi-Analytic Solution of Fractional Fokker-Planck and Fornburg-Whitham Equations Using a Unique Technique

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Abstract In this work, a new hybrid semi-analytical method is presented, merging the Kamal Transform and the Homotopy Perturbation Method (HPM), for solving fractional differential equations (FDEs) involving the Caputo derivative. The method is used to solve complex systems such as the Fokker-Planck and Fornberg-Whitham equations. This method beats classical iterative techniques in both computational efforts and simplicity. Numerous tables and graphics display the numerical results, establishing the method's accuracy by reference to the residual power series method and exact solutions. Our results illustrate the strength and the precision of this technique to be very efficient with a minor cost against numerical solution techniques for fractional partial differential equations.

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Keywords: Fokker-Planck and Fornberg-Whitham equation; Homotopy Perturbation Method; Kamal transform

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1. Introduction

The extension of classical differential equations to arbitrary orders is essential for examining the intrinsic properties of dynamical systems. This has led to a growing interest in fractional calculus, which allows researchers to study and solve accurate modeling of non-local phenomena due to its inherent properties. Over the past few years, fractional operators have become instrumental in capturing complex behaviors in real-world systems. Their applications have expanded across various fields, with fractional calculus proving highly effective in describing phenomena in engineering and sciences. This versatility has solidified its role as a powerful tool in both theoretical and practical applications [1].

Over the past two decades, several integral transformations utilizing exponential-type kernels have been developed to address diverse mathematical difficulties. Fractional order integral equations, as well as ordinary and partial fractional differential equations, are resolved utilizing these transforms [2]. Nonetheless, these integral transforms are insufficient to address non-linear equations due to the complexities arising from non-linear parts. To address these challenges, several approaches were developed through the hybridization of integral transforms of exponential order with distinct fractional operators [3–5].

In this article, we will discuss the new methodology on hybridization of Kamal transform with homotopy perturbation method. Further, we have shown that the introduced method can easily solves the fractional ordered differential equations viz. Kolmogorov equation, Fokker-Planck equation, and Fornberg-Whitham equation. Also, we discuss the efficiency of HPKTM by comparing the results with exact solution and residual power series method.

The generalized non-linear differential equation is,

$$D\varphi(\ominus, t) + R\varphi(\ominus, t) + N\varphi(\ominus, t) = f(\ominus, t), \tag{1.1}$$

where D is differential operator, $f(\ominus, t)$ is continuous function, $R(\varphi)$ and $N(\varphi)$ denotes the linear and non-linear terms respectively. Various semi-analytical methods like Adomian Decomposition Method [6], Residual Power Series Approach (RPSM) [7, 8], Variational Iteration Method (VIM) [9], Homotopy Perturbation method (HPM) [10], Homotopy Perturbation Transform Method [11, 12] were developed to solve FDEs.

Within last decades various work is carried out to enhance the research in fractional calculus and various methodology were developed [13–15] and numerous hybrid approaches that combine semi-analytic techniques with integral transforms have been presented in recent years like VIM [16], HPM [17] and ADM [18] to solve fractional partial differential equations (FPDEs). In 1999, Ji-Huan He constructed the homotopy perturbation method (HPM) [10, 23]. The above mentioned techniques were utilized by several engineers and mathematicians to deal with various fractional ordered equations arising in real world problems [19–22]. Furthermore, this technique is applied to oscillatory equations with discontinuities, various nonlinear wave equations, and boundary value problems. Hes HPM is applicable to solve the different class of non-linear equations. The solution by this approach is evolved as a sum of series terms, which generally converges fast to its precise solution [23].

Present work showcases a new hybrid approach that combines the HPM [10] and the Kamal transform (KT) [32] named Homotopy Perturbation Kamal Transform Method (HPKTM) to deal with the general FPDEs as it has superior convergence properties. Additionally, the proposed method has an effective computational algorithm for non-linear systems involving Fractional Ordinary Differential Equations (FODEs). With HPKTM, which is more appropriate, satisfactory findings are obtained than other semi-analytical and numerical methods. The main advantage of HPKTM over other methods is to solve complicated problems conveniently.

TABLE 1. Notations used

Symbol	Description
$\ominus$	space variable
t	time variable
φ	function of space and time
$\varphi_{\ominus} = \frac{\partial \varphi}{\partial \ominus}$	Partial derivative wrt "$\ominus$"
$\varphi_t = \frac{\partial \varphi}{\partial t}$	Partial derivative wrt "t"

The prime motive of HPKTM is to achieve precise results with less iterations compared to HPM. We solve the time-fractional Fokker-Planck (F-P) equation [33] using HPKTM,

$$\frac{\partial^{\omega} \varphi}{\partial t} = \left[-A(\ominus, t) \frac{\partial \varphi}{\partial \ominus} + B(\ominus, t) \frac{\partial^2 \varphi}{\partial \ominus^2} \right], \tag{1.2}$$

with the initial condition

$$\varphi(\ominus, 0) = f(\ominus), \ \ominus \in R.$$

F-P equation has several applications in various fields like theoretical biology, chemical physics, circuit theory, solid-state physics, and quantum optics in natural science [33]. Further, HPKTM technique is equally established and demonstrated on non-linear type fractional-order Fornberg Whitham (F-W) equation [34] given by

$$\varphi_t^{\omega} - \varphi_{\ominus\ominus t} + \varphi_{\ominus} = \varphi\varphi_{\ominus\ominus\ominus} - \varphi\varphi_{\ominus} + 3\varphi_{\ominus}\varphi_{\ominus\ominus}, t > 0, \text{ with } \varphi(\ominus, 0) = \frac{4}{3} e^{\frac{\ominus}{2}}. \tag{1.3}$$

The application of the F-W equation has notable significance in the study of the non-linear dispersive wave equation and the qualitative behaviour of wave breakage. It is exhibited that the F-W equation enables peak-on results as an occurrence of wave breaks and the numerical simulation to the limit wave heights.

The current work is organized in the following manner: Some well-defined definitions of terms like the Mittag-Leffler function, Kamal transform, and Caputo derivative and theorems are found in the section under "Preliminaries." The section headed "Homotopy Perturbation Kamal Transform Method (HPKTM)" discusses the Proposed Method, a hybrid approach that has been suggested. The "Applications" section discusses the solution of the linear and non-linear FPDEs using the suggested approach, and the "Results and discussion" section is next. Finally, the summary is covered in the part entitled "Conclusion".

2. Preliminaries

Definition 2.1. [35] Riemann-Liouville (RL) fractional integral of order ω of a function $\varphi(\ominus, t)$ is

$$I_t^{\omega}[\varphi(\ominus, t)] = {}_aD_t^{-\omega}[\varphi(\ominus, t)] = \frac{1}{\Gamma(\omega)} \int_a^t (t-t)^{\omega-1}\varphi(\ominus, t)dt. \tag{2.1}$$

Definition 2.2. [35] Caputo fractional derivative of order $\omega > 0$ of the function $\varphi(\ominus, t)$ can be written as

$$D_t^{\omega}[\varphi(\ominus, t)] = \begin{cases} \frac{1}{\Gamma(n-\omega)} \int_0^t (t-t)^{n-\omega-1} \frac{\partial^n \varphi(\varphi, t)}{\partial t^n} dt, & n-1 < \omega < n \\ \frac{\partial^n \varphi(\ominus, t)}{\partial t^n}, & \omega = n \in N. \end{cases} \tag{2.2}$$

Definition 2.3. [32] For $A = \{\varphi(t) : \exists M, b_1, b_2 > 0, |\varphi(t)| < Me^{\frac{|t|}{b_j}}$, if $t \in (-1)^j \times [0, \infty)\}$ with b_1, b_2 can be any values and M is finite value, Kamal transform (KT) of a function $\varphi(t) \in A$ is

$$K\{\varphi(t)\} = \int_0^{\infty} \varphi(t)e^{-\frac{t}{s}}dt, t \geq 0, b_1 \leq s \leq b_2. \tag{2.3}$$

The Kamal transform of some basic functions is listed in Table 2.

Table 2. Kamal transform $K\{\varphi(t)\}$ [32] of some basic function $\varphi(t)$.

Function $\varphi(t)$	Kamal transform $K\{\varphi(t)\}$
$K\{1\}$	s
$K\{t\}$	s^2
$K\{t^n\}$	$\Gamma(n+1)s^{n+1}, n > 0$
$K\{\sin(at)\}$	$\frac{as^2}{1+a^2s^2}$
$K\{\cos(at)\}$	$\frac{s}{1+a^2s^2}$
$K\{e^{at}\}$	$\frac{s}{1-as}$
$K\{\sinh(at)\}$	$\frac{as^2}{1-a^2s^2}$

Theorem 2.4 (Existence Theorem [32])**.** *A piece wise continuous function $\varphi(t)$ is said to be of exponential order $q \geq 0$, if it satisfies $|\varphi(t)| \leq Me^{qt}$, then $K\{\varphi(t)\}$ exist for all $\frac{1}{s} > q$.*

Proof. We have

$$\begin{aligned} |K\{\varphi(t)\}| &= |\int_0^{\infty} \varphi(t)e^{-\frac{t}{s}}dt| \leq \int_0^{\infty} |\varphi(t)|e^{-\frac{t}{s}}dt \\ &\leq \int_0^{\infty} Me^{qt}e^{-\frac{t}{s}}dt \leq \frac{Ms}{1-qs}. \end{aligned} \tag{2.4}$$

Theorem 2.5. [32] *Kamal transform of integer order derivative of $\varphi(\ominus, t)$ is*

$$K\{D_t^n \varphi(\ominus, t)\} = \frac{1}{s^n} K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1} \frac{1}{s^{n-m-1}} D_t^m \varphi(\ominus, 0). \tag{2.5}$$

Theorem 2.6. [32] *Kamal transform of fractional order derivative of $\varphi(\ominus, t)$ is*

$$K\{D_t^{\omega} \varphi(\ominus, t)\} = \frac{1}{s^{\omega}} K\{\varphi(\ominus, t)\} - \frac{1}{s^{\omega-1}} \varphi(\ominus, 0) \tag{2.6}$$

$$K\{D_t^{2\omega} \varphi(\ominus, t)\} = \frac{1}{s^{2\omega}} K\{\varphi(\ominus, t)\} - \frac{1}{s^{2\omega-1}} \varphi(\ominus, 0) - \frac{1}{s^{2\omega-2}} D_t \varphi(\ominus, 0) \tag{2.7}$$

$$\vdots$$

$$K\{D_t^{n\omega} \varphi(\ominus, t)\} = \frac{1}{s^{n\omega}} K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1} \frac{1}{s^{n\omega-m-1}} D_t^m \varphi(\ominus, 0). \tag{2.8}$$

Proof. Let consider the Caputo derivative definition as state in (2.2), we get

$$D_t^{\alpha} \varphi(\ominus, t) = D_t^{-(n-\alpha)} h(\ominus, t), \text{ where } h(\ominus, t) = \varphi^n(\ominus, t),\ n-1 < \alpha \leq n, \tag{2.9}$$

now operating the Kamal transform on RL fractional derivative, leads to

$$K\{D_t^{\alpha} \varphi(\ominus, t)\} = \left(\frac{1}{s}\right)^{-(n-\alpha)} K\{\varphi(\ominus, t)\}, \tag{2.10}$$

Implementing the result of equation (2.9) into equation (2.10), it yields

$$K\{D_t^{\alpha} \varphi(\ominus, t)\} = K\left\{D_t^{-(n-\alpha)} h(\ominus, t)\right\} = \left(\frac{1}{s}\right)^{-(n-\alpha)} K\{\varphi^n(\ominus, t)\}, \tag{2.11}$$

using equation (2.5), we have

$$K\{h(\ominus, t)\} = K\{\varphi^n(\ominus, t)\} = \frac{1}{s^n} K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1} \frac{1}{s^{n-m-1}} D_t^m \varphi(\ominus, 0), \tag{2.12}$$

substituting the output of equation (2.12) into equation (2.11), it leads to

$$\begin{aligned} K\{D_t^{\alpha} \varphi(\ominus, t)\} &= \left(\frac{1}{s}\right)^{-(n-\alpha)} \left[\frac{1}{s^n} K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1} \frac{1}{s^{n-m-1}} D_t^m(\ominus, t)\right] \\ &= \frac{1}{s^{\alpha}} K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1} \frac{1}{s^{\alpha-m-1}} D_t^m(\ominus, t), \end{aligned} \tag{2.13}$$

$\forall \omega,\ 0 < \omega \leq 1$ and taking $\alpha = n\omega$, we obtain the result (2.6) as desired.

3. Homotopy Perturbation Kamal Transform Method (HPKTM)

Let us begin with a generalized form of nonlinear FDEs as

$$D_t^{n\omega} \varphi(\ominus, t) + R\varphi(\ominus, t) + N\varphi(\ominus, t) = f(\ominus, t), \tag{3.1}$$

subject to required initial conditions

$$\varphi(\ominus, 0) = \phi_0(\ominus), \frac{\partial \varphi(\ominus, 0)}{\partial t} = \phi_1(\ominus), \ldots, \frac{\partial^{n-1} \varphi(\ominus, 0)}{\partial t^{n-1}} = \phi_{n-1}(\ominus), \tag{3.2}$$

in which $f(\ominus, t)$ is a continuous function, $D_t\ (\equiv \frac{\partial}{\partial t})$ is partial derivative wrt time, R denote linear terms and N denote non-linear terms.
First we start by operating Kamal transform on equation (3.1), we have

$$K\{D_t^{n\omega}\varphi(\ominus, t)\} = -K\{R\varphi(\ominus, t)\} - K\{N\varphi(\ominus, t)\} + K\{f(\ominus, t)\}. \tag{3.3}$$

Using differentiation properties as given in (2.6), it reach to

$$\begin{aligned}&\frac{1}{s^{n\omega}}K\{\varphi(\ominus, t)\} - \sum_{m=0}^{n-1}\frac{1}{s^{n\omega-m-1}}D_t^m u(\ominus, 0)\\ &= -K\{R\varphi(\ominus, t)\} - K\{N \ominus (\ominus, t)\} + K\{f(\ominus, t)\}\end{aligned} \tag{3.4}$$

or

$$\begin{aligned}K\{\varphi(\ominus, t)\} = s^{n\omega}\sum_{m=0}^{n-1}&\frac{1}{s^{n\omega-m-1}}D_t^m\varphi(\ominus, 0) - s^{n\omega}K\{R\varphi(\ominus, t)\}\\ &- s^{n\omega}K\{N\varphi(\ominus, t)\} + s^{n\omega}K\{f(\ominus, t)\},\end{aligned} \tag{3.5}$$

Incorporating the initial conditions given in equation (3.2)

$$\begin{aligned}K\{\varphi(\ominus, t)\} = \big[s\phi_0(\ominus) + s^2\phi_1(\ominus) + \cdots + s^n\phi_{n-1}(\ominus)\big] &- s^{n\omega}K\{R\varphi(\ominus, t)\}\\ - s^{n\omega}K\{N\varphi(\ominus, t)\} &+ s^{n\omega}K\{f(\ominus, t)\}.\end{aligned} \tag{3.6}$$

Taking Inverse Kamal transform (IKT), we have

$$\varphi(\ominus, t) = G(\ominus, t) - K^{-1}\left[s^{n\omega}K\{R\varphi(\ominus, t)\} - s^{n\omega}K\{N\varphi(\ominus, t)\}\right]. \tag{3.7}$$

in equation (3.7), $G(\ominus, t)$ donote IKT of first and last term of equation (3.6).
Implementing the HPM [10] to equation (3.7), yields

$$\begin{aligned}\sum_{i=0}^{\infty}p^i\varphi_i(\ominus, t) =& G(\ominus, t) - p\left[K^{-1}\left[s^{n\omega}K\{R\sum_{i=0}^{\infty}p^i\varphi_i(\ominus, t)\}\right.\right.\\ &\left.\left. + \ s^{n\omega}K\{N\sum_{i=0}^{\infty}p^i\varphi_i(\ominus, t)\}\right]\right].\end{aligned} \tag{3.8}$$

Eventually He's Polynomial [23] is used to decompose non-linear parts of equation (3.8) as

$$N\varphi(\ominus, t) = \sum_{n=0}^{\infty}p^n H_n(\varphi), \tag{3.9}$$

where $H_n(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n) = \dfrac{1}{n!}\left[\frac{\partial^n}{\partial p^n}N\left(\sum_{i=0}^{\infty}p^i\varphi_i\right)\right]_{p=0}, n = 0, 1, 2, \ldots$
Using equations (3.8) and (3.9), we get

$$\sum_{i=0}^{\infty}p^i\varphi_i(\ominus, t) = G(\ominus, t) - p\left[K^{-1}\left[s^{n\omega}K\{R\sum_{i=0}^{\infty}p^i\varphi_i(\ominus, t)\} + s^{n\omega}K\{\sum_{i=0}^{\infty}p^iH_i(\varphi)\}\right]\right]. \tag{3.10}$$

Comparing the terms of p on both sides of equation (3.10)

$$p^0 : \varphi_0(\ominus, t) = G(\ominus, t), \tag{3.11}$$
$$p^1 : \varphi_1(\ominus, t) = -K^{-1}\left[s^{n\omega} K\{R\varphi_0(\ominus, t)\} + s^{n\omega} K\{H_0(\varphi)\}\right], \tag{3.12}$$
$$p^2 : \varphi_2(\ominus, t) = -K^{-1}\left[s^{n\omega} K\{R\varphi_1(\ominus, t)\} + s^{n\omega} K\{H_1(\varphi)\}\right], \tag{3.13}$$
$$\vdots$$
$$p^n : \varphi_n(\ominus, t) = -K^{-1}\left[s^{n\omega} K\{R\varphi_{n-1}(\ominus, t)\} + s^{n\omega} K\{H_{n-1}(\varphi)\}\right]. \tag{3.14}$$

Ultimately, the result of application (3.1) is obtained as

$$\varphi(\ominus, t) = \lim_{p\to 1} \varphi_n(\ominus, t) = \varphi_0(\ominus, t) + p^1\varphi_1(\ominus, t) + p^2\varphi_2(\ominus, t) + \cdots \tag{3.15}$$
$$= \varphi_0(\ominus, t) + \varphi_1(\ominus, t) + \varphi_2(\ominus, t) + \cdots. \tag{3.16}$$

3.1. Convergence theorem of the approximate solution by HPKTM technique

Theorem 3.1. [36] *Assume that the Banach space of all continuous real-valued functions considered on the rectangular area $[a, b] \times [0, T]$ is defined as $B \equiv C([a, b] \times [0, T])$. Then, $\varphi(\ominus, t) = \sum_{k=0}^{\infty} \varphi_k(\ominus, t)$ is convergent as equation (3.16), if $\varphi_0 \in B$ is bounded and $\|\varphi_{k+1}\| \leq \sigma\|\varphi_k\|, \forall\varphi_k \in B$, for $0 < \sigma < 1$.*

Proof. Assuming $\{\mathbb{J}_q\}$ be the partial sum of the sequence given equation (3.16) as

$$\mathbb{J}_0 = \varphi_0(\ominus, t),$$
$$\mathbb{J}_1 = \varphi_0(\ominus, t) + \varphi_1(\ominus, t),$$
$$\mathbb{J}_2 = \varphi_0(\ominus, t) + \varphi_1(\ominus, t) + \varphi_2(\ominus, t),$$
$$\vdots$$
$$\mathbb{J}_q = \varphi_0(\ominus, t) + \varphi_1(\ominus, t) + \varphi_2(\ominus, t) + \cdots + \varphi_q(\ominus, t). \tag{3.17}$$

We will illustrate that $\{\mathbb{J}_q\}_{q=0}^{\infty}$ generates a Cauchy sequence in Banach space B in order to obtain the intended outcome. Additionally, let's take

$$\begin{aligned}\|\mathbb{J}_{q+1} - \mathbb{J}_q\| &= \|\varphi_{q+1}(\ominus, t)\| \\ &\leq \sigma\,\|\varphi_q(\ominus, t)\| \\ &\leq \sigma^2\,\|\varphi_{q-1}(\ominus, t)\| \\ &\leq \sigma^3\,\|\varphi_{q-2}(\ominus, t)\| \\ &\ \ \vdots \\ &\leq \sigma^{q+1}\,\|\varphi_0(\ominus, t)\|.\end{aligned} \tag{3.18}$$

Considering each $q, n \in N$, for $q \geq n,$, we obtain

$$\begin{aligned}
\|\mathbb{J}_q - \mathbb{J}_n\| &= \|(\mathbb{J}_q - \mathbb{J}_{q-1}) + (\mathbb{J}_{q-1} - \mathbb{J}_{q-2}) + (\mathbb{J}_{q-2} - \mathbb{J}_{q-3}) + \cdots + (\mathbb{J}_{n+1} - \mathbb{J}_n)\| \\
&\leq \|\mathbb{J}_q - \mathbb{J}_{q-1}\| + \|\mathbb{J}_{q-1} - \mathbb{J}_{q-2}\| + \|\mathbb{J}_{q-2} - \mathbb{J}_{q-3}\| + \cdots + \|\mathbb{J}_{n+1} - \mathbb{J}_n\| \\
&\leq \sigma^q \|\varphi_0(\ominus, t)\| + \sigma^{q-1} \|\varphi_0(\ominus, t)\| + \sigma^{q-2} \|\varphi_0(\ominus, t)\| + \cdots + \sigma^{n+1} \|\varphi_0(\ominus, t)\| \\
&\leq \left(\sigma^q + \sigma^{q-1} + \sigma^{q-2} + \cdots + \sigma^{n+1}\right) \|\varphi_0(\ominus, t)\| \\
&\leq \frac{(1 - \sigma^{q-n})}{(1 - \sigma)} \sigma^{n+1} \|\varphi_0(\ominus, t)\| \\
&\leq \beta \|\varphi_0(\ominus, t)\| .
\end{aligned} \tag{3.19}$$

in which $\beta = \frac{(1-\sigma^{q-n})}{(1-\sigma)}\sigma^{n+1}$. Given the limited nature of $\varphi_0(\ominus, t)$, $\|\varphi_0(\ominus, t)\| < \infty$. Additionally, if $q \to \infty$ is taken into consideration for any finite value of n, then $\beta \to 0$, which indicates

$$\lim_{q \to \infty} \|\mathbb{J}_q - \mathbb{J}_n\| = 0. \tag{3.20}$$

Therefore, the Cauchy sequence in B is $\{\mathbb{J}_q\}_{q=0}^{\infty}$. Thus, the equation (3.1) likewise as its series solution in the form of (3.16) become convergent.

Theorem 3.2. *If the estimated solution of equation (3.1) is* $\sum_{k=0}^{n} \varphi_k(\ominus, t)$, *then the maximum absolute error can be approximated as*

$$\left\| \varphi(\ominus, t) - \sum_{k=0}^{n} \varphi_k(\ominus, t) \right\| \leq \frac{\sigma^{n+1}}{1 - \sigma} \|\varphi_0(\ominus, t)\| . \tag{3.21}$$

Proof. From equation (3.19) as written in Theorem 3.1 is

$$\|\mathbb{J}_q - \mathbb{J}_n\| \leq \beta \|\varphi_0(\ominus, t)\| , \text{ where } \beta = \frac{(1 - \sigma^{\mathrm{q-n}})}{(1 - \sigma)} \sigma^{\mathrm{n+1}}. \tag{3.22}$$

Here, $\{\mathbb{J}_q\}_{q=0}^{\infty} \to \varphi(\ominus, t)$ as $q \to \infty$ along with equation (3.17), we could obtain $\mathbb{J}_n = \sum_{k=0}^{n} \varphi_k(\ominus, t)$,

$$\left\| \varphi(\ominus, t) - \sum_{k=0}^{n} \varphi_k(\ominus, t) \right\| \leq \beta \|\varphi_0(\ominus, t)\| . \tag{3.23}$$

Given that $0 < \sigma < 1$, $(1 - \sigma^{q-n}) < 1$ hence

$$\left\| \varphi(\ominus, t) - \sum_{k=0}^{n} \varphi_k(\ominus, t) \right\| \leq \frac{\sigma^{n+1}}{1 - \sigma} \|\varphi_0(\ominus, t)\| . \tag{3.24}$$

This proves the theorem.

Theorem 3.3. *The generalized form of the FDE as equation (3.1) could be solved analytically if the series solution* $\sum_{n=0}^{\infty} \varphi_n$ *presented as equation (3.16) is converges.*

Proof. We determine the sequence $\mathbb{J}_n = \varphi_0 + \varphi_1 + \varphi_2 + \cdots + \varphi_{n-1}$. by using iterative scheme,

$$\mathbb{J}_0 = \varphi_0, \tag{3.25}$$

$$\mathbb{J}_1 = \varphi_0 + \varphi_1, \tag{3.26}$$

$$\vdots$$

$$\mathbb{J}_n = \varphi(\ominus, 0) - K^{-1}\left[s^\omega K\left\{\varphi_1 + \varphi_2 + \cdots + \varphi_{n-1}\right\}\right] + K^{-1}\left[s^\omega K\left\{f(\ominus, t)\right\}\right]. \tag{3.27}$$

Assume that the solution of the series (3.16) converges, and $\varphi = \sum_{n=0}^{\infty} \varphi_n$, then we have

$$\begin{aligned}\varphi &= \lim_{n\to\infty} \mathbb{J}_n \\ &= \varphi(\ominus, 0) - K^{-1}\left[s^\omega K\left\{R \lim_{n\to\infty} \varphi_n + \lim_{n\to\infty} H_n(\varphi)\right\}\right] + K^{-1}\left[s^\omega K\left\{f(\ominus, t)\right\}\right],\end{aligned} \tag{3.28}$$

where $H_n(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n) = \dfrac{1}{n!} N\left(\sum_{i=0}^{\infty} \varphi_i\right), n = 0, 1, 2, \ldots$.
Implementing the HPM, it leads to

$$\begin{aligned}\varphi &= \sum_{n=0}^{\infty} p^n \varphi_n \\ &= \varphi(\ominus, 0) - pK^{-1}\left[s^\omega K\left\{R\left(\sum_{n=0}^{\infty} p^n \varphi_n\right) + N\left(\sum_{n=0}^{\infty} p^n \varphi_n\right)\right\}\right] \\ &\quad + K^{-1}\left[s^\omega K\left\{f(\ominus, t)\right\}\right]\end{aligned} \tag{3.29}$$

for homotopy parameter $p = 1$,

$$\begin{aligned}\varphi &= \sum_{n=0}^{\infty} \varphi_n \\ &= \varphi(\ominus, 0) - K^{-1}\left[s^\omega K\left\{R\left(\sum_{n=0}^{\infty} \varphi_n\right) + N\left(\sum_{n=0}^{\infty} \varphi_n\right)\right\}\right] \\ &\quad + K^{-1}\left[s^\omega K\left\{f(\ominus, t)\right\}\right],\end{aligned} \tag{3.30}$$

or,

$$\begin{aligned}\varphi(\ominus, t) =& \varphi(\ominus, 0) - K^{-1}\left[s^\omega K\left\{R(\varphi(\ominus, t)) + N(\varphi(\ominus, t))\right\}\right] \\ &+ K^{-1}\left[s^\omega K\left\{f(\ominus, t)\right\}\right],\end{aligned} \tag{3.31}$$

applying KT on (3.25),

$$\frac{K\left\{\varphi(\ominus, t)\right\} - s\varphi(\ominus, 0)}{s^\omega} = K\left\{(R\varphi(\ominus, t)\right\} + K\left\{(N\varphi(\ominus, t)\right\} + K\left\{(f(\ominus, t)\right\}, \tag{3.32}$$

or,

$$K\left\{(D^\omega \varphi(\ominus, t)\right\} = K\left\{R\varphi(\ominus, t)\right\} + K\left\{N\varphi(\ominus, t)\right\} + K\left\{f(\ominus, t)\right\}, \tag{3.33}$$

and by using IKT,

$$D^{\omega}\varphi(\ominus, t) = R\varphi(\ominus, t) + N\varphi(\ominus, t) + f(\ominus, t). \tag{3.34}$$

The outcome demonstrates that the employed method's solution as (3.10) is convergent and unconditionally stable [23].

4. Application of HPKTM and Numerical Discussions

Example 4.1. In class of Fokker-Planck equation, the time-fractional backward Kolmogorov equation [33] defined as:

$$\frac{\partial^{\omega}\varphi}{\partial t} = \left[-A(\ominus, t)\frac{\partial\varphi}{\partial\ominus} + B(\ominus, t)\frac{\partial^2\varphi}{\partial\ominus^2}\right], \tag{4.1}$$

subject to,

$$\varphi(\ominus, 0) = f(\ominus), \ \ominus \in R.$$

Consider $A(\ominus, t) = (\ominus + 1)$ and $B(\ominus, t) = \ominus^2 e^t$, (4.1) becomes

$$\varphi_t^{\omega} - (\ominus + 1)\varphi_{\ominus} - \ominus^2 e^t \varphi_{\ominus\ominus} = 0, \tag{4.2}$$

subject to

$$\varphi(\ominus, 0) = \ominus + 1. \tag{4.3}$$

Applying KT on (4.2),

$$K\{\varphi_t^{\omega}\} - K\{(\ominus + 1)\varphi_{\ominus}\} - K\{\ominus^2 e^t \varphi_{\ominus\ominus}\} = K\{0\}. \tag{4.4}$$

Using differentiation properties (2.3),

$$\left[\frac{1}{s^{\omega}}K\{\varphi(\ominus, t)\} - \frac{1}{s^{\omega-1}}\varphi(\ominus, 0)\right] - K\{(\ominus + 1)\varphi_{\ominus}\} - K\{\ominus^2 e^t \varphi_{\ominus\ominus}\} = 0. \tag{4.5}$$

Substituting equation (4.3),

$$K\{\varphi(\ominus, t)\} = s(\ominus + 1) + s^{\omega}K\{(\ominus + 1)\varphi_{\ominus}\} + s^{\omega}K\{\ominus^2 e^t \varphi_{\ominus\ominus}\}, \tag{4.6}$$

taking IKT on both sides gives,

$$\varphi(\ominus, t) = K^{-1}\left[s(\ominus + 1)\right] + K^{-1}\left[s^{\omega}K\{(\ominus + 1)\varphi_{\ominus}\} + s^{\omega}K\{\ominus^2 e^t \varphi_{\ominus\ominus}\}\right], \tag{4.7}$$

$$\varphi(\ominus, t) = (\ominus + 1) + K^{-1}\left[s^{\omega}K\{(\ominus + 1)\varphi_{\ominus}\} + s^{\omega}K\{\ominus^2 e^t \varphi_{\ominus\ominus}\}\right]. \tag{4.8}$$

By applying the HPM to (3.8) gives

$$\begin{aligned}\sum_{i=0}^{\infty} p^i \varphi_i(\ominus, t) =& (\ominus + 1) + pK^{-1}\left[s^{\omega}K\left\{(\ominus + 1)\frac{\partial}{\partial\ominus}\left(\sum_{i=0}^{\infty} p^i \varphi_i(\ominus, t)\right)\right\}\right] \\ &+ pK^{-1}\left[+s^{\omega}K\left\{\ominus^2 e^t \frac{\partial^2}{\partial\ominus^2}\left(\sum_{i=0}^{\infty} p^i u_i(\ominus, t)\right)\right\}\right].\end{aligned} \tag{4.9}$$

Comparing the power of \p” in equation (4.9),

$$
\begin{aligned}
p^0 : \varphi_0(\ominus, t) &= \ominus + 1, \\
p^1 : \varphi_1(\ominus, t) &= K^{-1}\left[s^\omega K\left\{(\ominus+1)\frac{\partial \varphi_0}{\partial \ominus} + \ominus^2 e^t \frac{\partial^2 \varphi_0}{\partial \ominus^2}\right\}\right] \\
&= (\ominus+1)\frac{t^\omega}{\Gamma(\omega+1)}, \\
p^2 : \varphi_2(\ominus, t) &= K^{-1}\left[s^\omega K\left\{(\ominus+1)\frac{\partial \varphi_1}{\partial \ominus} + \ominus^2 e^t \frac{\partial^2 \varphi_1}{\partial \ominus^2}\right\}\right] \\
&= (\ominus+1)\frac{t^{2\omega}}{\Gamma(2\omega+1)}, \\
p^3 : \varphi_3(\ominus, t) &= K^{-1}\left[s^\omega K\left\{(\ominus+1)\frac{\partial \varphi_2}{\partial \ominus} + \ominus^2 e^t \frac{\partial^2 \varphi_2}{\partial \ominus^2}\right\}\right] \\
&= (\ominus+1)\frac{t^{3\omega}}{\Gamma(3\omega+1)}, \\
&\vdots
\end{aligned}
\tag{4.10}
$$

Thus, using (3.16) the exact solution of (4.2) obtained as

$$
\begin{aligned}
\varphi(\ominus, t) &= (\ominus+1)\left[1 + \frac{t^\omega}{\Gamma(\omega+1)} + \frac{t^{2\omega}}{\Gamma(2\omega+1)} + \frac{t^{3\omega}}{\Gamma(3\omega+1)} + \cdots\right] \\
&= (\ominus+1)E_\omega(t^\omega).
\end{aligned}
\tag{4.11}
$$

where $E_\omega(t^\omega)$ is Mittag-Leffler function [35].

Example 4.2. The time-fractional Fokker-Planck equation[33] defined as

$$
\frac{\partial^\omega \varphi}{\partial t} = \left[-\frac{\partial}{\partial \ominus}A(\ominus, t)\varphi + \frac{\partial^2}{\partial \ominus^2}B(\ominus, t)\varphi\right], \tag{4.12}
$$

subject to initial condition,

$$
\varphi(\ominus, 0) = f(\ominus), \ \ominus \in R.
$$

Considering $A(\ominus, t) = e^t\cosh(\ominus)\coth(\ominus) + e^t\sinh(\ominus) - \coth(\ominus)$ and $B(\ominus, t) = e^t\cosh(\ominus)$, in (4.12) yields

$$
\begin{aligned}
&\frac{\partial^\omega \varphi}{\partial t} + \frac{\partial}{\partial \ominus}\left[\left(e^t\cosh(\ominus)\coth(\ominus) + e^t\sinh(\ominus) - \coth(\ominus)\right)\varphi\right] \\
&\qquad - \frac{\partial^2}{\partial \ominus^2}\left[\left(e^t\cosh(\ominus)\right)\varphi\right] = 0,
\end{aligned}
\tag{4.13}
$$

subject to,

$$
\varphi(\ominus, 0) = \sinh(\ominus). \tag{4.14}
$$

Operating KT, (4.13) is written as,

$$K\left\{\frac{\partial^{\omega}\varphi}{\partial t}\right\}+K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)-\coth(\ominus)\right)\varphi\right]\right\}$$
$$-K\left\{\frac{\partial^{2}}{\partial\ominus^{2}}\left[\left(e^{t}\cosh(\ominus)\right)\varphi\right]\right\}=K\left\{0\right\}. \tag{4.15}$$

Using differentiation properties (2.3),

$$\left[\frac{1}{s^{\omega}}K\left\{\varphi(\ominus,t)\right\}-\frac{1}{s^{\omega-1}}\varphi(\ominus,0)\right]$$
$$+K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)-\coth(\ominus)\right)\varphi\right]\right\}$$
$$-K\left\{\frac{\partial^{2}}{\partial\ominus^{2}}\left[\left(e^{t}\cosh(\ominus)\right)\varphi\right]\right\}=K\left\{0\right\}. \tag{4.16}$$

Substituting equation (4.14),

$$K\left\{\varphi(\ominus,t)\right\}=s(\sinh(\ominus))+s^{\omega}K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)\right.\right.\right.$$
$$\left.\left.\left.-\ \coth(\ominus)\right)\varphi\right]\right\}-s^{\omega}K\left\{\frac{\partial^{2}}{\partial\ominus^{2}}\left[\left(e^{t}\cosh(\ominus)\right)\varphi\right]\right\}. \tag{4.17}$$

Taking IKT,

$$\varphi(\ominus,t)=\sinh(\ominus)+K^{-1}\left[s^{\omega}K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\ \coth(\ominus)\right)\varphi\right]\right\}\right]-K^{-1}\left[s^{\omega}K\left\{\frac{\partial^{2}}{\partial\ominus^{2}}\left[\left(e^{t}\cosh(\ominus)\right)\varphi\right]\right\}\right]. \tag{4.18}$$

By applying the HPM,

$$\sum_{i=0}^{\infty}p^{i}\varphi_{i}(\ominus,t)=\sinh(\ominus)+pK^{-1}\left[s^{\omega}K\left\{\frac{\partial}{\partial\ominus}\left(\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\ \coth(\ominus)\right)\sum_{i=0}^{\infty}p^{i}\varphi_{i}(\ominus,t)\right)\right\}\right]$$
$$-pK^{-1}\left[s^{\omega}K\left\{\frac{\partial^{2}}{\partial\ominus^{2}}\left(\left(e^{t}\cosh(\ominus)\right)\sum_{i=0}^{\infty}p^{i}\varphi_{i}(\ominus,t)\right)\right\}\right]. \tag{4.19}$$

By comparing the power of \p” term’s of equation (4.19),

$$p^{0}:\varphi_{0}(\ominus,t)=\sinh(\ominus),$$
$$p^{1}:\varphi_{1}(\ominus,t)=K^{-1}\left[s^{\omega}K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^{t}\cosh(\ominus)\coth(\ominus)+e^{t}\sinh(\ominus)-\coth(\ominus)\right)\varphi_{0}\right]\right.\right.$$
$$\left.\left.-\ \frac{\partial^{2}}{\partial\ominus^{2}}\left[\left(e^{t}\cosh(\ominus)\right)\varphi_{0}\right]\right\}\right]$$
$$=\sinh(\ominus)\frac{t^{\omega}}{\Gamma(\omega+1)},$$

$$
\begin{aligned}
p^2: \varphi_2(\ominus,t) =& K^{-1}\left[s^\omega K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^t\cosh(\ominus)\coth(\ominus)+e^t\sinh(\ominus)-\coth(\ominus)\right)\varphi_1\right]\right.\right.\\
&\left.\left. -\frac{\partial^2}{\partial\ominus^2}\left[\left(e^t\cosh(\ominus)\right)\varphi_1\right]\right\}\right]\\
=&\sinh(\ominus)\frac{t^2\omega}{\Gamma(2\omega+1)},\\
p^3: \varphi_3(\ominus,t) =& K^{-1}\left[s^\omega K\left\{\frac{\partial}{\partial\ominus}\left[\left(e^t\cosh(\ominus)\coth(\ominus)+e^t\sinh(\ominus)-\coth(\ominus)\right)\varphi_2\right]\right.\right.\\
&\left.\left. -\frac{\partial^2}{\partial\ominus^2}\left[\left(e^t\cosh(\ominus)\right)\varphi_2\right]\right\}\right]\\
=&\sinh(\ominus)\frac{t^3\omega}{\Gamma(3\omega+1)},\\
&\vdots
\end{aligned}
\tag{4.20}
$$

Proceeding in the same way, the exact solution of (4.13), using (3.16) is given by

$$
\begin{aligned}
\varphi(\ominus,t) &= \sinh(\ominus)\left[1+\frac{t^\omega}{\Gamma(\omega+1)}+\frac{t^{2\omega}}{\Gamma(2\omega+1)}+\frac{t^{3\omega}}{\Gamma(3\omega+1)}+\cdots\right]\\
&=\sinh(\ominus)E_\omega(t^\omega).
\end{aligned}
\tag{4.21}
$$

where $E_\omega(t^\omega)$ is Mittag-Leffler function [35].

Example 4.3. Consider non-linear time-fractional Fornberg-Whitham equation [34] as follows:

$$\varphi_t^\omega-\varphi_{\ominus\ominus t}+\varphi_\ominus=\varphi\varphi_{\ominus\ominus\ominus}-\varphi\varphi_\ominus+3\varphi_\ominus\varphi_{\ominus\ominus} \tag{4.22}$$

with initial condition

$$\varphi(\ominus,0)=\frac{4}{3}e^{\frac{\ominus}{2}} \tag{4.23}$$

when $\omega=1$, the exact solution of (4.22) given by Zhang et al. [7],

$$\varphi(\ominus,t)=\frac{4}{3}e^{\left(\frac{\ominus}{2}-\frac{2t}{3}\right)}. \tag{4.24}$$

Operating KT on both sides of equation (4.22), we get

$$K\{\varphi_t^\omega\}-K\{\varphi_{\ominus\ominus t}\}+K\{\varphi_\ominus\}=K\{\varphi\varphi_{\ominus\ominus\ominus}\}-K\{\varphi\varphi_\ominus\}+3K\{\varphi_\ominus\varphi_{\ominus\ominus}\}. \tag{4.25}$$

Using differentiation properties (2.3), we have

$$
\begin{aligned}
&\left[\frac{1}{s^\omega}K\{\varphi(\ominus,t)\}-\frac{1}{s^{\omega-1}}\varphi(\ominus,0)\right]-K\{\varphi_{\ominus\ominus t}\}+K\{\varphi_\ominus\}\\
&=K\{\varphi\varphi_{\ominus\ominus\ominus}\}-K\{\varphi\varphi_\ominus\}+3K\{\varphi_\ominus\varphi_{\ominus\ominus}\}.
\end{aligned}
\tag{4.26}
$$

Applying the initial conditions (4.23),

$$K\{\varphi(\ominus,t)\}=\frac{4s}{3}e^{\frac{\ominus}{2}}+s^\omega K\left\{\varphi_{\ominus\ominus t}-\varphi_\ominus+\varphi\varphi_{\ominus\ominus\ominus}-\varphi\varphi_\ominus+3\varphi_\ominus\varphi_{\ominus\ominus}\right\}, \tag{4.27}$$

$$\varphi(\ominus,t)=\frac{4}{3}e^{\frac{\ominus}{2}}+K^{-1}\left[s^\omega K\left\{\varphi_{\ominus\ominus t}-\varphi_\ominus+\varphi\varphi_{\ominus\ominus\ominus}-\varphi\varphi_\ominus+3\varphi_\ominus\varphi_{\ominus\ominus}\right\}\right]. \tag{4.28}$$

Executing the HPM,

$$\sum_{i=0}^{\infty} p^i \varphi_i(\ominus, t) = \frac{4}{3} e^{\frac{\ominus}{2}} + pK^{-1}\left[s^{\omega} K\left\{R\left(\sum_{i=0}^{\infty} p^i \varphi_i\right) + \sum_{i=0}^{\infty} p^i H_i(\varphi)\right\}\right], \tag{4.29}$$

in which non-linear part of equation (4.21) has been decomposed using He's Polynomial $H_i(\varphi)$ as follows

$$\begin{aligned}
H_0(\varphi) &= \varphi_0\varphi_{0\ominus\ominus\ominus} - \varphi_0\varphi_{0\ominus} + 3\varphi_{0\ominus}\varphi_{0\ominus\ominus},\\
H_1(\varphi) &= \varphi_1\varphi_{0\ominus\ominus\ominus} + \varphi_0\varphi_{1\ominus\ominus\ominus} - \varphi_1\varphi_{0\ominus} - \varphi_0\varphi_{1\ominus} + 3\varphi_{1\ominus}\varphi_{0\ominus\ominus} + 3\varphi_{0\ominus}\varphi_{1\ominus\ominus},\\
H_2(\varphi) &= \varphi_2\varphi_{0\ominus\ominus\ominus} + 2\varphi_1\varphi_{1\ominus\ominus\ominus} + \varphi_0\varphi_{2\ominus\ominus\ominus} - \varphi_2\varphi_{0\ominus} - 2\varphi_1\varphi_{1\ominus} - \varphi_0\varphi_{2\ominus}\\
&\quad + 3\varphi_{2\ominus}\varphi_{0\ominus\ominus} + 6\varphi_{1\ominus}\varphi_{1\ominus\ominus} + 3\varphi_{0\ominus}\varphi_{2\ominus\ominus},\\
&\ \ \vdots
\end{aligned} \tag{4.30}$$

Comparing the power of p in (4.30),

$$\begin{aligned}
p^0 : \varphi_0(\ominus, t) &= \frac{4}{3} e^{\frac{\ominus}{2}},\\
p^1 : \varphi_1(\ominus, t) &= K^{-1}\left[s^{\omega} K\{R\varphi_0(\ominus, t)\} + s^{\omega} K\{H_0(\varphi)\}\right]\\
&= -\frac{2}{3} e^{\frac{\ominus}{2}} \frac{t^{\omega}}{\Gamma(\omega+1)},\\
p^2 : \varphi_2(\ominus, t) &= K^{-1}\left[s^{\omega} K\{R\varphi_1(\ominus, t)\} + s^{\omega} K\{H_1(\varphi)\}\right]\\
&= -\frac{1}{6} e^{\frac{\ominus}{2}} \left[\frac{t^{\omega}}{\Gamma(\omega+1)} - \frac{t^{2\omega}}{\Gamma(2\omega+1)}\right],\\
p^3 : \varphi_3(\ominus, t) &= K^{-1}\left[s^{\omega} K\{R\varphi_2(\ominus, t)\} + s^{\omega} K\{H_2(\varphi)\}\right]\\
&= -\frac{1}{24} e^{\frac{\ominus}{2}} \left[\frac{t^{\omega}}{\Gamma(\omega+1)} - 2\frac{t^{2\omega}}{\Gamma(2\omega+1)} + \frac{2}{3}\frac{t^{3\omega}}{\Gamma(3\omega+1)}\right],\\
p^4 : \varphi_4(\ominus, t) &= K^{-1}\left[s^{\omega} K\{R\varphi_3(\ominus, t)\} + s^{\omega} K\{H_3(\varphi)\}\right]\\
&= -\frac{1}{96} e^{\frac{\ominus}{2}} \left[\frac{t^{\omega}}{\Gamma(\omega+1)} - 3\frac{t^{2\omega}}{\Gamma(2\omega+1)} + 2\frac{t^{3\omega}}{\Gamma(3\omega+1)} - \frac{1}{3}\frac{t^{4\omega}}{\Gamma(4\omega+1)}\right],\\
&\ \ \vdots
\end{aligned} \tag{4.31}$$

Similarly, proceeding in the same way, we get approximate solution of (4.22) as follows,

$$\begin{aligned}
\varphi(\ominus, t) =& \frac{1}{3} e^{\frac{\ominus}{2}} \left[4 - \frac{85}{32}\frac{t^{\omega}}{\Gamma(\omega+1)} + \frac{27}{32}\frac{t^{2\omega}}{\Gamma(2\omega+1)}\right.\\
&\left. - \frac{7}{48}\frac{t^{3\omega}}{\Gamma(3\omega+1)} + \frac{1}{3}\frac{t^{4\omega}}{\Gamma(4\omega+1)} - \cdots\right].
\end{aligned} \tag{4.32}$$

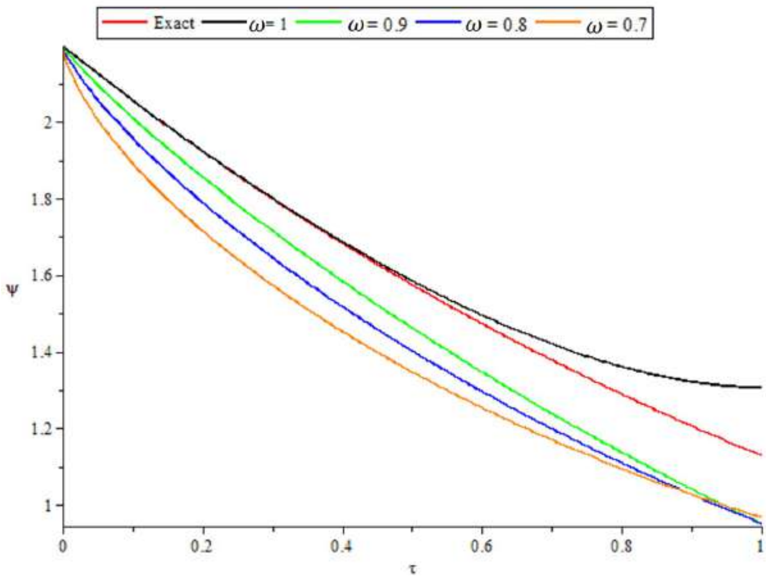
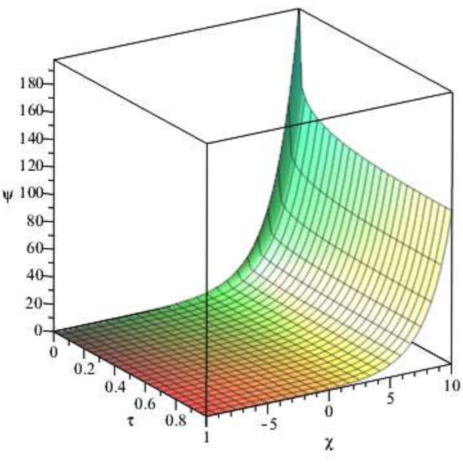
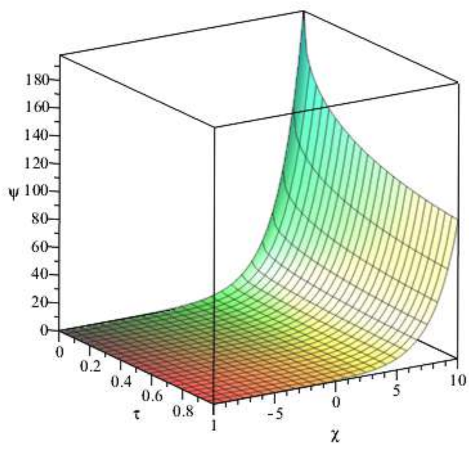


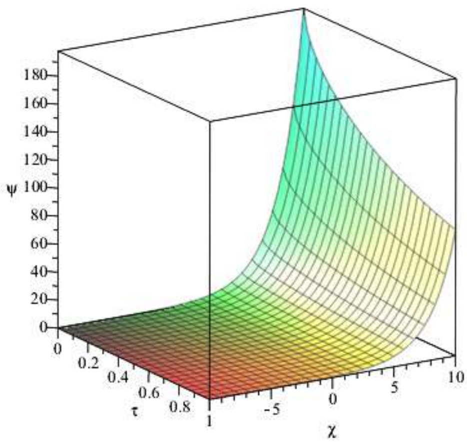
FIGURE 1. Comparison of Fornberg-Whitham equation at some specific values of ω, $\ominus = 1$ and its Exact result.



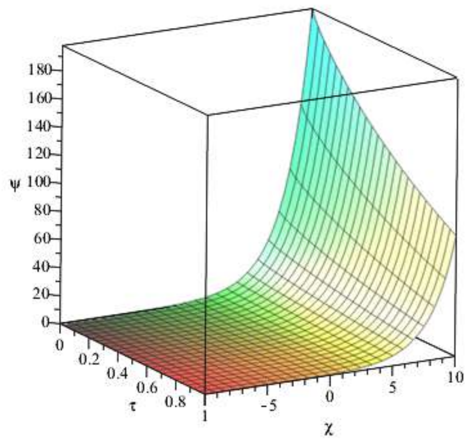
(A) $\omega = 0.2$



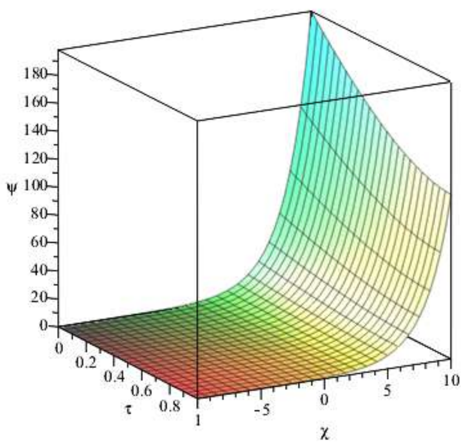
(B) $\omega = 0.4$



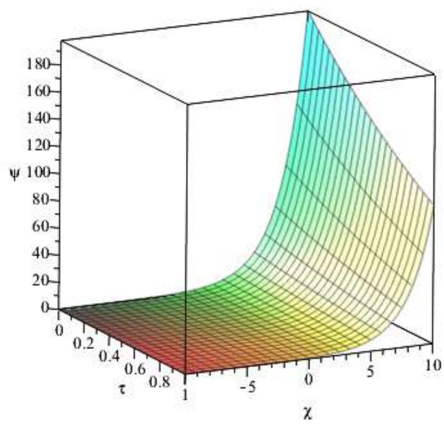
(C) $\omega = 0.6$



(D) $\omega = 0.8$



(E) $\omega = 1$



(F) Exact solution

FIGURE 1. The behavior of the outcome of F-W equation by HPKTM technique with certain order ω and its Exact Solution.

Table 3. The result of F-W equation at certain fixed value of $\ominus$ and t using Maple tool, suggested HPKTM technique, and RPSM [7] method.

$\ominus$	t	Exact	HPKTM	RPSM	|Exact-HPKTM|	|Exact-RPSM|
	0.1	0.00840452864	0.008406038080	0.008402202315	1.50944E-06	2.32632E-06
	0.2	0.00786249525	0.007865130666	0.007820475300	2.63541E-06	4.20200E-05
-10	0.3	0.00735541923	0.007361937038	0.007238748282	6.51781E-06	1.16671E-04
	0.4	0.00688104606	0.006898983932	0.006657021266	1.79379E-05	2.24025E-04
	0.5	0.00643726666	0.006480594855	0.006075294249	4.33282E-05	3.61972E-04
	0.1	0.10238811940	0.102406508144	0.102359778960	1.83887E-05	2.83404E-05
	0.2	0.09578480094	0.095816906847	0.095272893116	3.21059E-05	5.11908E-04
-5	0.3	0.08960735032	0.089686753511	0.088186007271	7.94032E-05	1.42134E-03
	0.4	0.08382830210	0.084046830091	0.081099121369	2.18528E-04	2.72918E-03
	0.5	0.07842196221	0.078949807683	0.074012235501	5.27845E-04	4.40973E-03
	0.1	2.05652035300	2.056889700499	2.055951119752	3.69347E-04	5.69233E-04
	0.2	1.92388915600	1.924534020346	1.913607212453	6.44864E-04	1.02819E-02
1	0.3	1.79981174400	1.801406599161	1.771263305154	1.59486E-03	2.85484E-02
	0.4	1.68373645700	1.688125709069	1.628919396701	4.38925E-03	5.48171E-02
	0.5	1.57514721700	1.585749277301	1.486575488908	1.06021E-02	8.85717E-02
	0.1	15.19574425000	15.198473386302	15.191538160505	2.72914E-03	4.20609E-03
	0.2	14.21572491000	14.220489840638	14.139751044134	4.76493E-03	7.59739E-02
5	0.3	13.29890994000	13.310694418187	13.087963927762	1.17845E-02	2.10946E-01
	0.4	12.44122314000	12.473655566364	12.036176802863	3.24324E-02	4.05046E-01
	0.5	11.63885116000	11.717190368813	10.984389682836	7.83392E-02	6.54461E-01

5. Results and Discussion

This paper presents a semi-analytical solution for the time-fractional Fokker-Planck (F-P) and Fokker-Whitham (F-W) equations utilizing the HPKTM method. Figure-1 illustrates the behavior of various fractional-order ω. The dynamic solution of the F-W equation for various fractional orders, specifically $\omega = 0.2,\ 0.4,\ 0.6,\ 0.8$, as well as $\omega = 1$ and the exact solution, is graphically presented in Figures 2a to 2f. The solution of the F-W equation, derived using HPKTM and the Residual Power Series Method alongside an exact solution, is presented in Table 3. The graph illustrates the hereditary and intelligibility of the system's dynamic behavior. The method can also address the F-W equation type with various initial condition problems. The result obtained in class of F-P equation as backward Kolmogorov equation is expressed in terms of the Mittag-Leffler function, indicating the convergence of the method.

6. Conclusion

The new semi-analytical method, HPKTM, is proposed in the present paper. We have applied it to the F-P equation and F-W equation in arbitrary order to analyze the concise behaviour of the system. We have showcased that the HPKTM can reduce the computational effort compared to the classic method while at the same time maintaining the high accuracy of the numerical results. We conclude that HPKTM represents a significant refinement of the existing method and has the potential for numerous possible applications. The proposed method demonstrates reliability and reduced the variability in computation, making it applicable across various domains in science and technology for addressing functional equations. The graphical representation of these problem types indicates that the HPKTM is an effective tool for solving fractional partial differential equations and, in particular circumstances, yields exact solutions as well.

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