APPROXIMATION METHOD FOR COMMON FIXED POINT OF A COUNTABLE FAMILY OF MULTI-VALUED QUASI-ϕ-NONEXPANSIVE MAPPINGS IN BANACH SPACES

Sani Saleh Musa[1]**, Adamu Yusuf Inuwa**[2,*]

[1] *Department of Mathematics, Jigawa state college of education and legal studies, Ringim, Nigeria*
E-mails: sanisale32@yahoo.com
[2] *Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University Kano. Kano, Nigeria*
E-mails: ibnyusufkoki@gmail.com

*Corresponding author.

Abstract In this paper, we study a new iterative algorithm to approximate a common fixed point of a countable family of multivalued quasi-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex real Banach space. We prove a strong convergence theorem, which improves and generalizes recently announced results in the literature. We also give some examples to illustrate our theorem, and we also give application to the zero-point problem of maximal monotone mappings.

1. INTRODUCTION

The theory of nonexpansive multi-valued mappings is more challenging to study than the corresponding theory of nonexpansive single-valued mappings. However, it has important applications. As a result of this, algorithms for multi-valued nonexpansive type mappings have attracted the interest of researchers and have become a flourishing area of research, especially for numerous mathematicians in the field of nonlinear operator theory and functional analysis. Hence, a lot of research interest is now devoted to this type of mappings due to their remarkable utility and wide applicability in applied health sciences, modern mathematics, and other research areas. The theory of multi-valued mappings has applications in control theory, convex optimization, game theory, market economy, differential equations, and other medical fields.

Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A space E is said to be strictly convex if

$$\forall x, y \in B, \ x \neq y, \ then, \ \left\|\frac{x+y}{2}\right\| < 1.$$

A real Banach space E is called uniformly convex if *for each* $\epsilon \in (0, 2]$, $\exists\ \delta > 0$

$$such\ that\ x \neq y\ \forall\ x, y \in B\ \|x - y\| \geq \epsilon,\ then,\ \left\|\frac{x+y}{2}\right\| < 1 - \delta.$$

A real Banach space E is called smooth [8] if for every $x \in E$ with $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that

$$\|x^*\| = 1\ and\ \langle x, x^* \rangle = \|x\|.$$

The space E is called uniformly smooth, if given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$ then

$$\|x + y\|^2 + \|x - y\|^2 < 2 + \epsilon\|y\|.$$

The normalized duality mapping [2] $J : E \to 2^{E^*}$ is defined as

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|f^*\|^2 : \|x\| = \|f^*\|, \ \forall x \in E\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between the element of E and E^*. The following properties of the normalized duality map will be needed in the sequel, (see for example [8]).
- If E is a reflexive, strictly convex and smooth real Banach space, then J is surjective, injective and single-valued.

Let C be a closed convex subset of a real Hilbert space H. The operator P_C is called a metric projection operator [2] if it assigns to each $x \in H$ its nearest point $y \in C$ such that,

$$\|x - y\| = \min\{\|x - z\| : z \in C\}.$$

Let E be a smooth, strictly convex and reflexive real Banach space, and C be a nonempty closed and convex subset of E. A map $\prod_C : E \to C$ [8] defined by $\prod_C x = u^0 \in C$ such that $\phi(u^0, x) = inf_{y \in C}\phi(y, x)$ is called the generalized projection, where $inf_{y \in C}\phi(y, x)$ is the set of all the minimizers of $\phi(\cdot, x)$.

Remark 1.1. In Hilbert spaces the generalized projection and the metric projection coincide.

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subset of C, respectively. The Hausdorff metric on $CB(C)$ [2] is defined by

$$d_H(A_1, A_2) = \max \left\{ \sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_2) \right\}, \text{ for } A_1, A_2 \in CB(C), \quad (1.2)$$

where

$$d(x, A_2) = \inf\{\|x - y\|, y \in A_2\}. \quad (1.3)$$

A point $p \in C$ is called a fixed point of the map $T : C \to N(C)$ if $p \in T(p)$, the set of fixed point of T is denoted by $F(T)$.

Remark 1.2. Yakov Alber[2] introduced a generalized projection map Π_C in a smooth Banach space E which is an analogue of the metric projection operator in Hilbert spaces. This map has been studied by many authors (see for example [4, 6, 15, 24, 25, 29, 32]).

A map $\phi : E \times E \to [0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2$$

is called Lyapunov functional [2], where $x, y \in E$ and j is the normalized duality mapping from E to E^* and E is a real Banach space and E^* it's dual space. Observed that in a real Hilbert space j is an identity map and

$$\phi(x, y) = \|x - y\|^2, \ \forall x, y \in H.$$

Thus the following important proposition holds for the Lyapunov functional:

Proposition 1.3. *(see for example [7]) Let E be a real Banach space. Then, for any $x, y, z \in E$ and $\alpha \in (0, 1)$ we have the following:*
(N_0)

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2 \quad (1.4)$$

(N_1)

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \le \alpha\phi(x, y) + (1 - \alpha)\phi(x, z). \quad (1.5)$$

Definition 1.4. A multi-valued mapping $T : C \to CB(C)$ is called
(i) nonexpansive if

$$d_H(T(x), T(y)) \le \|x - y\| \ for \ all \ x, y \in C.$$

(ii) quasi nonexpansive if $F(T) \ne \emptyset$ and

$$d_H(T(x), q) \le \|x - q\| \ for \ all \ x \in C, \ q \in F(T).$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let T be a mapping from C into itself. A point $p \in C$ is said to be an asymptotic fixed point of T, if $\exists \ \{u_n\} \in C$ such that $u^n \rightharpoonup p$ and $\|u^n - Tu^n\| \to 0$ as $n \to \infty$ (see for example Chang et, al. [5]). The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping T is said to be multi-valued relatively nonexpansive if $\hat{F}(T) = F(T)$ and

$$\phi(p, y) \le \phi(p, x) \ y \in T(x), \ p \in F(T). \quad (1.6)$$

A multi - valued mapping T is called relatively quasi nonexpansive(quasi-ϕ-nonexpansive) if

$$\phi(p,y) \leq \phi(p,x) \ y \in T(x), \ p \in F(T) \ and \ F(T) \neq \emptyset. \tag{1.7}$$

The class of multi-valued relatively quasi nonexpansive(quasi-ϕ-nonexpansive) mappings is more general than the class of multi-valued relatively nonexpansive mappings, which requires the strong condition $\hat{F}(T) = F(T) \neq \emptyset$.

Several authors have also studied fixed point problems of nonexpansive mappings in the context of both multi - valued and single valued mapping. Among thes authors, in 2003, Nakajo and Takahashi [22] studied the following CQ iterative algorithm for approximating a fixed point of a nonexpansive map in a real Hilbert space:

$$\begin{cases} u^0 \in C, \\ v^n = \alpha_n u^n + (1 - \alpha_n)Tu^n, \\ C_n = \{z \in C : \|v^n - z\| \leq \|u^n - z\|\}, \\ Q_n = \{z \in C : \langle u^n - z, u^n - u^0 \rangle \leq 0\}, \\ u^{n+1} = P_{C_n \bigcap Q_n} u^0, n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ are sequences in the interval [0,1], C is a nonempty, closed and convex subset of a real Hilbert space, T is a projection from C onto $C_n \cap Q_n$. They proved that the sequence generated by algorithm (1.8) converges strongly to a fixed point of T.
In 2005, Matsushita and Takahashi [20] extended the result of Nakajo and Takahashi [22] by studying the following iterative algorithm for approximating a fixed point of relatively nonexpansive map in a uniformly smooth and uniformly convex real Banach space:

$$\begin{cases} u^0 \in C, \\ v^n = J^{-1}[\alpha_n Ju^n + (1 - \alpha_n)JTu^n], \\ C_n = \{z \in C : \phi(z,v^n) \leq \phi(z,u^n)\}, \\ Q_n = \{z \in C : \langle u^n - z, Ju^n - Ju^0 \rangle \leq 0\}, \\ u^{n+1} = \prod_{C_n \bigcap Q_n} u^0, n \geq 1. \end{cases} \tag{1.9}$$

They proved that the sequence generated by (1.9) converges strongly to a fixed point of T.
In 2005, Sastry and Babu [26] established the convergence theorem of multi-valued maps using Ishikawa and Mann iterates. They also provided an example that illustrates how the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point $'p'$ and the initial guess.
In 2007, Panyanak [23] wrote a paper that extended one of the results of Sastry and Babu [26] to uniformly convex Banach spaces. He proved a convergence theorem using Mann iterates for a mapping defined on a noncompact domain.
Later in 2008, Song and Wang [28] demonstrated the strong convergence of Mann and Ishikawa iterates to a fixed point of a multi-valued nonexpansive mapping 'T' under certain appropriate conditions.
Additionally, in 2008, Dong et al. [10] studied the following inertial CQ iterative algorithm

for approximating a fixed point of a nonexpansive map in a real Hilbert space:

$$
\begin{cases}
u^0, u^1 \in H, \\
w^n = u^n + \alpha_n(u^n - u^{n-1}), \\
v^n = (1 - \beta_n)w^n + \beta_n T w^n, \\
C_n = \{z \in H : \|v^n - z\| \leq \|u^n - z\|\}, \\
Q_n = \{z \in C : \langle u^n - z, u^n - u^0 \rangle \leq 0\}, \\
u^{n+1} = P_{C_n \cap Q_n} u^0, n \geq 0.
\end{cases}
\tag{1.10}
$$

They proved that the iterative sequence generated by (1.10) converges in norm to $P_{F(T)}u^0$. Furthermore, Diop et. al., [9] constructed a new iterative algorithm and proved strong convergence theorems for common fixed point of a finite family of multivalued quasi-nonexpansive mappings in uniformly convex real Banach spaces.

Several authors have also studied fixed point problems of a multi-valued nonexpansive mappings in the setting of real Banach spaces (see for example [21],[13],[26],[23],[28],[1] and the references therein).

Chidume et. al., [7] studied the following inertial type algorithm for approximating a common fixed point for a countable family of a relatively nonexpansive mapping in a uniformly smooth and uniformly convex real Banach space

$$
\begin{cases}
C_0 = E, \\
w^n = u^n + \alpha_n(u^n - u^{n-1}), \\
v^n = J^{-1}[(1 - \beta)Jw^n + \beta J T w^n], \\
c_{n+1} = \{z \in c_n : \phi(z, v^n) \leq \phi(z, w^n)\}, \\
u^{n+1} = \prod_{C_{n+1}} u^0, n \geq 0.
\end{cases}
\tag{1.11}
$$

They proved that the sequence generated by (1.11) converges strongly to a point $p = \prod_{F(T)} u^0$

Remark 1.5. C_n and Q_n are halfspaces, that must be computed at each iterative stage in [10, 20, 22] and the host of other authors,), thus Chidume et. al, [7] reduces the computation by dispensing with Q_n and studied inertial-type algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps which is an extension of some recent existing results. Motivated by the work of Chidume et. al., [7] we propose a new iterative scheme in which both Q_n and C_n are dispensed with. Furthermore the condition $F(T) = \hat{F}(T)$ is dispensed with i.e we consider the class of quasi-ϕ-nonexpansive multi-valued maps which is more general than the class of relatively nonexpansive.

In addition, our theorem also improve the result of Diop et. al., [9], from the case of finite family of multi-valued quasi-ϕ-nonexpansive to Countable family of multi-valued quasi-ϕ-nonexpansive mappings.

We prove a strong convergence of our new iterative scheme to a common fixed point for a countable family of multi-valued quasi-ϕ-nonexpansive maps.

2. PRELIMINARIES

The following notions and results are very important in our subsequent discussion.

Definition 2.1. Let E be a reflexive, strictly convex and smooth real Banach space. The duality mapping J^* from E^* onto $E^{**} = E$ coincides with the inverse of the duality mapping J from E onto E^* that is, $J^* = J^{-1}$.
The following mapping will be needed in the sequel $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \ \forall (x, x^*) \in E \times E^*. \tag{2.1}$$

Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \forall (x, x^*) \in E \times E^*$.

We will also use the following lemmas.

Lemma 2.2. [15] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.3. [2, 16] *Let E be a reflexive, strictly convex and smooth Banach space. Then for any $x \in E$, and $x^*, y^* \in E^*$, we have,*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*). \tag{2.2}$$

Lemma 2.4. [15] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following hold:*

  (a) $\phi(x, \Pi_c y) + \phi(\Pi_c y, y) \leq \phi(x, y)$ *for all $x \in C$ and $y \in E$;*
  (b) *If $x \in E$ and $z \in C$, then $z = \Pi_c x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$;*
  (c) *for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.5. [12] *Let E be a smooth and strictly convex real Banach space, Let C be a nonempty closed and convex subset of E. Suppose $T : C \to N(C)$ is a quasi ϕ nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.*

Lemma 2.6. [19] *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that*

$$a_{n_j} < a_{n_{j+1}} \ \forall j \geq 0.$$

Also consider the sequence of integers $\{m_k\}$ defined by

$$m_k = \max\{i \leq k : a_i < a_{i+1}\}.$$

Then $\{m_k\}$ is a nondecreasing sequence satisfying

$$\lim_{k \to \infty} m_k = \infty,$$

there exists some $n_0 \in \mathbb{N}$, for some , the following two estimates hold;

$$a_{m_k} \leq a_{m_{k+1}} \ and \ a_k \leq a_{m_{k+1}}, \ \forall \ k \geq n_0.$$

Lemma 2.7. [31] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \tag{2.3}$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

Lemma 2.8. [17] *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $T_i : C \to E$ $i = 1, 2, 3...$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\alpha_n \in (0, 1), \beta_i \in (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1 - \beta_i)JT_i x))$ for each $x \in C$. Then T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.9. [30] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, n \geq 0, \tag{2.4}$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

> *i. $\{\alpha_n\} \subset [0, 1], \sum_{n=0}^{\infty} \alpha_n = \infty,$*
>
> *ii. $\limsup_{n \to \infty} \delta_n \leq 0.$*

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.10. [18] *Let E be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{\lambda_i\}_{i=1}^{\infty} \subset (0, 1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in E such that*

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2, \tag{2.5}$$

then $x_1 = x_2 = x_3 = \cdots$.

3. Main Result

In this section we present the main results of this paper.

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{\lambda_i\}_{i=1}^{\infty} \subset (0, 1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in E such that*

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^{\infty} \lambda_i Jx_i\right)\right) = \sum_{i=1}^{\infty} \lambda_i \phi(z, x_i), \tag{3.1}$$

then $x_1 = x_2 = x_3 = \cdots$.

Proof. We follow the proof line of Kohsaka and Takahashi[18] from the case of finite family to Countable family.

Let $z \in E$, we have from our assumption that

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^{\infty} \lambda_i Jx_i\right)\right) = \sum_{i=1}^{\infty} \lambda_i \phi(z, x_i),$$

$$\|z\|^2 - 2\left\langle z, \sum_{i=1}^{\infty} \lambda_i Jx_i \right\rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \left(\|z\|^2 - 2\langle z, x_i \rangle + \|x_i\|^2\right)$$

$$\|z\|^2 - 2\sum_{i=1}^{\infty} \lambda_i \langle z, Jx_i \rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \|z\|^2 - 2\sum_{i=1}^{\infty} \lambda_i \langle z, x_i \rangle + \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2$$

$$\sum_{i=1}^{\infty} \lambda_i \|z\|^2 - 2\sum_{i=1}^{\infty} \lambda_i \langle z, Jx_i \rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \|z\|^2 - 2\sum_{i=1}^{\infty} \lambda_i \langle z, x_i \rangle + \sum_{i=1}^{\infty} \lambda_i \|Jx_i\|^2.$$

This implies

$$\left\| \sum_{i=1}^{\infty} \lambda_i J x_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \| J x_i \|^2.$$

Since E is smooth and reflexive, E^* is strictly convex. Thus, Lemma(2.10) implies that $Jx_1 = Jx_2 = Jx_3 = \cdots$. By the strict convexity of E, J is one-to-one. Hence we have the desired result i.e $x_1 = x_2 = x_3 = \cdots$. ∎

We also need the following results for completion of our theorem

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space. Let $T_i : C \to CB(E)$ $i = 1, 2, 3, \ldots$ be a countable family of quasi-ϕ-nonexpansive multi-valued maps with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\alpha_n \in (0, 1), \beta_i \in (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=1}^{\infty} \alpha_i(\beta_i J x + (1 - \beta_i) J y_i))$ with $T_i(p) = \{p\}$ for each $x \in C, y_i \in T_i x$. Then T is quasi nonexpansive mapping and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Proof. We first show that T is well defined. Then $\forall x \in C$, $y_i \in T_i x$ and $\forall i \geq 0$, the mapping T can be rewritten as:

$$T(x) = J^{-1} \left(\sum_{i=1}^{\infty} \alpha_i(\beta_i J x + (1 - \beta_i) J y_i) \right)$$

$$= J^{-1} \left(\left(\sum_{i=1}^{\infty} \alpha_i \beta_i \right) J x + \left(\sum_{i=1}^{\infty} \alpha_i (1 - \beta_i) J y_i \right) \right)$$

$$= J^{-1} \left(\gamma_0 J y_0 + \sum_{i=1}^{\infty} \gamma_i J y_i \right)$$

$$= J^{-1} \left(\sum_{i=0}^{\infty} \gamma_i J y_i \right),$$

where $y_0 = x$ and T_0 is the identity mapping, $\gamma_0 = \sum_{i=1}^{\infty} \alpha_i \beta_i > 0$, $\gamma_i = \alpha_i(1 - \beta_i) > 0$ for all $i \geq 1$ and $\sum_{i=1}^{\infty} \gamma_i = 1$. Now for any fixed $p \in \bigcap_{i=1}^{\infty} F(T_i)$, $y_i \in T_i x$, T_i quasi nonexpansive multi-valued map and $x \in C$, we have

$$(\|p\| - \|y_i\|)^2 \leq \phi(p, y_i) \leq \phi(p, x) \leq (\|p\| + \|x\|)^2,$$

which implies,

$$(\|p\| - \|y_i\|)^2 \leq (\|p\| + \|x\|)^2$$
$$\|p\|^2 - 2\|p\|\|y_i\| + \|y_i\|^2 \leq \|p\|^2 + 2\|p\|\|x\| + \|x\|^2$$
$$\|y_i\|^2 - \|x\|^2 \leq 2\|p\|(\|y_i\| + \|x\|)$$
$$(\|y_i\| - \|x\|)(\|y_i\| + \|x\|) \leq 2\|p\|(\|y_i\| + \|x\|).$$

In particular, $\|y_i\| \leq 2\|p\| + \|x\|$ *for each $i \geq 0$ and $x \in C$.*
Thus,

$$
\left\| J^{-1} \sum_{i=0}^{\infty} \gamma_i J y_i \right\| \leq \sum_{i=0}^{\infty} \gamma_i \|y_i\|
$$

$$
\leq \sum_{i=0}^{\infty} \gamma_i (2\|p\| + \|x\|) < \infty.
$$

Hence T is well defined.

Next we show that T is quasi nonexpansive mapping.

Let $p \in F(T)$, by Lemma 2.7 and from the fact that $\{T_i\}$ is a countable family of quasi nonexpansive multi-valued mapping, that $y_i \in T_i x$ for each $i \geq 1$, $x \in C$, we have

$$
\phi(p, Tx) = \phi\left(p, J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i(1-\beta_i) Jy_i \right) \right)
$$

$$
= \|p\|^2 - 2\left\langle p, \sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i(1-\beta_i) Jy_i \right\rangle + \left\| \sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i(1-\beta_i) Jy_i \right\|^2
$$

$$
\leq \|p\|^2 - 2\sum_{i=1}^{\infty} \alpha_i \langle p, \beta_i Jx + (1-\beta_i) Jy_i \rangle + \sum_{i=1}^{\infty} \alpha_i \|\beta_i Jx + (1-\beta_i) Jy_i\|^2
$$

$$
\leq \sum_{i=1}^{\infty} \alpha_i \left(\|p\|^2 - 2\langle p, \beta_i Jx + (1-\beta_i) Jy_i \rangle + \beta_i \|x\|^2 + (1-\beta_i)\|y_i\|^2 \right)
$$

$$
- \sum_{i=1}^{\infty} \alpha_i \beta_i (1-\beta_i) g^*(\|Jx - Jy_i\|)
$$

$$
\leq \sum_{i=1}^{\infty} \alpha_i \left(\beta_i \phi(p, x) + (1-\beta_i)\phi(p, y_i) \right)
$$

$$
\leq \sum_{i=1}^{\infty} \alpha_i (\beta_i \phi(p, x) + (1-\beta_i)\phi(p, x))
$$

$$
= \sum_{i=1}^{\infty} \alpha_i \phi(p, x)
$$

$$
= \phi(p, x).
$$

Hence T is quasi-ϕ nonexpansive mapping.

Finally we show that $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. It suffices to show the inclusion $F(T) \subset \bigcap_{i=1}^{\infty} F(T_i)$.

Let $z \in F(T)$ be given and fix $p \in \bigcap_{i=1}^{\infty} F(T_i)$, we have,

$$\phi(p, z) = \phi(p, Tz)$$

$$= \phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i J y_i\right)\right)$$

$$= \|p\|^2 - 2\left\langle p, \sum_{i=0}^{\infty} \gamma_i J y_i\right\rangle + \left\|\sum_{i=0}^{\infty} \gamma_i J y_i\right\|^2$$

$$\leq \|p\|^2 - 2\sum_{i=0}^{\infty} \gamma_i \langle p, J y_i\rangle + \sum_{i=0}^{\infty} \gamma_i \|J y_i\|^2$$

$$= \sum_{i=0}^{\infty} \gamma_i \left(\|p\|^2 - 2\langle p, J y_i\rangle + \|J y_i\|^2\right)$$

$$= \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i)$$

$$\leq \phi(p, z), \ \forall i \geq 0.$$

Thus, $\phi(p, z) = \phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i J y_i\right)\right) \leq \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i) \leq \phi(p, z)$. Therefore, we have, $\phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i J y_i\right)\right) = \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i)$. By applying Lemma 3.1 we obtain that, $y_0 = y_1 = y_2 = y_3 = \cdots$.
But also $y_0 = z$ and T_0 is the identity mapping, then $y_0 = z$. Hence $z = y_0 = y_1 = y_2 = y_3 = \cdots$. Clearly $\bigcap_{i=1}^{\infty} F(T_i) \subset F(T)$ follows, Hence $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$. ∎

Theorem 3.3. *Let C be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space E. Let $T_i : E \to CB(E)$ be a countable family of quasi-ϕ-nonexpansive multi-valued maps such that $\bigcap_{i=0}^{\infty} F(T_i) \neq \emptyset$. Suppose $\eta_i, \lambda_i \in (0, 1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=0}^{\infty} \lambda_i(\eta_i Jx + (1 - \eta_i)J y_i))$ for each $x \in C$, $y_i \in T_i x$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm : $u^0, u^1 \in C$ and*

$$\begin{cases} w^n = J^{-1}\left[\alpha_n J u^0 + (1 - \alpha_n)J u^n\right], \\ v^n = J^{-1}\left[(1 - \beta)J w^n + \beta J T w^n\right], \\ u^{n+1} = \prod_C v^n, n \geq 1, \end{cases} \tag{3.2}$$

where β is $\in (0, 1)$ and $\{\alpha_n\}_n$ is a sequence in (0,1) satisfying

$(A_1) \ \lim_{n \to \infty} \alpha_n = 0 \ ;$
$(A_2) \ \sum_{n=1}^{\infty} \alpha_n = \infty.$
   Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

Proof. We partition the proof into three steps.

Step 1 : From Lemma 3.2 $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$ and by Lemma 2.1, $F(T_i)$ is closed and convex for each $i \in \mathbb{N} \cup \{0\}$. Thus $\bigcap_{i=0}^{\infty} F(T_i)$ is closed and convex so $\prod_{\bigcap_{i=0}^{\infty} F(T_i)}$ is well defined generalized projection.

Step 2 : We show that $\{u^n\}_{n=1}^{\infty}$, $\{v^n\}_{n=1}^{\infty}$ and $\{w^n\}_{n=1}^{\infty}$ are bounded. Let $u^* = \prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$, using Lemma 2.4(a) and (1.5)

$$\phi(u^*, u^{n+1}) = \phi(u^*, \Pi_C v^n)$$
$$\leq \phi(u^*, v^n)$$
$$= \phi\left(u^*, J^{-1}\left[(1-\beta)Jw^n + \beta JTw^n\right]\right)$$
$$\leq (1-\beta)\phi(u^*, w^n) + \beta\phi(u^*, Tw^n)$$
$$\leq (1-\beta)\phi(u^*, w^n) + \beta\phi(u^*, w^n)$$
$$= \phi(u^*, w^n).$$

Therefore,

$$\phi(u^*, u^{n+1}) \leq \phi(u^*, v^n) \leq \phi(u^*, w^n), \tag{3.3}$$

but,

$$\phi(u^*, w^n) = \phi(u^*, J^{-1}[(1-\alpha_n)Ju^n + \alpha_n Ju^0)])$$
$$\leq (1-\alpha_n)\phi(u^*, u^n) + \alpha_n \phi(u^*, u^0)$$
$$\leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}.$$

This implies,

$$\phi(u^*, w^n) \leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}. \tag{3.4}$$

From (3.3) and (3.4) we get

$$\phi(u^*, u^{n+1}) \leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}. \tag{3.5}$$

Hence, using induction we get

$$\phi(u^*, u^n) \leq \max\{\phi(u^*, u^1), \phi(u^*, u^0)\}, \ \forall n \geq 0.$$

This implies that $\{\phi(u^*, u^n)\}_{n=0}^{\infty}$ is bounded, therefore by (1.4) $\{u^n\}_{n=0}^{\infty}$ is bounded, also by (3.4) and (3.3) $\{\phi(u^*, v^n)\}_{n=0}^{\infty}$ and $\{\phi(u^*, w^n)\}_{n=0}^{\infty}$ are bounded. Thus $\{v^n\}_{n=0}^{\infty}$ and $\{w^n\}_{n=0}^{\infty}$ are also bounded by (1.4)

Step 3 : Finally we show $\{u^n\}_{n=1}^{\infty}$ converges to u^*.

From (3.3) we have,

$$\phi(u^*, u^{n+1}) \leq \phi(u^*, w^n) = V(u^*, Jw^n).$$

Using Lemma 2.3, that is taking $y^* = \alpha_n Ju^* - \alpha_n Ju^0$, $x = u^*$ and $x^* = Jw^n$, we have,

$$V(u^*, Jw^n) \leq V(u^*, Jw^n - \alpha_n(Ju^0 - Ju^*)) - 2\langle w^n - u^*, -\alpha_n(Ju^0 - Ju^*)\rangle$$
$$= V(u^*, (1-\alpha_n)Ju^n + \alpha_n Ju^0 - \alpha_n Ju^0 + \alpha_n Ju^*)$$
$$+ 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle$$
$$= V(u^*, (1-\alpha_n)Ju^n + \alpha_n Ju^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle.$$

By (2.1) we have,

$$V(u^*, (1-\alpha_n)Ju^n + \alpha_n Ju^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle$$
$$= \phi(u^*, J^{-1}(1-\alpha_n)Ju^n + \alpha_n Ju^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle.$$

Also applying (1.5) we have,

$$\phi(u^*, J^{-1}(1-\alpha_n)Ju^n + \alpha_n Ju^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle$$
$$\leq (1-\alpha_n)\phi(u^*, u^n) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle,$$

which implies,

$$\phi(u^*, u^{n+1}) \leq (1-\alpha_n)\phi(u^*, u^n) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^*\rangle \ \forall \ n \geq 0. \quad (3.6)$$

We next divide the proof into two cases

Case 1 :

Suppose that there exists n_0 such that $\{\phi(u^*, u^n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation $\{\phi(u^*, u^n)\}_n$ is then convergent. Then,

$$\lim_{n \to \infty}(\phi(u^*, u^n) - \phi(u^*, u^{n+1})) = 0. \quad (3.7)$$

Also since $\{w^n\}_{n=1}^{\infty}$ is bounded, we choose $\{w^{n_k}\}_{k=1}^{\infty}$ subsequence of $\{w^n\}_{n=1}^{\infty}$ such that, $w^{n_k} \rightharpoonup z$ and

$$\limsup_{n \to \infty}\langle w^n - u^*, Ju^0 - Ju^*\rangle = \lim_{k \to \infty}\langle w^{n_k} - u^*, Ju^0 - Ju^*\rangle \quad = \langle z - u^*, Ju^0 - Ju^*\rangle,$$

by Lemma 2.4, we have

$$\limsup_{n \to \infty}\langle w^n - u^*, Ju^0 - Ju^*\rangle = \langle z - u^*, Ju^0 - Ju^*\rangle \leq 0.$$

From (3.6) and Lemma 2.9 we have

$$\lim_{n \to \infty}\phi(u^*, u^n) = 0,$$

which implies $u^n \to u^*$, $as \ n \to \infty$, where $u^* = \prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

Case 2 : Suppose $\{\phi(u^*, u^n)\}_{n=1}^{\infty}$ is not nonincreasing, then that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(u^*, u^{n_i}) \leq \phi(u^*, u^{n_{i+1}}) \ for \ all \ i \geq 0.$$

By Lemma **??** there exists a nondecreasing sequence $\{m_k\}_{k=0}^{\infty}$ of nonnegative integers such that $m_k \to \infty$, $as \ k \to \infty$,

$$\phi(u^*, u^{m_k}) \leq \phi(u^*, u^{m_{k+1}}) \quad (3.8)$$

and

$$\phi(u^*, u^k) \leq \phi(u^*, u^{m_{k+1}}) \ for \ all \ k \geq 0. \quad (3.9)$$

Following the same argument as in **case 1**, we obtain

$$\limsup_{k \to \infty}\langle w^{m_k} - u^*, Ju^0 - Ju^*\rangle \leq 0. \quad (3.10)$$

From (3.6) we have

$$\phi(u^*, u^{m_k+1}) \leq (1-\alpha_{m_k})\phi(u^*, u^{m_k}) + 2\alpha_{m_k}\langle w^{m_k} - u^*, Ju^0 - Ju^*\rangle. \quad (3.11)$$

Rearranging we have,

$$\alpha_{m_k}\phi(u^*, u^{m_k}) \leq \phi(u^*, u^{m_k}) - \phi(u^*, u^{m_{k+1}}) + 2\alpha_{m_k}\langle w^{m_k} - u^*, Ju^0 - Ju^*\rangle,$$

employing (3.8) we have,

$$\alpha_{m_k}\phi(u^*, u^{m_k}) \leq 2\alpha_{m_k}\langle w^{m_k} - u^*, Ju^0 - Ju^*\rangle,$$

Since $\alpha_{m_k} \geq 0$; we get

$$0 \leq \phi(u^*, u^{m_k}) \leq 2\langle w^{m_k} - u^*, Ju^0 - Ju^* \rangle,$$

it follows from (3.10) that

$$\lim_{k \to \infty} \phi(u^*, u^{m_k}) = 0.$$

This together with (3.11) gives

$$\lim_{k \to \infty} \phi(u^*, u^{m_{k+1}}) = 0. \tag{3.12}$$

But by (3.9)

$$\phi(u^*, u^k) \leq \phi(u^*, u^{m_{k+1}}) \ for \ all \ k \geq 0.$$

By (3.12) we obtain that $\phi(u^*, u^k) \to 0$ as $k \to \infty$. Then from **case 1** and **case 2** we conclude that $\lim_{n \to \infty} \phi(u^*, u^n) = 0$, which implies $u^n \to \prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$ ∎

Corollary 3.4. *Let C be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space E. Let $T_i : C \to E$ be a countable family of quasi-ϕ-nonexpansive maps such that $\bigcap_{i=0}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\eta_i\}, \{\lambda_i\} \in (0,1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=0}^{\infty} \lambda_i(\eta_i Jx + (1 - \eta_i)JT_i x))$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm : $u^0, u^1 \in C$ and*

$$\begin{cases} w^n = J^{-1}\left[\alpha_n Ju^0 + (1 - \alpha_n)Ju^n\right], \\ v^n = J^{-1}\left[(1 - \beta)Jw^n + \beta JTw^n\right], \\ u^{n+1} = \prod_C v^n, n \geq 1, \end{cases} \tag{3.13}$$

where $\beta \in (0,1)$ and $\{\alpha_n\}_n$ is a sequence in (0,1) satisfying

$(A_1) \ \lim_{n \to \infty} \alpha_n = 0;$
$(A_2) \ \sum_{n=1}^{\infty} \alpha_n = \infty.$
  Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of $L_p(\text{or } l_p)$, $1 < p < \infty$. Let $T_i : C \to CB(E)$ be a countable family of quasi-ϕ-nonexpansive multi-valued maps such that $\bigcap_{i=0}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\eta_i\}, \{\lambda_i\} \in (0,1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=0}^{\infty} \lambda_i(\eta_i Jx + (1 - \eta_i)Jy_i))$ for each $x \in C$, $y_i \in T_i x$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm : $u^0, u^1 \in C$ and*

$$\begin{cases} w^n = J^{-1}\left[\alpha_n Ju^0 + (1 - \alpha_n)Ju^n\right], \\ v^n = J^{-1}\left[(1 - \beta)Jw^n + \beta JTw^n\right], \\ u^{n+1} = \prod_C v^n, n \geq 1. \end{cases} \tag{3.14}$$

where $\beta \in (0,1)$ and $\{\alpha_n\}_n$ is a sequence in (0,1) satisfying

$(A_1) \ \lim_{n \to \infty} \alpha_n = 0 \ ;$
$(A_2) \ \sum_{n=1}^{\infty} \alpha_n = \infty.$
  Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\prod_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of real Hilbert space H. Let $T_i : C \to CB(H)$ be a countable family of quasi nonexpansive multi-valued map such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\eta_i, \lambda_i \in (0,1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = (\sum_{i=0}^{\infty} \lambda_i(\eta_i x + (1 - \eta_i)y_i))$ for each $x \in C$, $y_i \in T_i x$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm : $u^0, u^1 \in C$ and*

$$\begin{cases} w^n = \left[\alpha_n u^0 + (1 - \alpha_n)u^n\right], \\ v^n = \left[(1 - \beta)w^n + \beta T w^n\right], \\ u^{n+1} = P_C v^n, n \geq 1, \end{cases} \tag{3.15}$$

where $\beta \in (0,1)$ and $\{\alpha_n\}_n$ is a sequence in (0,1) satisfying

(A_1) $\lim\limits_{n \to \infty} \alpha_n = 0$;
(A_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.
    Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $P_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

4. Application to Zero Point Problem of Maximal Monotone Mappings

Let E be a smooth, strictly convex and reflexive Banach space.

Definition 4.1. [2] An operator $A : E \to 2^{E^*}$ is said to be monotone, if whenever $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$, then

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

We denote the zero point set $\{x \in E : 0 \in Ax\}$ of A by $A^{-1}(0)$. A monotone operator A is said to be maximal if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Remark 4.2. For each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rA(x_r)$. We define the resolvent of A by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$, $\forall r > 0$. We know that J_r is a single valued quasi nonexpansive mapping and $A^{-1}(0) = F(J_r)$, $\forall r > 0$ (see for example [2], where $F(J_r)$ is the set of fixed points of J_r.

Theorem 4.3. *Let E, $\{\alpha_n\}_n$ as in theorem 3.3. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$, $\forall r > 0$ such that $A^{-1}(0) \neq \emptyset$. Let $\{u^n\}$ be a sequence generated by $u^0, u^1 \in E$ and*

$$\begin{cases} w^n = \left[\alpha_n u^0 + (1 - \alpha_n)u^n\right], \\ u^{n+1} = J^{-1}[(1 - \beta)Jw^n + \beta JJ_r w^n]. \end{cases}$$

Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\Pi_{A^{-1}(0)} u^0$.

Proof. In theorem (3.3), taking $C = E$, $T = J_r$, $r > 0$, then $T : E \to E$ is a single-valued quasi nonexpansive mapping and $A^{-1}0 = F(T) = \bigcap_{i=0}^{\infty} F(T_i) = F(J_r), \forall r > 0$ is a nonempty closed convex subset of E. Therefore all the conditions in theorem (3.3) are satisfied. The conclusion follows immediately. ∎

4.1. EXAMPLE

Example 4.4. Consider $E = \mathbb{R}$ with the standard norm $\| \cdot \| = | \cdot |$ and $C = [-1, 1]$. For $m \in \mathbb{N}$, we define a mapping $\{T_m\}_m$ on $\mathbb{R}$ by

$$T_m(x) = \begin{cases} 0, & x \le \frac{1}{m^2} \\ \frac{1}{m^2}, & x > \frac{1}{m^2} \end{cases} \tag{4.1}$$

for all $x \in \mathbb{R}$. Then $F(T_m) = \bigcap_{m=1}^{3} F(T_m) = \{0\}$ and

$$|T_m x - 0| \le |x - 0| \ \forall x \in \mathbb{R}.$$

So $\{T_m\}_m$ is a sequence of quasi nonexpansive. Finally we show that each $\{T_m\}$ is not relatively nonexpansive. Indeed we take $x_n = \frac{1}{m^2} + \frac{1}{n}$ then

$$x_n \to \frac{1}{m^2} \ , x_n - T_m x_n = \frac{1}{n} \to 0 \ as \ n \to \infty.$$

This implies $\frac{1}{m^2} \in F(\hat{T}_m)$ and $\frac{1}{m^2} \notin F(T_m)$. Hence T_m is not relatively nonexpansive.

Also in theorem (3.3), Fix $u^0 = 0.5$ and choose $\alpha_n = \frac{1}{2n+1}, \lambda_n = \frac{1}{3}$ and $\eta_n = \frac{1}{2n^2+1}$ clearly $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\sum_{n=1}^{3} \lambda_n = 1$ and $\{\eta_n\} \in (0, 1)$. Thus all conditions in our proposed Algorithm are satisfied, thus algorithm (3.2) will reduces to

$$\begin{cases} w^n = \frac{0.5 + 2nu^n}{2n+1} \\ v^n = (\beta w^n + (1 - \beta)Tw^n) \ , \ T(w^n) = \sum_{m=1}^{3} \frac{1}{3} \left(\frac{1}{2m^2+1} w^n + \frac{2m^2}{2m^2+1} \cdot T_m(w^n) \right) \\ u^{n+1} = \begin{cases} -1, v^n < -1 \\ v^n, v^n \in [-1, 1] \\ 1, v^n > 1. \end{cases} \end{cases}$$
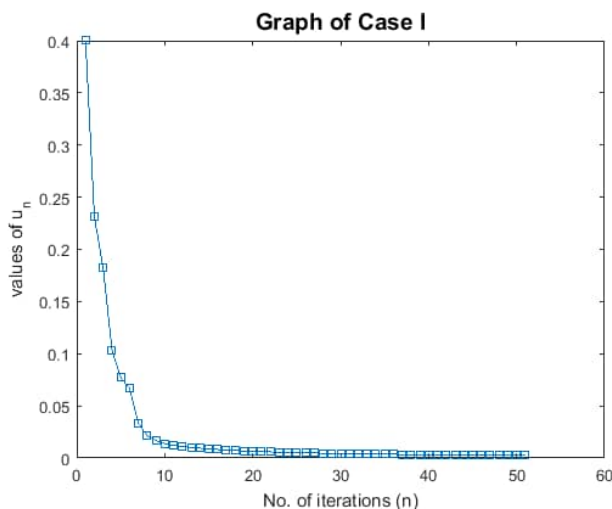
Case I. Take $u^1 = 0.4$ and $\beta = 0.8$

FIGURE 1. The graph of sequence $\{u_n\}$ generated by algorithm (3.2) versus number of iterations(Case I)

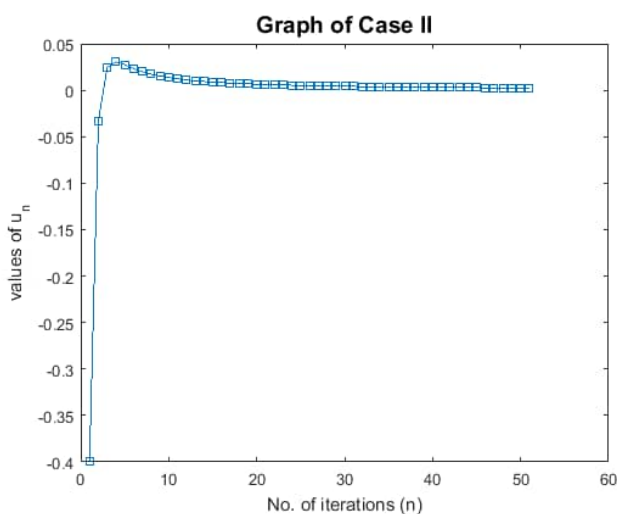**Case II.** Take $u^1 = -0.4$ and $\beta = 0.8$

FIGURE 2. The graph of sequence $\{u_n\}$ generated by algorithm (3.2) versus number of iterations(Case II)

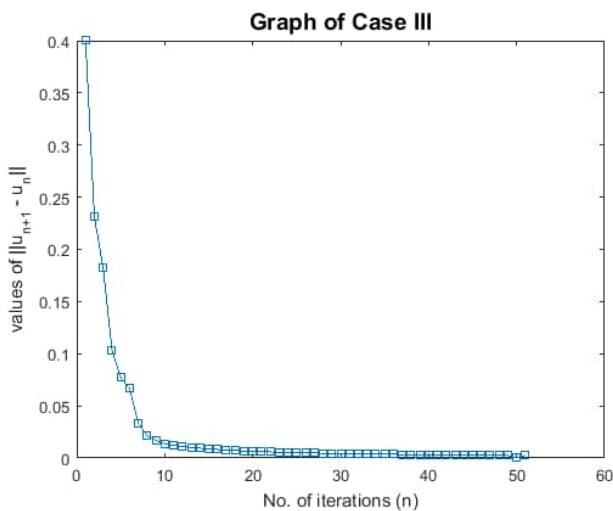**Case III.** Take $u^1 = 0.8$ and $\beta = 0.8$

FIGURE 3. The graph of $\|u_{n+1} - u_n\|$. versus number of iterations(Case III) where $\{u_n\}$ is generated by algorithm (3.2)

MATLAB version R2015a is used to obtain the graph of $\{u_n\}$ against number of iterations for different initial values as well as the graph of errors.

References

[1] M. Abbas, S.H. Khan, A.R. Khan, R.P. Agarwal, Common fixed points of two multi - valued nonexpansive mappings by one-step iterative scheme, Applied Mathematics Letters, 24(2)(2011) 97–102.

[2] Y. Alber, Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Applied Mathematics, 178(1996) 15–50. Dekker, New York.

[3] A. Bunyawat, S. Suantai, Common fixed points of a countable family of multivalued quasi-nonexpansive mappings in uniformly convex Banach spaces, International Journal of Computer Mathematics, 89(16)(2012) 2274–2279.

[4] C.E. Chidume, A.U. Bello, B. Usman, Krasnoselskii-type algorithm for zeros of strongly monotone Lipschitz maps in classical Banach spaces, SpringerPlus 4, Article ID 297 (2015). https://doi.org/10.1186/s40064-015-1044-1.

[5] S.S. Chang, J.K. Kim, X.R. Wang, Modified block iterative algorithm for solving convex feasibility problems in Banach spaces, J. Inequal. Appl. Article ID 869684 (2010). https://doi.org/10.1155/2010/869984.

[6] C.E. Chidume, U.V. Nnyaba, O.M. Romanus, C.G. Ezea, Convergence theorems for strictly J-pseudocontractions with application to zeros of Gamma-inverse strongly monotone maps. Panam. Math. J. 26(4)(2016) 57–76.

[7] C.E. Chidume, S.I. Ikechukwu, A. Adamu, Inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps, Fixed point Theory and Appl. DOI:10.1186/s13663-018-0634-3(2018).

[8] C.E. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations. Lectures Notes in Mathematics, 1965. Springer, London (2009). https://doi.org/10.1007/978-1-84882-190-3.

[9] C. Diop, M. Sene, N. Djitt, Iterative Algorithms for a Finite Family of Multi - valued Quasi-Nonexpansive Mappings, Advances in Numerical Analysis Volume 2014, Article ID 181049, 6 pages http://dx.doi.org/10.1155/2014/181049.

[10] Q.L. Dong, H.B. Yuan, C.Y. Je, Th.M Rassias, Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings, Optim. Lett. 12(2008) 87–102.

[11] M. Eslamain, Shrinking projection method for sequences of relatively quasi nonexpansive multi-valued mappings and equilibrium problem in Banach spaces, U.P.B. Sci. Bull., series A, 76(2)(2014).

[12] S. Homaeipour, A. Razani, Weak and Strong Convergence Theorems for Relatively Nonexpansive Multi-Valued Mappings in Banach Spaces, Fixed Point Theory Appl. 73(2011), DOI:10.1186/1687-1812-2011-73.

[13] T. Hu, J.C. Huang, B.E. Rhoades, A general principle for Ishikawa iterations for multi - valued mappings, Indian Journal of Pure and Applied Mathematics, 28(8)(1997) 1091–1098.

[14] T. Ibaraki, W. Takahashi, A new projection and convergence theorems for projections in Banach spaces, J. Approx. Theory 149(1)(2007) 1–14.

[15] S. Kamimura, W. Takahashi, Strong Convergence of a Proximal-Type Algorithm in a Banach Space, SIAMJ. Optim. 13(2002) 938–945.

[16] F. Kohsaka, W. Takahashi, Strong Convergence of an Iterative Sequence for Maximal Monotone Operators in a Banach Space, Abstr. Appl. Anal. (2004) 239–249.

[17] F. Kohsaka, W. Takahashi, The set of common fixed points of an infinite family of relatively nonexpansive mappings, In Banach and Function Spaces II(2008) 361–373, Yokohama Publishers, Yokohama.

[18] F. Kohsaka, W. Takahashi, Block Iterative Methods for a Finite Family of Relatively Nonexpansive Mappings in Banach Spaces, Fixed Point Theory and Applications, Volume (2007), Article ID 21972, 18 pages DOI:10.1155/2007/21972.

[19] P.E. Mainge, Strong Convergence of Projected Subgradient Methods for Nonsmooth and Nonstrictly Convex Minimization, Set-Valued Anal. 16(2008) 899–912.

[20] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134(2005) 257–266.

[21] S.B. Nadler Jr. Multi-valued contraction mappings, Pacific Journal of Mathematics, 30(1969) 475–488.

[22] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279(2003) 372–379.

[23] B. Panyanak, Mann and Ishikawa iterative processes for multi - valued mappings in Banach spaces, Computers and Mathematics with Applications, 54(6)(2007) 872–877.

[24] S. Reich, A weak convergence theorem for the alternating methods with Bregman distance. In: Kartsatos, A.G. (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Applied Mathematics, 178(1996) 313–318. Dekker, New York.

[25] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75(1)(1980) 287–292.

[26] K.P.R. Sastry, G.V.R. Babu, Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, Czechoslovak Mathematical Journal, 4(2005) 817–826.

[27] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis: Theory, Methods and Applications, 71(3-4)(2009) 838–844.

[28] Y. Song, H. Wang, Mann and Ishikawa iterative processes for multi - valued mappings in Banach spaces, Comput. Math. Appl. 54(6)(2007) 872–877.

[29] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Anal. 70(2009) 45–57.

[30] H.K. Xu, Another Control Condition in an Iterative Method for Nonexpansive Mappings, Bull. Aust. Math. Soc. 65(2002) 109–113.

[31] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Analysis, 16(12)(1991) 1127–1138.

[32] H. Zegeye, Strong convergence theorems for maximal monotone maps in Banach spaces. J. Math. Anal. Appl. 343(2008) 663–671.