APPROXIMATION METHOD FOR COMMON FIXED POINT OF A COUNTABLE FAMILY OF MULTI-VALUED QUASI-ϕ-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract In this paper, we study a new iterative algorithm to approximate a common fixed point of a countable family of multivalued quasi-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex real Banach space. We prove a strong convergence theorem, which improves and generalizes recently announced results in the literature. We also give some examples to illustrate our theorem, and we also give application to the zero-point problem of maximal monotone mappings.

MSC: 47H09; 47H05; 47J25; 47J05
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1. INTRODUCTION

The theory of nonexpansive multi-valued mappings is more challenging to study than the corresponding theory of nonexpansive single-valued mappings. However, it has important applications. As a result of this, algorithms for multi-valued nonexpansive type mappings have attracted the interest of researchers and have become a flourishing area of research, especially for numerous mathematicians in the field of nonlinear operator theory and functional analysis. Hence, a lot of research interest is now devoted to this type of mappings due to their remarkable utility and wide applicability in applied health sciences, modern mathematics, and other research areas. The theory of multi-valued mappings has applications in control theory, convex optimization, game theory, market economy, differential equations, and other medical fields.

Let \( B = \{ x \in E : \| x \| = 1 \} \) be the unit sphere of \( E \). A space \( E \) is said to be strictly convex if

\[
\forall x, y \in B, \ x \neq y, \ \text{then}, \quad \left\| \frac{x + y}{2} \right\| < 1.
\]

A real Banach space \( E \) is called uniformly convex if for each \( \epsilon \in (0, 2] \), \( \exists \delta > 0 \) such that \( x \neq y \ \forall \ x, y \in B \ \| x - y \| \geq \epsilon \), then, \( \left\| \frac{x + y}{2} \right\| < 1 - \delta \).

A real Banach space \( E \) is called smooth if for every \( x \in E \) with \( \| x \| = 1 \), there exists a unique \( x^* \in E^* \) such that

\[
\| x^* \| = 1 \ \text{and} \ \langle x, x^* \rangle = \| x \|.
\]

The space \( E \) is called uniformly smooth, if given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x, y \in E \) with \( \| x \| = 1 \) and \( \| y \| \leq \delta \) then

\[
\| x + y \|^2 + \| x - y \|^2 < 2 + \epsilon \| y \|.
\]

The normalized duality mapping \( J : E \to 2^{E^*} \) is defined as

\[
Jx = \{ f^* \in E^* : \langle x, f^* \rangle = \| f^* \|^2 : \| x \| = \| f^* \|, \ \forall x \in E \},
\]

(1.1)

where \( \langle \cdot, \cdot \rangle \) is the duality pairing between the element of \( E \) and \( E^* \). The following properties of the normalized duality map will be needed in the sequel, (see for example [8]).

- If \( E \) is a reflexive, strictly convex and smooth real Banach space, then \( J \) is surjective, injective and single-valued.

Let \( C \) be a closed convex subset of a real Hilbert space \( H \). The operator \( P_C \) is called a metric projection operator if it assigns to each \( x \in H \) its nearest point \( y \in C \) such that

\[
\| x - y \| = \min \{ \| x - z \| : z \in C \}.
\]

Let \( E \) be a smooth, strictly convex and reflexive real Banach space, and \( C \) be a nonempty closed and convex subset of \( E \). A map \( \prod_C : E \to C \) defined by \( \prod_C x = u^0 \in C \) such that \( \phi(u^0, x) = \inf_{y \in C} \phi(y, x) \) is called the generalized projection, where \( \inf_{y \in C} \phi(y, x) \) is the set of all the minimizers of \( \phi(\cdot, x) \).

Remark 1.1. In Hilbert spaces the generalized projection and the metric projection coincide.
Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subset of $C$, respectively. The Hausdorff metric on $CB(C)$ [2] is defined by

$$d_H(A_1, A_2) = \max \left\{ \sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_2) \right\}, \text{ for } A_1, A_2 \in CB(C),$$

(1.2)

where

$$d(x, A_2) = \inf \{\|x - y\|, y \in A_2\}.$$  

(1.3)

A point $p \in C$ is called a fixed point of the map $T : C \to N(C)$ if $p \in T(p)$, the set of fixed point of $T$ is denoted by $F(T)$.

**Remark 1.2.** Yakov Alber[2] introduced a generalized projection map $\Pi_C$ in a smooth Banach space $E$ which is an analogue of the metric projection operator in Hilbert spaces. This map has been studied by many authors (see for example [4, 6, 15, 24, 25, 29, 32]).

A map $\phi : E \times E \to [0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, j(y) \rangle + \|y\|^2$$

is called Lyapunov functional [2], where $x, y \in E$ and $j$ is the normalized duality mapping from $E$ to $E^*$ and $E$ is a real Banach space and $E^*$ it’s dual space. Observed that in a real Hilbert space $j$ is an identity map and

$$\phi(x, y) = \|x - y\|^2, \ \forall x, y \in H.$$  

Thus the following important proposition holds for the Lyapunov functional:

**Proposition 1.3.** *(see for example [7])* Let $E$ be a real Banach space. Then, for any $x, y, z \in E$ and $\alpha \in (0, 1)$ we have the following:

\begin{align*}
(N_0) & \quad (\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \\
(N_1) & \quad \phi(x, J^{-1}(\alpha J y + (1 - \alpha) J z)) \leq \alpha \phi(x, y) + (1 - \alpha)\phi(x, z).
\end{align*}

(1.4) \hspace{1cm} (1.5)

**Definition 1.4.** A multi-valued mapping $T : C \to CB(C)$ is called

(i) nonexpansive if

$$d_H(T(x), T(y)) \leq \|x - y\| \text{ for all } x, y \in C.$$  

(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and

$$d_H(T(x), q) \leq \|x - q\| \text{ for all } x \in C, q \in F(T).$$

Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, and let $T$ be a mapping from $C$ into itself. A point $p \in C$ is said to be an asymptotic fixed point of $T$, if $\exists \ \{u_n\} \subset C$ such that $u_n \to p$ and $\|u_n - T u_n\| \to 0$ as $n \to \infty$ (see for example Chang et, al. [5]). The set of asymptotic fixed points of $T$ is denoted by $\hat{F}(T)$. A mapping $T$ is said to be multi-valued relatively nonexpansive if $\hat{F}(T) = F(T)$ and

$$\phi(p, y) \leq \phi(p, x) \ y \in T(x), \ p \in F(T).$$

(1.6)
A multi-valued mapping $T$ is called relatively quasi nonexpansive (quasi-$\phi$-nonexpansive) if

$$
\phi(p, y) \leq \phi(p, x) \text{ for } y \in T(x), \ p \in F(T) \text{ and } F(T) \neq \emptyset.
$$

The class of multi-valued relatively quasi nonexpansive (quasi-$\phi$-nonexpansive) mappings is more general than the class of multi-valued relatively nonexpansive mappings, which requires the strong condition $F(T) = F(T) \neq \emptyset$.

Several authors have also studied fixed point problems of nonexpansive mappings in the context of both multi-valued and single valued mapping. Among thes authors, in 2003, Nakajo and Takahashi [22] studied the following CQ iterative algorithm for approximating a fixed point of a nonexpansive map in a real Hilbert space:

$$
\begin{align*}
&u^0 \in C, \\
&v^n = \alpha_n u^n + (1 - \alpha_n) T u^n, \\
&C_n = \{z \in C : \|v^n - z\| \leq \|u^n - z\|\}, \\
&Q_n = \{z \in C : \langle u^n - z, u^n - u^0 \rangle \leq 0\}, \\
&u^{n+1} = P_{C_n \cap Q_n} u^0, n \geq 1,
\end{align*}
$$

(1.8)

where $\{\alpha_n\}$ are sequences in the interval $[0,1]$, $C$ is a nonempty, closed and convex subset of a real Hilbert space, $T$ is a projection from $C$ onto $C_n \cap Q_n$. They proved that the sequence generated by algorithm (1.8) converges strongly to a fixed point of $T$.

In 2005, Matsushita and Takahashi [20] extended the result of Nakajo and Takahashi [22] by studying the following iterative algorithm for approximating a fixed point of relatively nonexpansive map in a uniformly smooth and uniformly convex real Banach space:

$$
\begin{align*}
&u^0 \in C, \\
&v^n = J^{-1}[\alpha_nJu^n + (1 - \alpha_n)JT u^n], \\
&C_n = \{z \in C : \phi(z, v^n) \leq \phi(z, u^n)\}, \\
&Q_n = \{z \in C : \langle u^n - z, Ju^n - Ju^0 \rangle \leq 0\}, \\
&u^{n+1} = \prod_{C_n \cap Q_n} u^0, n \geq 1.
\end{align*}
$$

(1.9)

They proved that the sequence generated by (1.9) converges strongly to a fixed point of $T$.

In 2005, Sastry and Babu [26] established the convergence theorem of multi-valued maps using Ishikawa and Mann iterates. They also provided an example that illustrates how the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point $'p'$ and the initial guess.

In 2007, Panyanak [23] wrote a paper that extended one of the results of Sastry and Babu [26] to uniformly convex Banach spaces. He proved a convergence theorem using Mann iterates for a mapping defined on a noncompact domain.

Later in 2008, Song and Wang [28] demonstrated the strong convergence of Mann and Ishikawa iterates to a fixed point of a multi-valued nonexpansive mapping '$T'$ under certain appropriate conditions.

Additionally, in 2008, Dong et al. [10] studied the following inertial CQ iterative algorithm
for approximating a fixed point of a nonexpansive map in a real Hilbert space:

\[
\begin{aligned}
  u^0, u^1 &\in H, \\
  w^n &= u^n + \alpha_n(u^n - u^{n-1}), \\
  v^n &= (1-\beta_n)w^n + \beta_nT w^n, \\
  C_n &= \{ z \in H : \| v^n - z \| \leq \| u^n - z \| \}, \\
  Q_n &= \{ z \in C : \langle u^n - z, u^n - u^0 \rangle \leq 0 \}, \\
  u^{n+1} &= P_{C_n \cap Q_n} u^0, n \geq 0.
\end{aligned}
\]  

(1.10)

They proved that the iterative sequence generated by (1.10) converges in norm to \( P_{F(T)} u^0 \).

Furthermore, Diop et. al., [9] constructed a new iterative algorithm and proved strong convergence theorems for common fixed point of a finite family of multivalued quasi-nonexpansive mappings in uniformly convex real Banach spaces.

Several authors have also studied fixed point problems of a multi-valued nonexpansive mappings in the setting of real Banach spaces (see for example [21],[13],[26],[23],[28],[1] and the references therein).

Chidume et. al., [7] studied the following inertial type algorithm for approximating a common fixed point for a countable family of a relatively nonexpansive mapping in a uniformly smooth and uniformly convex real Banach space

\[
\begin{aligned}
  C_0 &= E, \\
  w^n &= u^n + \alpha_n(u^n - u^{n-1}), \\
  v^n &= J^{-1}[ (1-\beta) J w^n + \beta J T w^n ], \\
  c_{n+1} &= \{ z \in c_n : \phi(z, v^n) \leq \phi(z, w^n) \}, \\
  u^{n+1} &= \Pi_{C_{n+1}} u^0, n \geq 0.
\end{aligned}
\]  

(1.11)

They proved that the sequence generated by (1.11) converges strongly to a point \( p = \Pi_{F(T)} u^0 \).

**Remark 1.5.** \( C_n \) and \( Q_n \) are halfspaces, that must be computed at each iterative stage in \([10, 20, 22]\) and the host of other authors,.) thus Chidume et. al, [7] reduces the computation by dispensing with \( Q_n \) and studied inertial-type algorithm for approximating a common fixed point for a countable family of relatively nonexpansive maps which is an extension of some recent existing results. Motivated by the work of Chidume et. al., [7] we propose a new iterative scheme in which both \( Q_n \) and \( C_n \) are dispensed with. Furthermore the condition \( F(T) = \bar{F}(T) \) is dispensed with i.e we consider the class of quasi-\( \phi \)-nonexpansive multi-valued maps which is more general than the class of relatively nonexpansive.

In addition, our theorem also improve the result of Diop et. al., [9], from the case of finite family of multi-valued quasi-\( \phi \)-nonexpansive to Countable family of multi-valued quasi-\( \phi \)-nonexpansive mappings.

We prove a strong convergence of our new iterative scheme to a common fixed point for a countable family of multi-valued quasi-\( \phi \)-nonexpansive maps.
2. PRELIMINARIES

The following notions and results are very important in our subsequent discussion.

Definition 2.1. Let $E$ be a reflexive, strictly convex and smooth real Banach space. The duality mapping $J^*$ from $E^*$ onto $E^{**} = E$ coincides with the inverse of the duality mapping $J$ from $E$ onto $E^*$ that is, $J^* = J^{-1}$.

The following mapping will be needed in the sequel $V : E \times E^* \to \mathbb{R}$ defined by

$$ V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \forall (x, x^*) \in E \times E^*. \quad (2.1) $$

Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*)) \forall (x, x^*) \in E \times E^*$.

We will also use the following lemmas.

Lemma 2.2. [15] Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$ such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$ as $n \to \infty$.

Lemma 2.3. [2, 16] Let $E$ be a reflexive, strictly convex and smooth Banach space. Then for any $x \in E$, and $x^*, y^* \in E^*$, we have,

$$ V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*). \quad (2.2) $$

Lemma 2.4. [15] Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Then the following hold:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;

(b) If $x \in E$ and $z \in C$, then $z = \Pi_C x \iff \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$;

(c) for $x, y \in E, \phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.5. [12] Let $E$ be a smooth and strictly convex real Banach space, Let $C$ be a nonempty closed and convex subset of $E$. Suppose $T : C \to N(C)$ is a quasi $\phi$ nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.

Lemma 2.6. [19] Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that

$$ a_{n_j} < a_{n_{j+1}} \forall j \geq 0. $$

Also consider the sequence of integers $\{m_k\}$ defined by

$$ m_k = \max\{i \leq k : a_i < a_{i+1}\}. $$

Then $\{m_k\}$ is a nondecreasing sequence satisfying

$$ \lim_{k \to \infty} m_k = \infty, $$

there exists some $n_0 \in \mathbb{N}$, for some , the following two estimates hold;

$$ a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{m_{k+1}}, \forall k \geq n_0. $$

Lemma 2.7. [31] Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$ \|\lambda x + (1 - \lambda) y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \quad (2.3) $$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$. 

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Lemma 2.8. [17] Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $T_i : C \to E$ for $i = 1, 2, 3, \ldots$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\alpha_n \in (0, 1), \beta_i \in (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=1}^{\infty} \alpha_i (\beta_i Jx + (1 - \beta_i)JT_i x))$ for each $x \in C$. Then $T$ is relatively nonexpansive and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.

Lemma 2.9. [30] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying
\[
a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq 0, \tag{2.4}
\]
where $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

i. $\{\alpha_n\} \subset [0, 1], \sum_{n=0}^{\infty} \alpha_n = \infty$,

ii. $\limsup_{n \to \infty} \delta_n \leq 0$.

Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.10. [18] Let $E$ be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{\lambda_i\}_{i=1}^{\infty} \subset (0, 1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in $E$ such that
\[
\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2, \tag{2.5}
\]
then $x_1 = x_2 = x_3 = \cdots$.

3. Main Result

In this section we present the main results of this paper.

Lemma 3.1. Let $E$ be a smooth, strictly convex and reflexive Banach space, let $z \in E$ and let $\{\lambda_i\}_{i=1}^{\infty} \subset (0, 1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is a sequence in $E$ such that
\[
\phi \left( z, J^{-1} \left( \sum_{i=1}^{\infty} \lambda_i Jx_i \right) \right) = \sum_{i=1}^{\infty} \lambda_i \phi(z, x_i), \tag{3.1}
\]
then $x_1 = x_2 = x_3 = \cdots$.

Proof. We follow the proof line of Kohsaka and Takahashi[18] from the case of finite family to Countable family.

Let $z \in E$, we have from our assumption that
\[
\phi \left( z, J^{-1} \left( \sum_{i=1}^{\infty} \lambda_i Jx_i \right) \right) = \sum_{i=1}^{\infty} \lambda_i \phi(z, x_i),
\]
\[
\|z\|^2 - 2 \left\langle z, \sum_{i=1}^{\infty} \lambda_i Jx_i \right\rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i (\|z\|^2 - 2 \langle z, x_i \rangle + \|x_i\|^2)
\]
\[
\|z\|^2 - 2 \sum_{i=1}^{\infty} \lambda_i \langle z, Jx_i \rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i (\|z\|^2 - 2 \sum_{i=1}^{\infty} \lambda_i \langle z, x_i \rangle + \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2)
\]
\[
\sum_{i=1}^{\infty} \lambda_i \|z\|^2 - 2 \sum_{i=1}^{\infty} \lambda_i \langle z, Jx_i \rangle + \left\| \sum_{i=1}^{\infty} \lambda_i Jx_i \right\|^2 = \sum_{i=1}^{\infty} \lambda_i (\|z\|^2 - 2 \sum_{i=1}^{\infty} \lambda_i \langle z, x_i \rangle + \sum_{i=1}^{\infty} \lambda_i \|Jx_i\|^2).
\]
This implies

$$\sum_{i=1}^{\infty} \lambda_i Jx_i^2 = \sum_{i=1}^{\infty} \lambda_i \|Jx_i\|^2.$$  

Since $E$ is smooth and reflexive, $E^*$ is strictly convex. Thus, Lemma (2.10) implies that $Jx_1 = Jx_2 = Jx_3 = \cdots$. By the strict convexity of $E$, $J$ is one-to-one. Hence we have the desired result i.e $x_1 = x_2 = x_3 = \cdots$. 

We also need the following results for completion of our theorem

**Lemma 3.2.** Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space. Let $T_i : C \to CB(E)$ $i = 1, 2, 3, \ldots$ be a countable family of quasi-$\phi$-nonexpansive multi-valued maps with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\alpha_n \in (0, 1), \beta_i \in (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}(\sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1 - \beta_i)Jy_i))$ with $T_i(p) = \{p\}$ for each $x \in C, y_i \in T_i x$. Then $T$ is quasi nonexpansive mapping and $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.

**Proof.** We first show that $T$ is well defined. Then $\forall x \in C$, $y_i \in T_i x$ and $\forall i \geq 0$, the mapping $T$ can be rewritten as:

$$T(x) = J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1 - \beta_i)Jy_i) \right)$$

$$= J^{-1} \left( \left( \sum_{i=1}^{\infty} \alpha_i \beta_i \right) Jx + \left( \sum_{i=1}^{\infty} \alpha_i (1 - \beta_i)Jy_i \right) \right)$$

$$= J^{-1} \left( \gamma_0 Jy_0 + \sum_{i=1}^{\infty} \gamma_i Jy_i \right)$$

where $y_0 = x$ and $T_0$ is the identity mapping, $\gamma_0 = \sum_{i=1}^{\infty} \alpha_i \beta_i > 0, \gamma_i = \alpha_i (1 - \beta_i) > 0$ for all $i \geq 1$ and $\sum_{i=1}^{\infty} \gamma_i = 1$. Now for any fixed $p \in \bigcap_{i=1}^{\infty} F(T_i), y_i \in T_i x$, $T_i$ quasi nonexpansive multi-valued map and $x \in C$, we have

$$(\|p\| - \|y_i\|)^2 \leq \phi(p, y_i) \leq \phi(p, x) \leq (\|p\| + \|x\|)^2,$$

which implies,

$$\|p\|^2 - 2\|p\||y_i| + |y_i|^2 \leq \|p\|^2 + 2\|p\||x| + |x|^2$$

$$\|y_i\|^2 - |x|^2 \leq 2\|p\||y_i| + |x|$$

$$\|y_i\| - |x| \leq 2\|p\||y_i| + |x|.$$
In particular, \( \|y_i\| \leq 2\|p\| + \|x\| \) for each \( i \geq 0 \) and \( x \in C \).

Thus,
\[
J^{-1}\sum_{i=0}^{\infty} \gamma_i Jy_i \leq \sum_{i=0}^{\infty} \gamma_i \|y_i\| \\
\leq \sum_{i=0}^{\infty} \gamma_i (2\|p\| + \|x\|) < \infty.
\]

Hence \( T \) is well defined.

Next we show that \( T \) is quasi nonexpansive mapping.

Let \( p \in F(T) \), by Lemma 2.7 and from the fact that \( \{T_i\} \) is a countable family of quasi nonexpansive multi-valued mapping, that \( y_i \in T_ix \) for each \( i \geq 1, x \in C \), we have
\[
\phi(p, Tx) = \phi \left( p, J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i (1 - \beta_i) Jy_i \right) \right) \\
= \|p\|^2 - 2 \left( p, \sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i (1 - \beta_i) Jy_i \right) + \left\| \sum_{i=1}^{\infty} \alpha_i \beta_i Jx + \alpha_i (1 - \beta_i) Jy_i \right\|^2 \\
\leq \|p\|^2 - 2 \alpha_i \langle p, \beta_i Jx + (1 - \beta_i) Jy_i \rangle + \sum_{i=1}^{\infty} \alpha_i \|\beta_i Jx + (1 - \beta_i) Jy_i\|^2 \\
\leq \sum_{i=1}^{\infty} \alpha_i \left( \|p\|^2 - 2 \langle p, \beta_i Jx + (1 - \beta_i) Jy_i \rangle + \beta_i \|x\|^2 + (1 - \beta_i) \|y_i\|^2 \right) \\
- \sum_{i=1}^{\infty} \alpha_i \beta_i (1 - \beta_i) g^\ast(\|Jx - Jy_i\|) \\
\leq \sum_{i=1}^{\infty} \alpha_i (\beta_i \phi(p, x) + (1 - \beta_i) \phi(p, y_i)) \\
\leq \sum_{i=1}^{\infty} \alpha_i (\beta_i \phi(p, x) + (1 - \beta_i) \phi(p, x)) \\
= \sum_{i=1}^{\infty} \alpha_i \phi(p, x) \\
= \phi(p, x).
\]

Hence \( T \) is quasi-\( \phi \) nonexpansive mapping.

Finally we show that \( F(T) = \bigcap_{i=1}^{\infty} F(T_i) \). It suffices to show the inclusion \( F(T) \subset \bigcap_{i=1}^{\infty} F(T_i) \).
Thus, $\phi(p, z) = \phi(p, Tz)
\begin{align*}
&= \phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i Jy_i\right)\right) \\
&= \|p\|^2 - 2 \left\langle p, \sum_{i=0}^{\infty} \gamma_i Jy_i\right\rangle + \left\| \sum_{i=0}^{\infty} \gamma_i Jy_i\right\|^2 \\
&\leq \|p\|^2 - 2 \sum_{i=0}^{\infty} \gamma_i \left\langle p, Jy_i\right\rangle + \sum_{i=0}^{\infty} \gamma_i \|Jy_i\|^2 \\
&= \sum_{i=0}^{\infty} \gamma_i \left(\|p\|^2 - 2 \left\langle p, Jy_i\right\rangle + \|Jy_i\|^2\right) \\
&= \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i) \\
&\leq \phi(p, z), \forall i \geq 0.
\end{align*}

Thus, $\phi(p, z) = \phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i Jy_i\right)\right) \leq \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i) \leq \phi(p, z)$. Therefore, we have, $\phi\left(p, J^{-1}\left(\sum_{i=0}^{\infty} \gamma_i Jy_i\right)\right) = \sum_{i=0}^{\infty} \gamma_i \phi(p, y_i)$. By applying Lemma 3.1 we obtain that, $y_0 = y_1 = y_2 = y_3 = \cdots$. Clearly $\bigcap_{i=1}^{\infty} F(T_i) \subset F(T)$ follows, Hence $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$.

**Theorem 3.3.** Let $C$ be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space $E$. Let $T_i : E \to CB(E)$ be a countable family of quasi-$\phi$-nonexpansive multi-valued maps such that $\bigcap_{i=0}^{\infty} F(T_i) \neq \emptyset$. Suppose $\eta_i, \lambda_i \in (0, 1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = J^{-1}\left(\sum_{i=0}^{\infty} \lambda_i (\eta_i Jx + (1 - \eta_i) Jy_i)\right)$ for each $x \in C$, $y_i \in T_i x$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm: $u^0, u^1 \in C$ and
\begin{align*}
\begin{cases}
w^n = J^{-1}\left[\alpha_n Ju^0 + (1 - \alpha_n) Ju^n\right], \\
v^n = J^{-1}\left[(1 - \beta) Ju^n + \beta JTW^n\right], \\
u^{n+1} = \Pi_C u^n, n \geq 1,
\end{cases}
\end{align*}

where $\beta \in (0, 1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying
\begin{align*}
(A_1) \quad &\lim_{n \to \infty} \alpha_n = 0; \\
(A_2) \quad &\sum_{n=1}^{\infty} \alpha_n = \infty.
\end{align*}

Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\Pi_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

**Proof.** We partition the proof into three steps.

**Step 1:** From Lemma 3.2 $F(T) = \bigcap_{i=1}^{\infty} F(T_i)$ and by Lemma 2.1, $F(T_i)$ is closed and convex for each $i \in \mathbb{N} \cup \{0\}$. Thus $\bigcap_{i=0}^{\infty} F(T_i)$ is closed and convex so $\Pi_{\bigcap_{i=0}^{\infty} F(T_i)}$ is well defined generalized projection.

**Step 2:** We show that $\{u^n\}_{n=1}^{\infty}$, $\{v^n\}_{n=1}^{\infty}$ and $\{u^n\}_{n=1}^{\infty}$ are bounded. Let $u^* = \Pi_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$, using Lemma 2.4(a) and (1.5)
\[\phi(u^*, u^{n+1}) = \phi(u^*, \Pi_C v^n) \]
\[\leq \phi(u^*, v^n)\]
\[= \phi(u^*, J^{-1}[(1 - \beta)Jw^n + \beta JTW^n])\]
\[\leq (1 - \beta)\phi(u^*, w^n) + \beta\phi(u^*, Tw^n)\]
\[\leq (1 - \beta)\phi(u^*, w^n) + \beta\phi(u^*, w^n)\]
\[= \phi(u^*, w^n).\]

Therefore,
\[\phi(u^*, u^{n+1}) \leq \phi(u^*, v^n) \leq \phi(u^*, w^n), \tag{3.3}\]
but,
\[\phi(u^*, w^n) = \phi(u^*, J^{-1}[(1 - \alpha_n)Ju^n + \alpha_nJu^0])]\]
\[\leq (1 - \alpha_n)\phi(u^*, u^n) + \alpha_n\phi(u^*, u^0)\]
\[\leq \max\{\phi(u^*, u^n), \phi(u^*, u^0}\}.\]

This implies,
\[\phi(u^*, w^n) \leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}. \tag{3.4}\]

From (3.3) and (3.4) we get
\[\phi(u^*, u^{n+1}) \leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}. \tag{3.5}\]

Hence, using induction we get
\[\phi(u^*, u^n) \leq \max\{\phi(u^*, u^n), \phi(u^*, u^0)\}, \forall n \geq 0.\]

This implies that \(\{\phi(u^*, u^n)\}_{n=0}^{\infty}\) is bounded, therefore by (1.4) \(\{u^n\}_{n=0}^{\infty}\) is bounded, also by (3.4) and (3.3) \(\{\phi(u^*, v^n)\}_{n=0}^{\infty}\) and \(\{\phi(u^*, w^n)\}_{n=0}^{\infty}\) are bounded. Thus \(\{v^n\}_{n=0}^{\infty}\) and \(\{w^n\}_{n=0}^{\infty}\) are also bounded by (1.4)

**Step 3**: Finally we show \(\{u^n\}_{n=1}^{\infty}\) converges to \(u^*\).

From (3.3) we have,
\[\phi(u^*, u^{n+1}) \leq \phi(u^*, w^n) = V(u^*, Jw^n).\]

Using Lemma 2.3, that is taking \(y^* = \alpha_nJu^* - \alpha_nJu^0, x = u^* \) and \(x^* = Jw^n, \) we have,
\[V(u^*, Jw^n) \leq V(u^*, Ju^n - \alpha_n(Ju^0 - Ju^*)) - 2\langle w^n - u^*, -\alpha_n(Ju^0 - Ju^*) \rangle\]
\[= V(u^*, (1 - \alpha_n)Ju^n + \alpha_nJu^0 - \alpha_nJu^0 + \alpha_nJu^*)\]
\[+ 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^* \rangle\]
\[= V(u^*, (1 - \alpha_n)Ju^n + \alpha_nJu^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^* \rangle.\]

By (2.1) we have,
\[V(u^*, (1 - \alpha_n)Ju^n + \alpha_nJu^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^* \rangle\]
\[= \phi(u^*, J^{-1}(1 - \alpha_n)Ju^n + \alpha_nJu^*) + 2\alpha_n\langle w^n - u^*, Ju^0 - Ju^* \rangle.\]
Also applying (1.5) we have,
\[ \phi(u^*, J^{-1}(1 - \alpha_n)Ju^n + \alpha_nJu^*) + 2\alpha_n \langle w^n - u^*, Ju^0 - Ju^* \rangle \]
\[ \leq (1 - \alpha_n)\phi(u^*, u^n) + 2\alpha_n \langle w^n - u^*, Ju^0 - Ju^* \rangle, \]
which implies,
\[ \phi(u^*, u^{n+1}) \leq (1 - \alpha_n)\phi(u^*, u^n) + 2\alpha_n \langle w^n - u^*, Ju^0 - Ju^* \rangle \forall n \geq 0. \] (3.6)

We next divide the proof into two cases.

**Case 1:** Suppose that there exists \( n_0 \) such that \( \{\phi(u^*, u^n)\}_{n=n_0}^{\infty} \) is nonincreasing. In this situation \( \{\phi(u^*, u^n)\}_n \) is then convergent. Then,
\[ \lim_{n \to \infty} (\phi(u^*, u^n) - \phi(u^*, u^{n+1})) = 0. \] (3.7)
Also since \( \{w^n\}_{n=1}^{\infty} \) is bounded, we choose \( \{w^{n_k}\}_{k=1}^{\infty} \) subsequence of \( \{w^n\}_{n=1}^{\infty} \) such that,
\[ w^{n_k} \to z \quad \text{and} \quad \limsup_{n \to \infty} \langle w^n - u^*, Ju^0 - Ju^* \rangle = \lim_{k \to \infty} \langle w^{n_k} - u^*, Ju^0 - Ju^* \rangle = \langle z - u^*, Ju^0 - Ju^* \rangle, \]
by Lemma 2.4, we have
\[ \limsup_{n \to \infty} \langle w^n - u^*, Ju^0 - Ju^* \rangle = \langle z - u^*, Ju^0 - Ju^* \rangle \leq 0. \]

From (3.6) and Lemma 2.9 we have
\[ \lim_{n \to \infty} \phi(u^*, u^n) = 0, \]
which implies \( u^n \to u^* \), as \( n \to \infty \), where \( u^* = \prod_{n=0}^{\infty} F(T_i) u^0. \)

**Case 2:** Suppose \( \{\phi(u^*, u^n)\}_{n=1}^{\infty} \) is not nonincreasing, then that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that
\[ \phi(u^*, u^{n_i}) \leq \phi(u^*, u^{n_i+1}) \quad \text{for all} \quad i \geq 0. \]

By Lemma ?? there exists a nondecreasing sequence \( \{m_k\}_{k=0}^{\infty} \) of nonnegative integers such that \( m_k \to \infty \), as \( k \to \infty \),
\[ \phi(u^*, u^{m_k}) \leq \phi(u^*, u^{m_{k+1}}) \] (3.8)
and
\[ \phi(u^*, u^k) \leq \phi(u^*, u^{m_{k+1}}) \quad \text{for all} \quad k \geq 0. \] (3.9)

Following the same argument as in case 1, we obtain
\[ \limsup_{k \to \infty} \langle w^{m_k} - u^*, Ju^0 - Ju^* \rangle \leq 0. \] (3.10)

From (3.6) we have
\[ \phi(u^*, u^{m_{k+1}}) \leq (1 - \alpha_{m_k})\phi(u^*, u^{m_k}) + 2\alpha_{m_k} \langle w^{m_k} - u^*, Ju^0 - Ju^* \rangle. \] (3.11)
Rearranging we have,
\[ \alpha_{m_k} \phi(u^*, u^{m_k}) \leq \phi(u^*, u^{m_k}) - \phi(u^*, u^{m_{k+1}}) + 2\alpha_{m_k} \langle w^{m_k} - u^*, Ju^0 - Ju^* \rangle, \]
employing (3.8) we have,
\[ \alpha_{m_k} \phi(u^*, u^{m_k}) \leq 2\alpha_{m_k} \langle w^{m_k} - u^*, Ju^0 - Ju^* \rangle, \]
Since \(\alpha_{m_k} \geq 0\); we get
\[
0 \leq \phi(u^*, u^{m_k}) \leq 2(w^{m_k} - u^*, Ju^0 - Ju^*),
\]
it follows from (3.10) that
\[
\lim_{k \to \infty} \phi(u^*, u^{m_k}) = 0.
\]
This together with (3.11) gives
\[
\lim_{k \to \infty} \phi(u^*, u^{m_{k+1}}) = 0. \tag{3.12}
\]
But by (3.9)
\[
\phi(u^*, u^k) \leq \phi(u^*, u^{m_{k+1}}) \text{ for all } k \geq 0.
\]
By (3.12) we obtain that \(\phi(u^*, u^k) \to 0 \text{ as } k \to \infty\). Then from case 1 and case 2 we conclude that \(n \lim_{n \to \infty} \phi(u^*, u^n) = 0\), which implies \(u^n \to \prod_{i=n=0}^\infty F(T_i) u^0\).

**Corollary 3.4.** Let \(C\) be a nonempty, closed and convex subset of uniformly convex and uniformly smooth real Banach space \(E\). Let \(T_i : C \to E\) be a countable family of quasi-\(\phi\)-nonexpansive maps such that \(\bigcap_{i=0}^\infty F(T_i) \neq \emptyset\). Suppose \(\{\eta_i\}, \{\lambda_i\} \in (0, 1)\) are sequences such that \(\sum_{i=0}^\infty \lambda_i = 1\) and \(T : C \to E\) is defined by \(T(x) = J^{-1}(\sum_{i=0}^\infty \lambda_i(\eta_i Jx + (1 - \eta_i)JT_i x))\) for each \(i \geq 1\). Let \(\{u^n\}\) be generated by the following algorithm : \(u^0, u^1 \in C\) and
\[
\begin{align*}
w^n &= J^{-1} [\alpha_n J u^0 + (1 - \alpha_n) J u^n], \\
v^n &= J^{-1} [(1 - \beta) J w^n + \beta JT w^n], \\
u^{n+1} &= \Pi_C v^n, n \geq 1, 
\end{align*} \tag{3.13}
\]
where \(\beta \in (0, 1)\) and \(\{\alpha_n\}_n\) is a sequence in \((0,1)\) satisfying

(A1) \(\lim_{n \to \infty} \alpha_n = 0\);

(A2) \(\sum_{n=1}^\infty \alpha_n = \infty\). Then \(\{u^n\}_{n=1}^\infty\) converges strongly to \(\Pi_{\bigcap_{i=0}^\infty F(T_i)} u^0\).

**Corollary 3.5.** Let \(C\) be a nonempty, closed and convex subset of \(L_p\) or \(l_p\), \(1 < p < \infty\). Let \(T_i : C \to CB(E)\) be a countable family of quasi-\(\phi\)-nonexpansive multi-valued maps such that \(\bigcap_{i=0}^\infty F(T_i) \neq \emptyset\). Suppose \(\{\eta_i\}, \{\lambda_i\} \in (0, 1)\) are sequences such that \(\sum_{i=0}^\infty \lambda_i = 1\) and \(T : C \to E\) is defined by \(T(x) = J^{-1}(\sum_{i=0}^\infty \lambda_i(\eta_i Jx + (1 - \eta_i)JT_i x))\) for each \(x \in C, y_i \in T_i x\) for each \(i \geq 1\). Let \(\{u^n\}\) be generated by the following algorithm : \(u^0, u^1 \in C\) and
\[
\begin{align*}
w^n &= J^{-1} [\alpha_n J u^0 + (1 - \alpha_n) J u^n], \\
v^n &= J^{-1} [(1 - \beta) J w^n + \beta JT w^n], \\
u^{n+1} &= \Pi_C v^n, n \geq 1. 
\end{align*} \tag{3.14}
\]
where \(\beta \in (0, 1)\) and \(\{\alpha_n\}_n\) is a sequence in \((0,1)\) satisfying

(A1) \(\lim_{n \to \infty} \alpha_n = 0\);

(A2) \(\sum_{n=1}^\infty \alpha_n = \infty\). Then \(\{u^n\}_{n=1}^\infty\) converges strongly to \(\Pi_{\bigcap_{i=0}^\infty F(T_i)} u^0\).
Corollary 3.6. Let $C$ be a nonempty, closed and convex subset of real Hilbert space $H$. Let $T_i : C \to CB(H)$ be a countable family of quasi nonexpansive multi-valued map such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\eta_i, \lambda_i \in (0, 1)$ are sequences such that $\sum_{i=0}^{\infty} \lambda_i = 1$ and $T : C \to E$ is defined by $T(x) = \left(\sum_{i=0}^{\infty} \lambda_i \eta_i (x + (1 - \eta_i) y_i)\right)$ for each $x \in C$, $y_i \in T_i x$ for each $i \geq 1$. Let $\{u^n\}$ be generated by the following algorithm: $u^0, u^1 \in C$ and

$$
\begin{align*}
  w^n &= \left[\alpha_n u^0 + (1 - \alpha_n) u^n\right], \\
  v^n &= \left[(1 - \beta) w^n + \beta T u^n\right], \\
  u^{n+1} &= P_C v^n, 
\end{align*}
$$

where $\beta \in (0, 1)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

$$(A_1) \quad \lim_{n \to \infty} \alpha_n = 0;$$

$$(A_2) \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$ 

Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $P_{\bigcap_{i=0}^{\infty} F(T_i)} u^0$.

4. Application to Zero Point Problem of Maximal Monotone Mappings

Let $E$ be a smooth, strictly convex and reflexive Banach space.

Definition 4.1. [2] An operator $A : E \to 2^{E^*}$ is said to be monotone, if whenever $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$, then

$$\langle x - y, x^* - y^* \rangle \geq 0.$$ 

We denote the zero point set $\{x \in E : 0 \in Ax\}$ of $A$ by $A^{-1}(0)$. A monotone operator $A$ is said to be maximal if its graph $G(A) := \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Remark 4.2. For each $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that $Jx \in Jx_r + rA(x_r)$. We define the resolvent of $A$ by $J_r x = x_r$. In other words, $J_r = (J + rA)^{-1}J$, $\forall r > 0$. We know that $J_r$ is a single valued quasi nonexpansive mapping and $A^{-1}(0) = F(J_r)$, $\forall r > 0$ (see for example [2], where $F(J_r)$ is the set of fixed points of $J_r$).

Theorem 4.3. Let $E$, $\{\alpha_n\}_n$ as in theorem 3.3. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and $J_r = (J + rA)^{-1}J$, $\forall r > 0$ such that $A^{-1}(0) \neq \emptyset$. Let $\{u^n\}$ be a sequence generated by $u^0, u^1 \in E$ and

$$
\begin{align*}
  w^n &= \left[\alpha_n u^0 + (1 - \alpha_n) u^n\right], \\
  u^{n+1} &= J^{-1}\left[(1 - \beta) J w^n + \beta JJ_r w^n\right].
\end{align*}
$$

Then $\{u^n\}_{n=1}^{\infty}$ converges strongly to $\Pi_{A^{-1}(0)} u^0$.

Proof. In theorem (3.3), taking $C = E$, $T = J_r$, $r > 0$, then $T : E \to E$ is a single-valued quasi nonexpansive mapping and $A^{-1} = F(T) = \bigcap_{i=0}^{\infty} F(T_i) = F(J_r)$, $\forall r > 0$ is a nonempty closed convex subset of $E$. Therefore all the conditions in theorem (3.3) are satisfied. The conclusion follows immediately.
4.1. Example

**Example 4.4.** Consider $E = \mathbb{R}$ with the standard norm $\| \cdot \| = | \cdot |$ and $C = [-1, 1]$. For $m \in \mathbb{N}$, we define a mapping $\{T_m\}_m$ on $\mathbb{R}$ by

$$T_m(x) = \begin{cases} 
0, & x \leq \frac{1}{m^2} \\
\frac{1}{m^2}, & x > \frac{1}{m^2}
\end{cases}$$

(4.1)

for all $x \in \mathbb{R}$. Then $F(T_m) = \bigcap_{m=1}^{3} F(T_m) = \{0\}$ and

$$|T_m x - 0| \leq |x - 0| \quad \forall x \in \mathbb{R}.$$ 

So $\{T_m\}_m$ is a sequence of quasi nonexpansive. Finally we show that each $\{T_m\}$ is not relatively nonexpansive. Indeed we take $x_n = \frac{1}{m^2} + \frac{1}{n}$ then

$$x_n \to \frac{1}{m^2}, \quad x_n - T_m x_n = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.$$ 

This implies $\frac{1}{m^2} \in F(T_m)$ and $\frac{1}{m^2} \notin F(T_m)$. Hence $T_m$ is not relatively nonexpansive.

Also in theorem (3.3), Fix $u^0 = 0.5$ and choose $\alpha_n = \frac{1}{2n+1}$, $\lambda_n = \frac{1}{3}$ and $\eta_n = \frac{1}{2n^2+1}$ clearly $\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Also $\sum_{n=1}^{3} \lambda_n = 1$ and $\{\eta_n\} \in (0, 1)$. Thus all conditions in our proposed Algorithm are satisfied, thus algorithm (3.2) will reduces to

$$\begin{cases}
\begin{aligned}
u^n &= \frac{0.5+2nu^n}{2n+1} \\
v^n &= (\beta v^n + (1-\beta)T(w^n)) , \quad T(w^n) = \sum_{m=1}^{3} \frac{1}{3} \left( \frac{1}{2m^2+1} w^n + \frac{2m^2}{2m^2+1} T_m(w^n) \right) \\
u^{n+1} &= \begin{cases} 
-1, & v^n < -1 \\
v^n, & v^n \in [-1, 1] \\
1, & v^n > 1.
\end{cases}
\end{aligned}
\end{cases}$$

**Case I.** Take $u^1 = 0.4$ and $\beta = 0.8$

![Graph of Case I](image-url)
Case II. Take $u^1 = -0.4$ and $\beta = 0.8$

**FIGURE 2.** The graph of sequence $\{u_n\}$ generated by algorithm (3.2) versus number of iterations (Case II)

Case III. Take $u^1 = 0.8$ and $\beta = 0.8$

**FIGURE 3.** The graph of $\|u_{n+1} - u_n\|$ versus number of iterations (Case III) where $\{u_n\}$ is generated by algorithm (3.2)

MATLAB version R2015a is used to obtain the graph of $\{u_n\}$ against number of iterations for different initial values as well as the graph of errors.
REFERENCES


