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Fixed Point Theorems in the Fuzzy Supermetric Spaces and Application

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Abstract This article introduces a new mathematical concept that extends the notion of a supermetric space, termed a fuzzy supermetric space. An illustrative example is provided to validate the proposed concept, and a class of contractive mappings is developed and employed to establish the existence and uniqueness of fixed-point theorems within the framework of fuzzy supermetric spaces. These results generalize both classical and recent findings as special cases within a broader, more inclusive setting, offering a more suitable framework for modeling and solving real-world problems. In addition, the concept of $(\beta-\theta)$-contractive mappings is examined in this context. Finally, the theoretical results are applied to demonstrate the existence of a solution to the Fredholm integral equation.

MSC: 47H10, 45D05, 54H2

Keywords: Fixed Point; Contractive Mappings; Supermetric Spaces; Fuzzy Supermetric Spaces; Fuzzy Metric Spaces; Fredholm Integral Equation

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1. Introduction

The study of metric space was started in 1906 by French mathematician Fréchet [1], as it was one of the most helpful and extraordinary topics of nonlinear functional analysis. Fixed point theory is a motivating subject, having many applications in many branches of mathematics, such as differential equations, partial differential equations, integral equations, variational inequalities, etc. The concept of the fixed-point theorem was introduced to the field of metric space studies by the "Banach fixed-point theorem," which established and guaranteed the existence and uniqueness of fixed points of particular self-maps of complete metric spaces. Many authors refined, extended, modified, and generalized the Banach Contraction Principle and studied the existence, uniqueness, and Ulam stability of fixed point theorems. See [[2], [3], [4], [5], [6], [7], [8], and [9]] and references cited therein.

Following the foundational work of Karapnar and Fulga [10], who introduced the concept of supermetric spaces, the area has garnered significant interest among researchers in nonlinear analysis. Their formulation generalizes classical metric notions and has inspired a series of studies exploring various types of contractive conditions within this new setting. For instance, Shahi et al. [11] examined generalized fixed point results in supermetric spaces, while Hooda et al. [12] extended the theory by introducing common fixed point results using control functions and compatibility conditions. Similarly, Abodayeh et al. [13] advanced the theory through the development of Hardy-Rogers-type and rational $\mathcal{Z}$-contraction mappings in ordered supermetric spaces. Other notable contributions include the works of Shah et al. [14] and Saxena and Gairola [15], which demonstrate the behavior of supermetric space theory to different contractions.

Despite these substantial advances, much of the existing literature has focused on deterministic settings. A promising and relatively unexplored direction is the extension of supermetric structures to fuzzy metric spaces, where uncertainty and imprecision are inherently present. Integrating the notion of supermetrics with the fuzzy framework will provide new avenues for addressing problems in applied mathematics, information theory, and decision sciences, particularly where data is imprecise or probabilistic in nature. Thus, this work offers a novel direction by extending existing theories into fuzzy supermetric contexts, laying a foundational bridge between fixed point theory, fuzzy analysis, and fractional modeling under uncertainty.

After Zadeh investigated the fuzzy set concepts, a lot of researchers introduced the theory of a fixed point corresponding to fuzzy metric spaces for further research ([16], [17], [18], [19], [20]). The study of a fuzzy metric space permits the distance between two points to be a value in the range [0, 1], allowing for a more flexible and partitioned representation of distance than the study of an ordinary metric space, where the distance between two points is always a real number.

In recent years, significant attention has been given to the development of mathematical frameworks that better capture the uncertainty and complexity inherent in real-world phenomena. This is particularly evident in the modeling of epidemic dynamics like HIV, COVID-19, malaria, and Ebola virus, where classical deterministic models often fall short in representing the vagueness of transmission rates, recovery parameters, and public response behaviors. For details, see [21], [22], [23], [24], [25], and [26].

In this work, a new mathematical concept that extends the novel of supermetric space into fuzzy supermetric spaces is defined. Contractive mappings are developed and utilized to prove fixed point theorems in the framework of fuzzy supermetric space, which extends classical and recent results as special cases within a broader, more inclusive setting and provides a more suitable framework for modeling and solving real-life problems in physics and engineering, especially in systems characterized by nonlinearity, complex interactions, and uncertainty. In physics, they are particularly useful in modeling complex systems with inherent ambiguity, such as quantum systems, where precise state measurements are often uncertain. In engineering, fuzzy supermetric spaces are applied in control systems, robotics, and signal processing to handle imprecise sensor inputs, noise, and tolerance variations, enabling more robust and flexible designs. In the context of nonlinear analysis, fuzzy supermetric spaces find significant applications in the investigation of nonlinear integral equations, differential inclusions, and boundary value problems characterized by fuzzy parameters. In particular, the major theoretical results are employed to establish the existence of a solutions to the Fredholm integral equation.

2. Preliminary

This section presents the necessary terms, useful definitions, and related concepts that support the subsequent developments in this paper.

Definition 2.1. [27] A metric m on a nonempty set H is a function defined on the Cartesian product $H \times H$ and taking its values in the set $\mathbb{R}^+$ of reals such that the following conditions are valid:

(m-1) $m(\kappa, \varsigma) \geq 0$ for each $\kappa, \varsigma \in H$;
(m-2) $m(\kappa, \varsigma) = 0$ if and only if $\kappa = \varsigma$, for each $\kappa, \varsigma \in H$;
(m-3) $m(\kappa, \varsigma) = m(\varsigma, \kappa)$, for each $\kappa, \varsigma \in H$;
(m-4) $m(\kappa, \xi) \leq m(\kappa, \varsigma) + m(\varsigma, \xi)$, for each $\kappa, \varsigma, \xi \in H$.

The above definition defines the metric m at an ordered pair (κ, ς) as the distance between κ and ς. An ordered pair (H, m) is known as a metric space.
Mustafa and Sims [28] defined the generalized metric space by reducing the tetrahedral inequality to the triangle inequality in $\mathbb{R}^3$ as below.

Definition 2.2. A mapping G expressed on a nonempty set H given by $G : H \times H \times H \to \mathbb{R}$ is known as a G-metric (generalized metric) for every $\kappa, \varsigma, \xi, \varrho \in H$ fulfilling the next conditions:

(G-1) $G(\kappa, \varsigma, \xi) \geq 0$;
(G-2) $G(\kappa, \varsigma, \xi) = 0$ if and only if $\kappa = \varsigma = \xi$;
(G-3) $G(\kappa, \varsigma, \xi) = G(p\{\kappa, \varsigma, \xi\})$, where a parameter p is a permutation function;
(G-4) $G(\kappa, \varsigma, \xi) + G(\xi, \xi, \varrho) \geq G(\kappa, \varsigma, \varrho)$.

Then, a pair given by (H, G) is known as a G-metric space.

Bakhtin [29] introduced the concept of b-metric spaces as a generalization of classical metric spaces by modifying the standard triangle inequality to accommodate a broader class of distance functions.

Definition 2.3. The triplet (H, m, b) is known as a b-metric space for $H(\neq \emptyset)$ and $b \geq 1$ is a given real number. A function $m : H \times H \to [0, \infty)$ is a b-metric on H if the next conditions exist:

(B-1) $m(\kappa, \varsigma) = 0$ if and only if $\kappa = \varsigma$, for any $\kappa, \varsigma \in H$;
(B-2) $m(\kappa, \varsigma) = m(\varsigma, \kappa)$ for any $\kappa, \varsigma \in H$;
(B-3) $m(\kappa, \xi) \leq b[m(\kappa, \varsigma) + m(\varsigma, \xi)]$ for any $\kappa, \varsigma, \xi \in H$.

Remark 2.4. Setting $t = 1$ in the B-3 yields the definition of a usual metric space.

Karapinar and Fulga [10] announced a new notion of supermetric as below.

Definition 2.5. Assume that H is a nonempty set and $\delta : H \times H \to [1, \infty)$ is given. The function δ is called a supermetric if it fulfills the next conditions:

(S-1) If $\delta(\kappa, \varsigma) = 0$, then $\kappa = \varsigma$ for any $\kappa, \varsigma \in H$;
(S-2) For all $\kappa, \varsigma \in H, \delta(\kappa, \varsigma) = \delta(\varsigma, \kappa)$;
(S-3) For every $\varsigma \in H$, there are distinct sequences $\{\kappa_n\}, \{\varsigma_n\} \in H$, with $\delta(\kappa_n, \varsigma_n)$ going to zero when n approaches infinity, such that there exists $\alpha \geq 1$,

$$\lim_{n\to\infty} \sup \delta(\varsigma_n, \varsigma) \leq \alpha \lim_{n\to\infty} \sup \delta(\kappa_n, \varsigma). \tag{2.1}$$

The triplet (H, δ, α) is called a supermetric space.

Karapinar et al.[10] defined the ideas of Cauchy sequence, convergence, and completeness concerning a supermetric space as follows:

Definition 2.6. Suppose the triplet (H, δ, α) is a supermetric space. Then,

(i) a sequence $\{\kappa_n\}$ is called a convergent in H if and only if for all $\kappa \in H$, such that

$$\lim_{n\to\infty} \delta(\kappa_n, \kappa) = 0; \tag{2.2}$$

(ii) a sequence $\{\kappa_n\}$ is known as Cauchy sequence for every $\kappa \in H$ if and only if

$$\lim_{n\to\infty} \sup\{\delta(\kappa_n, \kappa_p) : p > n\} = 0. \tag{2.3}$$

Remark 2.7. In a supermetric space, the limit of a convergent sequence is unique.

Definition 2.8. [10] A supermetric space (H, δ, α) is said to be complete if and only if every Cauchy sequence in H converges to a point in H.

George and Veeramani [18] redefined the concept of fuzzy metric spaces, originally introduced by Kramosil and Michlek [27], by incorporating a triangular norm, as described below.

Definition 2.9. A triplet $(H, F, \diamondsuit)$ of any arbitrary set $H \neq \emptyset$, a continuous triangular norm $\diamondsuit$ with a fuzzy set F on $H \times H \times (0, \infty)$ is known as "fuzzy metric spaces" fulfilling the next situations for every $\kappa, \varsigma, \xi, \varrho$ in H and all $s, t > 0$:

(GV-1) $F(\kappa, \varsigma, s) > 0$;
(GV-2) $F(\kappa, \varsigma, s) = 1$ if and only if $\kappa = \varsigma$;
(GV-3) $F(\varsigma, \kappa, s) = F(\kappa, \varsigma, s)$;
(GV-4) $F(\kappa, \xi, s + t) \geq F(\kappa, \varsigma, s) \diamondsuit F(\varsigma, \xi, t)$;
(GV-5) $F(\kappa, \varsigma, \cdot) : [0, \infty) \longrightarrow [0, 1]$ is continuous.

Definition 2.10. [18] Consider $\{\kappa_n\}$ to be a sequence in any arbitrary nonempty set H and $(H, F, \diamondsuit)$ be "a fuzzy metric space." Next,

(1) For any κ in H and all $s > 0$, $\lim\limits_{n\to+\infty} F(\kappa_n, \kappa, s) = 1$, implies a sequence $\{\kappa_n\}$ converges to κ.
(2) For each $s > 0$ and $\epsilon \in (0, 1)$, there exists a positive integer n_0 such that $F(\kappa_n, \kappa_m, s) > 1 - \epsilon$, which implies that a sequence $\{\kappa_n\}$ is a Cauchy sequence for all $m, n \geq n_0$.
(3) A complete fuzzy metric space is a fuzzy metric space where all Cauchy sequences converge.

3. Main Results

This section introduces fuzzy supermetric spaces as an extension of classical supermetric structures within fuzzy set theory. An illustrative example supports the definition, followed by the development of suitable contractive mappings to establish fixed point theorems. The results generalize existing theorems and provide a unified framework for addressing problems under uncertainty.

Definition 3.1. A quadruple $(H, F, \diamondsuit, \delta)$ of any arbitrary set $H \neq \emptyset$, $\diamondsuit$ a continuous triangular norm, a real number $\delta \geq 1$ with a fuzzy set F on $H^2 \times (0, \infty)$ is called a fuzzy supermetric space if the next conditions are fulfilled for each $\kappa, \varsigma, \xi \in H$ and all $s, t > 0$:

(FSMS-1): $F(\kappa, \varsigma, s) > 0$;
(FSMS-2): $F(\kappa, \varsigma, s) = 1$, if and only if $\kappa = \varsigma$;
(FSMS-3): $F(\kappa, \varsigma, s) = F(\varsigma, \kappa, s)$;
(FSMS-4): $F(\kappa, \varsigma, s) \diamondsuit F(\varsigma, \xi, t) \leq F(\kappa, \xi, s + t)$;
(FSMS-5): There exist $\delta \geq 1$ such that for each $\kappa, \varsigma \in H$, there exist distinct sequences $\{\kappa_n\}, \{\varsigma_n\} \in H$, with $F(\kappa_n, \varsigma_n, s) \to 1$ when $n \to \infty$, $\lim_{n\to\infty} \sup F(\kappa_n, \kappa, s) \leq \delta \lim_{n\to\infty} \sup F(\varsigma_n, \kappa, s)$,
(FSMS-6): $F(\kappa, \varsigma, .) : [0, \infty) \to [0, 1]$ is continuous.

Remark 3.2. If the fifth condition in the above definition is omitted, we obtain GV-fuzzy metric spaces without the requirement that $\delta \geq 1$.

Remark 3.3. If the fifth condition is incorporated into the triangle inequality, our definition leads to fuzzy b-metric spaces.

Example 3.4. Consider $H = \mathbb{R}$ and defining a product continuous triangular norm $x \diamondsuit y = xy$, for each $\kappa, \varsigma, \xi \in H$ and all $s > 0$ by

$$F(\kappa, \varsigma, s) = \frac{1}{exp\Big(\frac{|\kappa - \varsigma|}{s}\Big)}.$$

Then $(H, F, \diamondsuit, \delta)$ is a fuzzy supermetric space.

Proof. We can demonstrate that:

1: $F(\kappa, \varsigma, s) = \dfrac{1}{exp\Big(\frac{|\kappa - \varsigma|}{s}\Big)} > 0$ as $0 < |\kappa - \varsigma|$.

2: $F(\kappa, \varsigma, s) = \dfrac{1}{exp\Big(\frac{|\kappa - \varsigma|}{s}\Big)} = 1$ if and only if $\kappa = \varsigma$.

3: $F(\kappa, \varsigma, s) = \dfrac{1}{exp\Big(\frac{|\kappa - \varsigma|}{s}\Big)} = \dfrac{1}{exp\Big(\frac{|\varsigma - \kappa|}{s}\Big)} = F(\varsigma, \kappa, s)$.

4: $F(\xi, \kappa, s + t) \geq F(\xi, \varsigma, s) \diamondsuit F(\varsigma, \kappa, t)$, from the fact

$$|\xi - \kappa| \leq \Big(\frac{s + t}{s}\Big)|\xi - \varsigma| + \Big(\frac{t + s}{t}\Big)|\varsigma - \kappa|,$$

we can drive

$$\frac{|\xi - \kappa|}{t + s} \leq \frac{|\xi - \varsigma|}{s} + \frac{|\varsigma - \kappa|}{t}.$$

Taking the exponents on both sides, we obtain,

$$exp\Big(\frac{|\xi-\kappa|}{s+t}\Big) \le exp\Big(\frac{|\xi-\varsigma|}{s}\Big)exp\Big(\frac{|\varsigma-\kappa|}{t}\Big)$$

and we have,

$$\frac{1}{exp\Big(\frac{|\xi-\kappa|}{s+t}\Big)} \ge \left(\frac{1}{exp\Big(\frac{|\xi-\varsigma|}{s}\Big)}\right)\left(\frac{1}{exp\Big(\frac{|\varsigma-\kappa|}{t}\Big)}\right).$$

Thus $F(\xi,\kappa,s+t) \ge F(\xi,\varsigma,s)\diamondsuit F(\varsigma,\kappa,t)$ for each $\kappa,\xi,\varsigma \in H$ and all $s,t>0$.

5: If there exist $\delta \ge 1$ such that for each $\kappa,\varsigma \in H$, then we have a distinct sequences $\{\kappa_n\},\{\varsigma_n\} \in H$, with $F(\kappa_n,\varsigma_n,s) \to 1$ when n goes to ∞,

$$\lim_{n\to\infty}\sup\left(\frac{1}{exp\Big(\frac{|\kappa_n-\kappa|}{s}\Big)}\right) \le \delta \lim_{n\to\infty}\sup\left(\frac{1}{exp\Big(\frac{|\varsigma_n-\kappa|}{s}\Big)}\right).$$

6: $F(\kappa,\varsigma,.) : [0,\infty) \to [0,1]$ is continuous. Hence, $(H,F,\diamondsuit,\delta)$ is a fuzzy supermetric space.

Remark 3.5. Every fuzzy supermetric space can be a fuzzy metric space, the reverse is false.

Definition 3.6. Consider $(H,F,\diamondsuit,\delta)$ to be a fuzzy supermetric space. Then, F is known as triangular if the next condition holds:

$$\frac{1}{F(\kappa,\xi,s)} - 1 \le (\frac{1}{F(\kappa,\varsigma,s)} - 1) + (\frac{1}{F(\varsigma,\xi,s)} - 1), \tag{3.1}$$

for each κ,ς,ξ in H and all $s>0$.

Definition 3.7. A function $\theta : [0,1] \to [0,1]$ is known as a δ-nondecreasing for any real number $\delta \ge 1$, there exist $\kappa < \varsigma$ in the domain such that $\theta(\kappa) \le \theta(\delta\varsigma)$.

Lemma 3.8. *A function $F(\kappa,\varsigma,.) : [0,\infty) \longrightarrow [0,1]$ is a δ-nondecreasing for every $\kappa,\varsigma \in H$ and for any real number $\delta \ge 1$.*

Proof. Assuming that $s>0, t>0$, and $s>t$, we ensure

$$F(\kappa,\varsigma,\delta t) \ge F(\kappa,\kappa,t-s)\diamondsuit F(\kappa,\varsigma,s) = 1\diamondsuit F(\kappa,\varsigma,s) = F(\kappa,\varsigma,s).$$

Hence, $F(\kappa,\varsigma,s) \le F(\kappa,\varsigma,\delta t)$, that is a δ-nondecreasing.

Lemma 3.9. *If a quadruple $(H,F,\diamondsuit,\delta)$ is a fuzzy supermetric space, then there exists a δ-nondecreasing $F(\kappa,\varsigma,s)$ for any $\kappa,\varsigma,\kappa \in H$, all $s,t>0$.*

Proof. For each positive parameter s,t with $t>s$, then from (FSMS-4) of fuzzy supermetric space we have

$$F(\kappa,\xi,t)\diamondsuit F(\xi,\varsigma,s) \le F(\kappa,\varsigma,t+s).$$

And if there exist distinct sequences $\{\kappa_n\},\{\varsigma_n\}$ in H such that $F(\kappa_n,\varsigma_n,s) \to 1$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\sup F(\kappa_n,\kappa,s) \ge \delta \lim_{n\to\infty}\sup F(\varsigma_n,\kappa,s).$$

Hence, $F(\kappa,\varsigma,s)$ is a δ-nondecreasing.

Definition 3.10. Assume that $(H,F,\diamondsuit,\delta)$ is a fuzzy supermetric space. Next

a) A sequence $\{\kappa_n\}$ in a fuzzy supermetric space converges to an element κ in H if and only if $\lim_{n\to\infty} \sup F(\kappa_n, \kappa, s) = 1$.
b) If there exists $\epsilon \in (0, 1)$ such that $F(\kappa_n, \kappa_m, s) > 1 - \epsilon$, for each $\kappa \in H$, then a sequence $\{\kappa_n\}$ in a fuzzy supermetric space is said to be a Cauchy sequence for each $n, m \geq n_0$ where n_0 is a positive integer.
c) A fuzzy supermetric space $(H, F, \diamondsuit, \delta)$ is said to be complete if every Cauchy sequence $\{\kappa_n\}$ in H converges to a point κ in H.

Definition 3.11. Suppose that a quadruple $(H, F, \diamondsuit, \delta)$ is a fuzzy supermetric space. A function $T : H \to H$ is called a fuzzy contractive if any μ in (0, 1) such that

$$\frac{1}{F(T\kappa, T\varsigma, s)} - 1 \leq \mu\Big(\frac{1}{F(\kappa, \varsigma, s)} - 1\Big), \tag{3.2}$$

for each $\kappa, \varsigma \in H$ and $s > 0$.

Theorem 3.12. *Let $T : H \to H$ be a fuzzy contractive mapping, and suppose that the quadruple $(H, F, \diamondsuit, \delta)$ forms a complete fuzzy supermetric space. Then T admits a unique fixed point κ in H such that $T\kappa = \kappa$.*

Proof. Consider $\kappa_0 \in H$ and define an iterative $\kappa_n = T^n\kappa_0$ for $n \in N$. For each $s > 0$ and using Equation (3.2), we have

$$\begin{aligned}
\frac{1}{F(\kappa_n, \kappa_{n+1}, s)} - 1 = \frac{1}{F(T\kappa_{n-1}, T\kappa_n, s)} - 1 &\leq \mu(\frac{1}{F(\kappa_{n-1}, \kappa_n, s)} - 1) \\
&\leq \mu(\mu(\frac{1}{F(\kappa_{n-2}, \kappa_{n-1}, h)} - 1)) = \mu^2(\frac{1}{F(\kappa_{n-2}, \kappa_{n-1}, s)} - 1) \\
&\ \vdots \\
&\leq \mu^n(\frac{1}{F(\kappa_0, \kappa_1, s)} - 1).
\end{aligned}$$

By (FSMS-6), proceeding the limit as n goes to ∞ with the point $\lim_{n\to\infty} \mu^n(t) = 0$, where $t = \dfrac{1}{F(\kappa_0, \kappa, s)} - 1$, we get $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+1}, s) = 1$. Again, using Equations (3.1) and (3.2) for any positive $n, m \in N$ by means of $n < m$, we get

$$\begin{aligned}
\frac{1}{F(\kappa_n, \kappa_m, s)} - 1 &\leq \Big(\frac{1}{F(\kappa_n, \kappa_{n+1}, s)} - 1\Big) + \Big(\frac{1}{F(\kappa_{n+1}, \kappa_m, s)} - 1\Big) \\
&= \Big(\frac{1}{F(T\kappa_{n-1}, T\kappa_n, s)} - 1\Big) + \Big(\frac{1}{F(T\kappa_n, T\kappa_{m-1}, s)} - 1\Big) \\
&\leq \mu\Big[\Big(\frac{1}{F(\kappa_{n-1}, \kappa_n, s)} - 1\Big) + \Big(\frac{1}{F(\kappa_n, \kappa_{m-1}, s)} - 1\Big)\Big] \\
&\leq \mu\Big[\mu\Big[\Big(\frac{1}{F(\kappa_{n-2}, \kappa_{n-1}, s)} - 1\Big) + \Big(\frac{1}{F(\kappa_{n-1}, \kappa_{m-2}, s)} - 1\Big)\Big]\Big] \\
&\leq \cdots \leq \mu^n\Big[\Big(\frac{1}{F(\kappa_0, \kappa_1, s)} - 1\Big) + \Big(\frac{1}{F(\kappa_1, \kappa_2, s)} - 1\Big)\Big]
\end{aligned}$$

goes to 0 as n goes to infinity yields that $\lim_{n\to\infty} F(\kappa_n, \kappa_m, s) = 1$. As a result, the $\{\kappa_n\}$ sequence is Cauchy. Given the completeness of the specified fuzzy supermetric space, the Cauchy sequence $\{\kappa_n\}$ is guaranteed to converge. Let's say it converges to a point κ in

H such that $\kappa_n \to \kappa$. Using (FSMS-4) and all $s, t > 0$, we have

$$\begin{aligned} F(T\kappa, \kappa, s+t)) &\geq F(T\kappa, T\kappa_n, s) \diamondsuit F(\kappa_{n+1}, \kappa, t) \\ &\geq F(\kappa, \kappa_n, s) \diamondsuit F(\kappa_{n+1}, \kappa, t) \\ &\geq 1 \diamondsuit 1 = 1. \end{aligned}$$

as n goes to infinity. By (FSMS-2), $T\kappa = \kappa$ implies κ is a fixed point of T. To show uniqueness, suppose that $T\varsigma = \varsigma \neq \kappa = T\kappa$. For any positive s, we have

$$\begin{aligned} \frac{1}{F(\kappa, \varsigma, s)} - 1 = \frac{1}{F(T\kappa, T\varsigma, s)} - 1 &\leq \mu\Big(\frac{1}{F(\kappa, \varsigma, s)} - 1\Big) \\ &\leq \cdots \leq \mu^n\Big(\frac{1}{F(\kappa, \varsigma, s)} - 1\Big) \to 0, \end{aligned}$$

as n goes to infinity. Therefore, $F(\kappa, \varsigma, s) = 1$ implies $\kappa = \varsigma$.

Remark 3.13. The above theorem generalizes particular instances of the fuzzy Banach contraction theorem presented in [30], Theorem 1 of [31], and Theorem 4.4 of [32], by extending the framework from a fuzzy metric space to that of a fuzzy supermetric space.

Example 3.14. Let $H = \mathbb{R}$, and define the mapping $T : \mathbb{R} \to \mathbb{R}$ by

$$T(\kappa) = \frac{\kappa}{2}.$$

Define a fuzzy supermetric $F : \mathbb{R} \times \mathbb{R} \times (0, \infty) \to [0, 1]$ by

$$F(\kappa, \varsigma, s) = \frac{s}{s + |\kappa - \varsigma|},$$

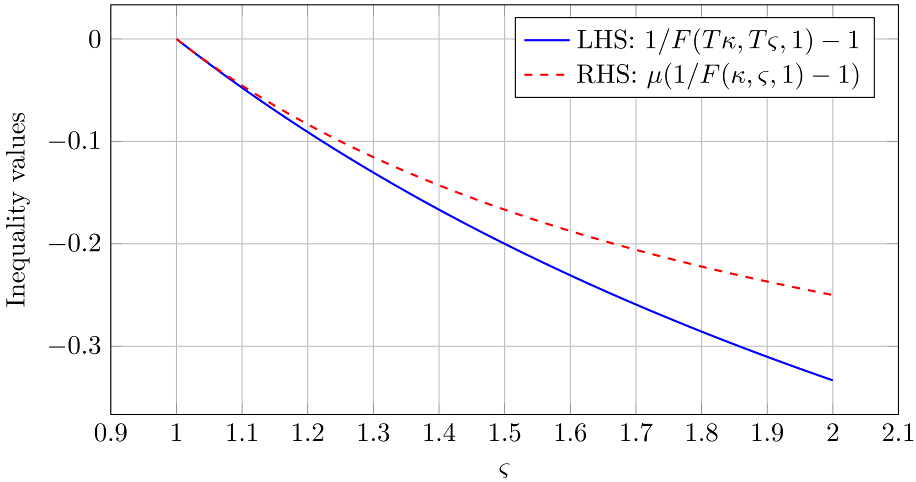
and $\diamondsuit$ is the product t-norm: $a \diamondsuit b = ab, \delta = 1$. Then $(H, F, \diamondsuit, \delta)$ is a fuzzy supermetric space.
To verify Theorem (3.12), first we verify that T is a fuzzy contractive mapping, that is, inequality (3.2) satisfied. Thus, there exists $\mu = 0.5 \in (0, 1)$ such that

$$\frac{1}{F(T\kappa, T\varsigma, s)} - 1 \leq \mu\left(\frac{1}{F(\kappa, \varsigma, s)} - 1\right).$$

Then taking $s = 1$, and sample values as follows

κ	ς	LHS: $\frac{1}{F(T\kappa, T\varsigma, 1)} - 1$	RHS: $\mu\left(\frac{1}{F(\kappa, \varsigma, 1)} - 1\right)$
1.0	2.0	≈ 0.500	$0.5 \cdot 1.0 = 0.500$
1.0	1.5	≈ 0.333	$0.5 \cdot 0.667 \approx 0.333$
1.0	1.25	≈ 0.200	$0.5 \cdot 0.400 = 0.200$
1.0	1.125	≈ 0.111	$0.5 \cdot 0.222 = 0.111$

This confirms the contractive inequality is satisfied for all $\kappa \neq \varsigma \in \mathbb{R}$, and T is fuzzy contractive. Now, we illustrate the left-hand side (LHS) and right-hand side (RHS) of the inequality (3.2) for $\kappa = 1, \varsigma \in [1, 2], \mu = 0.5$, and $s = 1$.

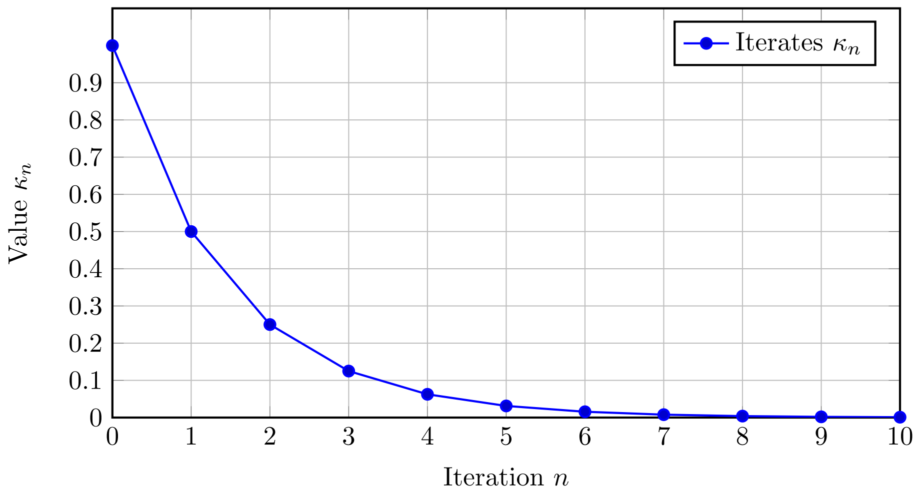


The blue curve (LHS) lies below or on the red curve (RHS), verifying the fuzzy contractive condition.
Finally, we demonstrate the fixed point approximation using the iterative process by

$$\kappa_{n+1} = T(\kappa_n) = \frac{\kappa_n}{2},$$

starting with $\kappa_0 = 1$, we generate the sequence

$$\kappa_0 = 1, \quad \kappa_1 = 0.5, \quad \kappa_2 = 0.25, \quad \kappa_3 = 0.125, \quad \ldots$$



Hence, all the assumptions of Theorem (3.12) are satisfied. Since $(\mathbb{R}, F, \cdot, 1)$ is a complete fuzzy supermetric space, we conclude that T has a unique fixed point, i.e., $\kappa^* = 0$.

4. A $(\beta - \theta)-$ Contractive Mappings for Fuzzy Supermetric Space

The $(\beta - \theta)-$ contractive mapping notion was announced by Samet et al. [33], who subsequently developed fixed point theorems for it. Consider Θ to be the family of all non-decreasing functions. A mapping from $[0, +\infty)$ to $[0, +\infty)$, where θ^n is the n^{th} iteration of θ such that for every $s > 0, \sum_{n=1}^{+\infty} \theta^n(s) < +\infty$.

Here, for each function $\theta : [0, +\infty) \to [0, +\infty)$, the next property holds: If θ is non-decreasing, then for each $s > 0$,

$$\lim_{n \to +\infty} \theta^n(s) = 0 \text{ implies } \theta(s) < s.$$

Definition 4.1. [33] Consider (H, m) to represent a metric space. If there are two functions $\beta : H \times H \to [0, +\infty)$ and $\theta \in \Theta$, then a mapping $T : H \to H$ is called $(\beta - \theta)$-contractive if

$$\beta(\kappa, \varsigma) m(T\kappa, T\varsigma) \leq \theta(m(\kappa, \varsigma)), \tag{4.1}$$

for all $\kappa, \varsigma \in H$.

Remark 4.2. Letting $\theta(s) = s$ and $\beta(\kappa, \varsigma) = 1$, definition (4.1) reduces to Banach Contraction for $0 \leq s < 1$.

Definition 4.3. [33] Let $\beta : H \times H \to [0, +\infty)$ and a function $T : H \to H$ be given. When $\kappa, \varsigma \in H$, then T is referred to as an β-admissible.

$$\beta(\kappa, \varsigma) \geq 1 \Rightarrow \beta(T\kappa, T\varsigma) \geq 1. \tag{4.2}$$

Example 4.4. [33] Consider $H = (0, +\infty)$. Define a functions: $T : H \to H$ and $\beta : H \times H \to [0, +\infty)$ as $T\kappa = \ln \kappa$ and $\beta(\kappa, \varsigma) = \begin{cases} 2, & \text{if } \kappa \geq \varsigma, \\ 0, & \text{if } \kappa < \varsigma, \end{cases}$ for each κ, ς in H. Then, a mapping T is β-admissible.

Samet et al. [33] formulated and proved a fixed point theorem pertaining to the notion of a $(\beta - \theta)$-contractive mapping, stated as follows:

Theorem 4.5. *Consider a pair (H, m) be a complete metric space. A mapping $T : H \to H$ be a $(\beta - \theta)$-contractive function satisfying the next conditions:*

(i) T is β-admissible;
(ii) there is one $\kappa_0 \in H$ such that $\beta(\kappa_0, T\kappa_0) \geq 1$;
(iii) T is continuous.

A mapping T has a fixed point, meaning that $T\kappa_0 = \kappa_0$ exists for all $\kappa_0 \in H$.

Karapnar [34] presented fixed point theorems for generalized metric spaces based on (β, θ)-contractive mappings, as outlined below.

Theorem 4.6. *Assume that a pair (H, m) represents a complete generalized metric space. Assume that a function $T : H \to H$ is an $(\beta - \theta)$-contractive and*

(i) T is β-admissible;
(ii) there exists $\kappa_0 \in H$ so that $\beta(\kappa_0, T\kappa_0) \geq 1$ & $\beta(\kappa_0, T^2\kappa_0) \geq 1$;
(iii) T is continuous.

Then, a function T admits a fixed point in H.

Aydi et al. [35] extended the concept of $(\beta - \theta)$-contractive mappings within the framework of generalized metric spaces.

Definition 4.7. [35] Let (H, m) be a pair representing a generalized metric space. For every $\kappa, \varsigma \in H$, a self-mapping T is referred to as a generalized $(\beta - \theta)$-contractive mapping of type I, if two functions $\beta : H \times H \to [0, \infty), \theta \in \Theta$ exist such that

$$\beta(\kappa, \varsigma) m(T\kappa, T\varsigma) \leq \theta(N(\kappa, \varsigma)), \tag{4.3}$$

where

$$N(\kappa, \varsigma) = \max\{m(\kappa, \varsigma), m(\kappa, T\kappa), m(\varsigma, T\varsigma)\}. \tag{4.4}$$

Definition 4.8. [35] Let (H, m) be a pair representing a generalized metric space. For any $\kappa, \varsigma \in H$, a self-mapping T is referred to as a generalized $(\beta - \theta)$-contractive mapping of type II, if two functions $\beta : H \times H \to [0, \infty), \theta \in \Theta$ exist such that

$$\beta(\kappa, \varsigma) m(T\kappa, T\varsigma) \leq \theta(D(\kappa, \varsigma)), \tag{4.5}$$

where

$$D(\kappa, \varsigma) = \max\{m(\kappa, \varsigma), \frac{m(\kappa, T\kappa) + m(\varsigma, T\varsigma)}{2}\}. \tag{4.6}$$

Theorem 4.9. [35] *Consider a pair (H, m) to be "a complete generalized metric space" and $T : H \to H$ be a generalized type I $(\beta - \theta)$-contractive mapping. Suppose that*

(i) T is β-admissible;
(ii) there exist $\kappa_0 \in H$ such that $\beta(\kappa_0, T\kappa_0) \geq 1, \beta(\kappa_0, T^2\kappa_0) \geq 1$;
(iii) T is continuous.

Then, T has a fixed point $\kappa \in H$.

By initiating with the works of Aydi et al. [35], Karapnar [34], and Samet et al. [33], we defined the $(\beta - \theta)$-contractive mappings for fuzzy supermetric space. Let Θ be the set of all functions $\theta : (0, 1] \to (0, 1]$ satisfying the following properties:

(i) θ is nondecreasing on $(0, 1]$;
(ii) $\theta(t) \leq t$ for all $t \in (0, 1]$ (i.e., θ is a generalized contraction);
(iii) $\lim_{n\to\infty} \theta^n(t) = 0$ for all $t \in (0, 1)$ (i.e., the iterates $\theta^n(t)$ converge to zero);
(iv) $\theta(1) < 1$.

Definition 4.10. Consider $T : H \to H$ to be a mapping and a quadruple $(H, F, \diamondsuit, \delta)$ to be a fuzzy supermetric space. If there are two functions $\beta : H \times H \times (0, +\infty) \to [0, 1]$ and $\theta \in \Theta$, then T is called a $(\beta - \theta)$-contractive mapping of the first type such that

$$\beta(\kappa, \varsigma, s) F(T\kappa, T\varsigma, s) \geq \theta(M(\kappa, \varsigma, s)), \tag{4.7}$$

where

$$M(\kappa, \varsigma, s) = \max\{F(\kappa, \varsigma, s), F(\kappa, T\kappa, s), F(\varsigma, T\varsigma, s)\},$$

for each $\kappa, \varsigma \in H$ and all $s > 0$.

Building on Definition 4.3 from [33, Definition 2.2], we introduce the notion of β-admissibility in the context of fuzzy supermetric spaces as follows.

Definition 4.11. Consider $T : H \to H$ and $\beta : H \times H \times [0, +\infty) \to [0, +\infty)$ to be functions. For each $\kappa, \varsigma \in H$ and $s > 0$, we say that T is β-admissible if

$$\beta(\kappa, \varsigma, s) \geq 1 \Rightarrow \beta(T\kappa, T\varsigma, s) \geq 1. \tag{4.8}$$

Theorem 4.12. *Let $(H, F, \diamondsuit, \delta)$ be a complete fuzzy supermetric space. Consider $T : H \to H$ be $(\beta - \theta)$-contractive mapping of the first type. Assume that*

(a) T is a β-admissible;
(b) there is one $\kappa_0 \in H$ such that $\beta(\kappa_0, T\kappa_0, s) \geq 1$ and $\beta(\kappa_0, T^2\kappa_0, s) \geq 1$;
(c) T is continuous.

Then, for all $\kappa, \varsigma \in Fix(T)$, a mapping T has a unique fixed point such that $\beta(\kappa, \varsigma, s) \geq 1$, where the set of fixed points of T is denoted by $Fix(T)$.

Proof. By assumption (b), there exists $\kappa_0 \in H$ such that $\beta(\kappa_0, T\kappa_0, s) \geq 1$ and $\beta(\kappa_0, T^2\kappa_0, s) \geq 1$ for any $s > 0$, thereby satisfying the initial condition required to guarantee the existence of a fixed point. Defining an iterative sequence $\{\kappa_n\}$ in H by

$$\kappa_{n+1} = T\kappa_n = T^{n+1}\kappa_0, \text{ for any } n \text{ in } N.$$

If we have $\kappa_{n_0} = \kappa_{n_0+1}$ for some n_0, then $\kappa_n = \kappa_{n_0}$ is a fixed point of T. Therefore, in the course of the proof, we consider that

$$\kappa_n \neq \kappa_{n+1}. \tag{4.9}$$

By assumption a), since T is a β-admissible, we have

$$\beta(\kappa_0, \kappa_1, s) = \beta(\kappa_0, T\kappa_0, s) \geq 1 \Rightarrow \beta(T\kappa_0, T^2\kappa_0, s) = \beta(\kappa_1, \kappa_2, s) \geq 1. \tag{4.10}$$

Recursively, we obtain,

$$\beta(\kappa_n, \kappa_{n+1}, s) \geq 1 \text{ for each } n = 0, 1, 2, \cdots. \tag{4.11}$$

And in parallel, we determines that

$$\beta(\kappa_0, \kappa_2, s) = \beta(\kappa_0, T^2\kappa_0, s) \geq 1 \Rightarrow \beta(T\kappa_0, T^3\kappa_0, s) = \beta(\kappa_1, \kappa_3, s) \geq 1. \tag{4.12}$$

Iteratively, we obtain

$$\beta(\kappa_n, \kappa_{n+2}, s) \geq 1 \text{ for each } n = 0, 1, 2, \cdots. \tag{4.13}$$

Now, to prove the Cauchy sequences, we need to show the following cases to be valid. These are

i) $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+1}, s) = 1.$

As the given mapping is $(\beta - \theta)$-contractive mapping, with Equation (4.11), we realize that

$$\begin{aligned} F(\kappa_n, \kappa_{n+1}, s) &= F(T\kappa_{n-1}, T\kappa_n, s) \\ &\geq \beta(\kappa_{n-1}, \kappa_n, s) F(T\kappa_{n-1}, T\kappa_n, s) \\ &\geq \theta(M(\kappa_{n-1}, \kappa_n, s)), \end{aligned} \tag{4.14}$$

for each $n \geq 1$, where

$$\begin{aligned} M(\kappa_{n-1}, \kappa_n, s) &= \max\{F(\kappa_{n-1}, \kappa_n, s), F(\kappa_{n-1}, T\kappa_{n-1}, s), F(\kappa_n, T\kappa_n, s)\} \\ &= \max\{F(\kappa_{n-1}, \kappa_n, s), F(\kappa_{n-1}, \kappa_n, s), F(\kappa_n, \kappa_{n+1}, s)\} \\ &= \max\{F(\kappa_{n-1}, \kappa_n, s), F(\kappa_n, \kappa_{n+1}, s)\}. \end{aligned} \tag{4.15}$$

Now, if $M(\kappa_{n-1}, \kappa_n, s) = F(\kappa_n, \kappa_{n+1}, s)(\neq 0)$, then Equation (4.14) becomes

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta(M(\kappa_{n-1}, \kappa_n, s)) = \theta(F(\kappa_n, \kappa_{n+1}, s)) < F(\kappa_n, \kappa_{n+1}, s),$$

which is a contradiction. Hence, $M(\kappa_{n-1}, \kappa_n, s) = F(\kappa_{n-1}, \kappa_n, s)$ for each $n \in N$ and Equation (4.14) becomes

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta(F(\kappa_{n-1}, \kappa_n, s)) \Rightarrow F(\kappa_n, \kappa_{n+1}, s) < F(\kappa_{n-1}, \kappa_n, s). \tag{4.16}$$

Then, by Equation (4.14), we obtain that

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta^n(F(\kappa_0, \kappa_1, s)) \text{ for each } n \in N. \tag{4.17}$$

So that case (i) is proved by the property of θ, that is, $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+1}, s) = 1$. In the same fashion, we can prove **(ii)**. That is, $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+2}, s) = 1$.

iii) We aim to show that $\lim_{n,m\to\infty} F(\kappa_n, \kappa_m, s) = 1$. Assume, for all $n \neq m$, that

$$\kappa_n \neq \kappa_m.$$

To proceed, we employ a proof by contradiction: Suppose there exist n, m in N with $n \neq m$ such that $\kappa_n = \kappa_m$. Since $F(\kappa_q, \kappa_{q+1}, s) > 0$ for each $q \in N$, without loss of generality, we suppose that $m > n+1$. Having $m > n$, we can derive $\kappa_m = T^{m-n}(T^n\kappa_0) = T^n\kappa_0 = \kappa_n$. Now, we can consider that

$$\begin{aligned} F(\kappa_n, \kappa_{n+1}, s) &= F(\kappa_n, T\kappa_n, s) = F(\kappa_m, T\kappa_m, s) \\ &= F(T\kappa_{m-1}, T\kappa_m, s) \geq \beta(\kappa_{m-1}, \kappa_m, s)F(T\kappa_{m-1}, T\kappa_m, s) \\ &\geq \theta(M(\kappa_{m-1}, \kappa_m, s)), \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} M(\kappa_{m-1}, \kappa_m, s) &= \max\{F(\kappa_{m-1}, \kappa_m, s), F(\kappa_{m-1}, T\kappa_{m-1}, s), F(\kappa_m, T\kappa_m, s)\} \\ &= \max\{F(\kappa_{m-1}, \kappa_m, s), F(\kappa_{m-1}, \kappa_m, s), F(\kappa_m, \kappa_{m+1}, s)\} \\ &= \max\{F(\kappa_{m-1}, \kappa_m, s), F(\kappa_m, \kappa_{m+1}, s)\}. \end{aligned} \tag{4.19}$$

If $M(\kappa_{m-1}, \kappa_m, s) = F(\kappa_{m-1}, \kappa_m, s)$, then Equation (4.18) becomes

$$\begin{aligned} F(\kappa_n, \kappa_{n+1}, s) &= F(\kappa_n, T\kappa_n, s) = F(\kappa_m, T\kappa_m, s) \\ &= F(\kappa_m, \kappa_{m+1}, s) \geq \beta(\kappa_m, \kappa_{m+1}, s)F(T\kappa_{m-1}, T\kappa_m, s) \\ &\geq \theta(M(\kappa_{m-1}, \kappa_m, s)) = \theta(F(\kappa_{m-1}, \kappa_m, s)) \\ &\geq \theta^{m-n}(F(\kappa_n, \kappa_{n+1}, s)). \end{aligned} \tag{4.20}$$

Using the property of θ, inequality (4.20) yields

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta^{m-n}(F(\kappa_n, \kappa_{n+1}, s)) < F(\kappa_n, \kappa_{n+1}, s), \tag{4.21}$$

which is a contradiction. Again, if $M(\kappa_{m-1}, \kappa_m, s) = F(\kappa_m, \kappa_{m+1}, s)$, then Equation (4.18) becomes

$$\begin{aligned} F(\kappa_n, \kappa_{n+1}, s) &= F(\kappa_n, T\kappa_n, s) = F(\kappa_m, T\kappa_m, s) \\ &= F(T\kappa_{m-1}, T\kappa_m, s) \geq \beta(\kappa_{m-1}, \kappa_m, s)F(T\kappa_{m-1}, T\kappa_m, s) \\ &\geq \theta(M(\kappa_{m-1}, \kappa_m, s)) = \theta(F(\kappa_m, \kappa_{m+1}, s)) \\ &\geq \theta^{m-n+1}(F(\kappa_n, \kappa_{n+1}, s)). \end{aligned} \tag{4.22}$$

By the property of θ, inequality (4.22) gives that

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta^{m-n+1}(F(\kappa_n, \kappa_{n+1}, s)) < F(\kappa_n, \kappa_{n+1}, s), \tag{4.23}$$

which is a contradiction. So, $\lim_{n,m\to\infty} F(\kappa_n, \kappa_m, s) = 1$.

iv) Now, if we prove $\lim_{n,m\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1$, then we confirm that $\{\kappa_n\}$ is a Cauchy sequence. We must consider two scenarios to achieve this.

Case 1: Suppose that $m = 2q+1$ where $q \geq 1, q \in N$. Then, using the triangle inequality repeatedly and letting $s_0, s_1, \cdots, s_{2q} > 0$ such that $\sum_{i=0}^{2q} s_i = s$, we have

$$\begin{aligned} F(\kappa_n, \kappa_{n+m}, s) &= F(\kappa_n, \kappa_{n+2q+1}, s) \\ &\geq F(\kappa_n, \kappa_{n+1}, s_0) \diamondsuit F(\kappa_{n+1}, \kappa_{n+2}, s_1) \diamondsuit \ldots \diamondsuit F(\kappa_{n+2q}, \kappa_{n+2q+1}, s_{2q}). \end{aligned} \tag{4.24}$$

Each term in the product satisfies $F(\kappa_i, \kappa_{i+1}, s_j) \to 1$ as $i \to \infty$. Hence, for large enough n, we have

$$F(\kappa_n, \kappa_{n+m}, s) \geq \prod_{j=0}^{2q} F(\kappa_{n+j}, \kappa_{n+j+1}, s_j) \to 1. \tag{4.25}$$

Therefore,

$$\lim_{n\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1.$$

Case 2: Assume that $m = 2q$ where $q \geq 1, q \in N$. Then, using the triangle inequality repetitively and choose $s_0, s_1, \cdots, s_{2q-1} > 0$ such that $\sum_{i=0}^{2q-1} s_i = s$. Then

$$\begin{aligned} F(\kappa_n, \kappa_{n+m}, s) &= F(\kappa_n, \kappa_{n+2q}, s) \\ &\geq F(\kappa_n, \kappa_{n+1}, s_0) \diamondsuit F(\kappa_{n+1}, \kappa_{n+2}, s_1) \diamondsuit \ldots \diamondsuit F(\kappa_{n+2q-1}, \kappa_{n+2q}, s_{2q-1}). \end{aligned} \tag{4.26}$$

Again, as $n \to \infty$, each term tends to 1, and by the continuity of the t-norm $\diamondsuit$, we get

$$\lim_{n\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1.$$

Since both even and odd cases yield the same result, we conclude that

$$\lim_{n,m\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1, \text{ for each } s > 0,$$

which implies that $\{\kappa_n\}$ is a Cauchy sequence. Since $(H, F, \diamondsuit, \delta)$ is a complete fuzzy supermetric space, there is a point κ in H such that,

$$\lim_{n\to\infty} F(\kappa_n, \kappa, s) = 1. \tag{4.27}$$

And, by assumption c), since T is continuous, we get from Equation (4.27)

$$\lim_{n\to\infty} F(\kappa_{n+1}, T\kappa, s) = \lim_{n\to\infty} F(T\kappa_n, T\kappa, s) = 1. \tag{4.28}$$

That is, $T\kappa = \kappa$ indicates that κ is a fixed point of T, as $\lim_{n\to\infty} \kappa_{n+1} = \kappa$, suggests. To show uniqueness, consider $T\kappa = \kappa \neq \varsigma = T\varsigma$ for each $\kappa, \varsigma \in H$. Then, we have

$$\begin{aligned} F(\kappa, \varsigma, s) &\geq \beta(\kappa, \varsigma, s) F(\kappa, \varsigma, s) = \beta(T\kappa, T\varsigma, s) F(T\kappa, T\varsigma, s) \\ &\geq \theta(M(\kappa, \varsigma, s)) = \theta(\max\{F(\kappa, \varsigma, s), F(\kappa, T\kappa, s), F(\varsigma, T\varsigma, s)\}) \\ &= \theta(F(\kappa, \varsigma, s)) < F(\kappa, \varsigma, s), \end{aligned} \tag{4.29}$$

which is a contradiction. Therefore, $\kappa = \varsigma$.

Remark 4.13. The foregoing theorem extends a particular case of Theorem 15 in [35] by generalizing the framework from a generalized metric space to that of a fuzzy supermetric spaces.

Definition 4.14. Let $T : H \to H$ be a mapping and let $(H, F, \diamondsuit, \delta)$ be a fuzzy supermetric space. Suppose there exist two functions $\beta : H \times H \times (0, +\infty) \to [0, 1]$ and $\theta \in \Theta$, then T is called a $(\beta - \theta)$-contractive mapping of the second type such that

$$\beta(\kappa, \varsigma, s)F(T\kappa, T\varsigma, s) \geq \theta(N(\kappa, \varsigma, s)), \tag{4.30}$$

where

$$N(\kappa, \varsigma, s) = \max\Big\{F(\kappa, \varsigma, s), \frac{F(\kappa, T\kappa, s)\diamondsuit F(\varsigma, T\varsigma, s)}{2}\Big\},$$

for each $\kappa, \varsigma \in H$. and all $s > 0$.

Theorem 4.15. *Let $(H, F, \diamondsuit, \delta)$ be a complete fuzzy supermetric space, and let $T : H \to H$ be a $(\beta - \theta)$-contractive mapping of the second type. Suppose that the following conditions hold:*

(a) T is a β-admissible;
(b) there is one $\kappa_0 \in H$ such that $\beta(\kappa_0, T\kappa_0, s) \geq 1$ and $\beta(\kappa_0, T^2\kappa_0, s) \geq 1$;
(c) T is continuous.

Then, for all $\kappa_0, \kappa_1 \in Fix(T)$, a mapping T has a unique fixed point such that $\beta(\kappa_0, \kappa_1, s) \geq 1$, where all the set of fixed points of T is denoted by $Fix(T)$.

Proof. By utilizing assumption (b), we aim to establish the existence of a fixed point. For any $\kappa_0 \in H$, this assumption guarantees that

$$\beta(\kappa_0, T\kappa_0, s) \geq 1 \quad \text{and} \quad \beta(\kappa_0, T^2\kappa_0, s) \geq 1 \quad \text{for all } s > 0.$$

Define an iterative sequence $\{\kappa_n\}$ in H by

$$\kappa_{n+1} = T\kappa_n = T^{n+1}\kappa_0, \quad \text{for each } n \geq 0. \tag{4.31}$$

The proof is complete if there exists some $n \in \mathbb{N}$ such that $\kappa_n = \kappa_{n+1}$, which implies that κ_n is a fixed point of T. Suppose that $\kappa_n \neq \kappa_{n+1}$ for each $n = 0, 1, 2, \cdots$.
By assumption (a), since T is a β-admissible, it helps to obtain

$$\beta(\kappa_0, \kappa_1, s) = \beta(\kappa_0, T\kappa_0, s) \geq 1 \Rightarrow \beta(T\kappa_0, T^2\kappa_0, s) = \beta(\kappa_1, \kappa_2, s) \geq 1. \tag{4.32}$$

Inductively, we can drive

$$\beta(\kappa_n, \kappa_{n+1}, s) \geq 1 \quad \text{for each } n = 0, 1, 2, \cdots . \tag{4.33}$$

Analogously, we realize that

$$\beta(\kappa_0, \kappa_2, s) = \beta(\kappa_0, T^2\kappa_0, s) \geq 1 \Rightarrow \beta(T\kappa_0, T^3\kappa_0, s) = \beta(\kappa_1, \kappa_3, s) \geq 1. \tag{4.34}$$

Iteratively, we obtain

$$\beta(\kappa_n, \kappa_{n+2}, s) \geq 1 \quad \text{for each } n = 0, 1, 2, \cdots . \tag{4.35}$$

To demonstrate that the sequence $\{\kappa_n\}$ in H is Cauchy, we proceed with the following steps.
Step 1: Show that $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+1}, s) = 1$.
Since T is $(\beta-\theta)$-contractive of the second type, we apply the condition

$$\beta(\kappa_{n-1}, \kappa_n, s)F(\kappa_n, \kappa_{n+1}, s) \geq \theta(N(\kappa_{n-1}, \kappa_n, s)), \tag{4.36}$$

where

$$N(\kappa_{n-1}, \kappa_n, s) = \max\left\{F(\kappa_{n-1}, \kappa_n, s), \frac{F(\kappa_{n-1}, \kappa_n, s)\diamondsuit F(\kappa_n, \kappa_{n+1}, s)}{2}\right\}. \tag{4.37}$$

Since $\beta(\kappa_{n-1}, \kappa_n, s) \geq 1$, we obtain

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta(N(\kappa_{n-1}, \kappa_n, s)). \tag{4.38}$$

Now, suppose that $N(\kappa_{n-1}, \kappa_n, s) = F(\kappa_n, \kappa_{n+1}, s)$, then we get

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta(F(\kappa_n, \kappa_{n+1}, s)), \tag{4.39}$$

which contradicts the properties of θ unless $F(\kappa_n, \kappa_{n+1}, s) = 0$. Hence,

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta(F(\kappa_{n-1}, \kappa_n, s)) < F(\kappa_{n-1}, \kappa_n, s). \tag{4.40}$$

Therefore, the sequence $\{F(\kappa_n, \kappa_{n+1}, s)\}$ is nondecreasing and bounded below, implying the limit exists. Using iteration and the properties of θ (i.e., $\theta^n(s) \to 0$), we conclude

$$F(\kappa_n, \kappa_{n+1}, s) \geq \theta^n(F(\kappa_0, \kappa_1, s)) \to 0 \Rightarrow \lim_{n\to\infty} F(\kappa_n, \kappa_{n+1}, s) = 1. \tag{4.41}$$

Step 2: Show that $\lim_{n\to\infty} F(\kappa_n, \kappa_{n+2}, s) = 1$.
Similarly, we proceed as in Step 1 by employing

$$F(\kappa_n, \kappa_{n+2}, s) \geq \theta\left(\max\left\{F(\kappa_n, \kappa_{n+2}, s), \frac{F(\kappa_n, \kappa_{n+1}, s) \diamondsuit F(\kappa_{n+1}, \kappa_{n+2}, s)}{2}\right\}\right).$$

By contradiction, we deduce that

$$\lim_{n\to\infty} F(\kappa_n, \kappa_{n+2}, s) = 1. \tag{4.42}$$

Step 3: Show that $\lim_{n,m\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1$.
Now, we consider two (odd and even) cases for $m \geq 3$:
Case 1: Suppose that $m = 2q + 1$ for $q \in \mathbb{N}$. Let $s_0, \ldots, s_{2q} > 0$ such that $\sum_{i=0}^{2q} s_i = s$. Applying the triangle inequality, we obtain

$$F(\kappa_n, \kappa_{n+m}, s) \geq \prod_{i=0}^{2q} F(\kappa_{n+i}, \kappa_{n+i+1}, s_i) \to 1 \text{ as } n \to \infty. \tag{4.43}$$

Case 2: Suppose that $m = 2q$. Let $s_0, \ldots, s_{2q-1} > 0$ with $\sum_{i=0}^{2q-1} s_i = s$. Utilizing the triangle inequality, we deduce that

$$F(\kappa_n, \kappa_{n+m}, s) \geq \prod_{i=0}^{2q-1} F(\kappa_{n+i}, \kappa_{n+i+1}, s_i) \to 1 \text{ as } n \to \infty. \tag{4.44}$$

From inequality (4.43) and (4.44), we get that $\lim_{n,m\to\infty} F(\kappa_n, \kappa_{n+m}, s) = 1$. Hence, $\{\kappa_n\}$ is Cauchy, and by completeness of $(H, F, \diamondsuit, \delta)$, there exists $\kappa \in H$ such that

$$\lim_{n\to\infty} F(\kappa_n, \kappa, s) = 1. \tag{4.45}$$

To show that κ is a fixed point, from continuity of T (assumption c)), we get

$$\lim_{n\to\infty} F(T\kappa_n, T\kappa, s) = \lim_{n\to\infty} F(\kappa_{n+1}, T\kappa, s) = 1. \tag{4.46}$$

Hence, $T\kappa = \kappa$.
To show the uniqueness, assume that $\kappa, \varsigma \in Fix(T)$ with $\kappa \neq \varsigma$ and $\beta(\kappa, \varsigma, s) \geq 1$. Then, we have

$$F(\kappa, \varsigma, s) \geq \theta\left(\max\left\{F(\kappa, \varsigma, s), \frac{F(\kappa, T\kappa, s) \diamondsuit F(\varsigma, T\varsigma, s)}{2}\right\}\right). \tag{4.47}$$

Since $T\kappa = \kappa$ and $T\varsigma = \varsigma$, inequality (4.47) becomes

$$F(\kappa, \varsigma, s) \geq \theta(F(\kappa, \varsigma, s)) < F(\kappa, \varsigma, s), \tag{4.48}$$

a contradiction. Hence, $\kappa = \varsigma$.

Remark 4.16. The above theorem is an extension of a special case of Theorem 16 of [35], where the generalized metric space is replaced by fuzzy supermetric spaces.

Example 4.17. Let $H = \mathbb{R}$, and define the mapping $T : \mathbb{R} \to \mathbb{R}$ by

$$T(\kappa) = \frac{\kappa}{2}.$$

Define a fuzzy supermetric $F : \mathbb{R} \times \mathbb{R} \times (0, \infty) \to [0, 1]$ by

$$F(\kappa, \varsigma, s) = \frac{s}{s + |\kappa - \varsigma|},$$

and $\diamondsuit$ is the product t-norm: $a \diamondsuit b = ab$, with real number $\delta = 1$ and fix $\theta(s) = \frac{s}{2}$, $\beta(\kappa, \varsigma, s) = 1$ for all $\kappa, \varsigma \in \mathbb{R}, s > 0$. Then $(H, F, \diamondsuit, \delta)$ is a fuzzy supermetric space. Now, to verify Theorem 4.15 first we verify the contractive inequality (second type) (4.30). Let $\kappa = 1,\ \varsigma = 1.5,\ s = 1$. Then, we compute

$$\begin{aligned}
&T(\kappa) = 0.5, \quad T(\varsigma) = 0.75\\
&F(T\kappa, T\varsigma, s) = \frac{1}{1 + |0.5 - 0.75|} = \frac{1}{1.25} = 0.8\\
&F(\kappa, \varsigma, s) = \frac{1}{1 + 0.5} = 0.6667\\
&F(\kappa, T\kappa, s) = \frac{1}{1 + 0.5} = 0.6667, \quad F(\varsigma, T\varsigma, s) = \frac{1}{1 + 0.75} \approx 0.5714\\
&F(\kappa, T\kappa, s) \diamondsuit F(\varsigma, T\varsigma, s) = 0.6667 \cdot 0.5714 \approx 0.381 \Rightarrow \frac{0.381}{2} \approx 0.190
\end{aligned}$$

where

$$N(\kappa, \varsigma, s) = \max\{0.6667,\ 0.190\} = 0.6667, \quad \theta(N) = \frac{0.6667}{2} = 0.3333.$$

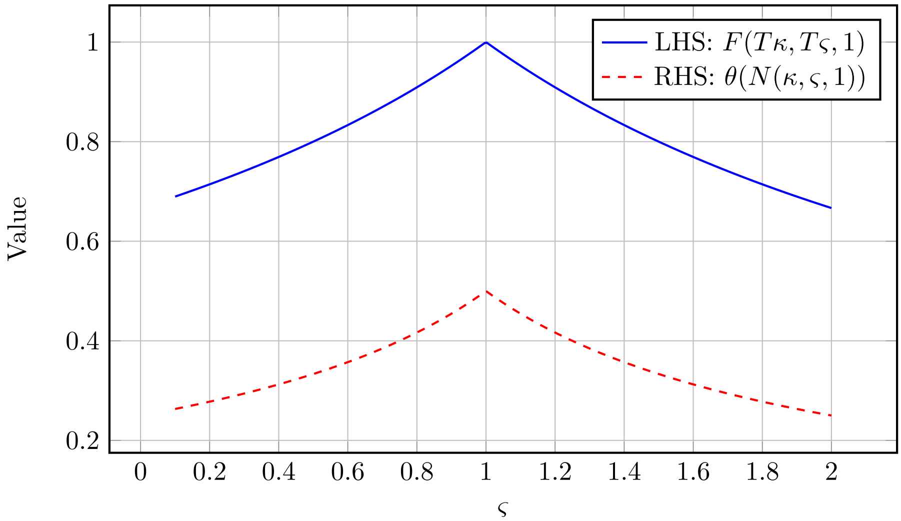
Hence, we obtain

$$\text{LHS} = \beta(\kappa, \varsigma, 1) \cdot F(T\kappa, T\varsigma, 1) = 1 \cdot 0.8 = 0.8 \geq 0.3333 = \text{RHS}.$$

Thus, contractive inequality (4.30) is satisfied. Next, we illustrate the comparison of LHS and RHS in the $(\beta{-}\theta)$-contractive inequality as

$$F(T\kappa, T\varsigma, 1) \geq \theta\left(N(\kappa, \varsigma, 1)\right)$$

for fixed $\kappa = 1, s = 1$, and varying $\varsigma \in [0.1, 2]$.



As observed, the blue curve (LHS) lies *above* the red curve (RHS), verifying the $(\beta-\theta)$-contractive condition (4.30). Then we verify conditions (a), (b), (c) of Theorem 4.15

(a) $\beta \equiv 1 \Rightarrow \beta$-admissibility holds trivially.
(b) Let $\kappa_0 = 1$. Then $\beta(\kappa_0, T(\kappa_0), 1) = 1, \quad \beta(\kappa_0, T^2(\kappa_0), 1) = 1$.
(c) $T(\kappa) = \dfrac{\kappa}{2}$ is continuous on $\mathbb{R}$.

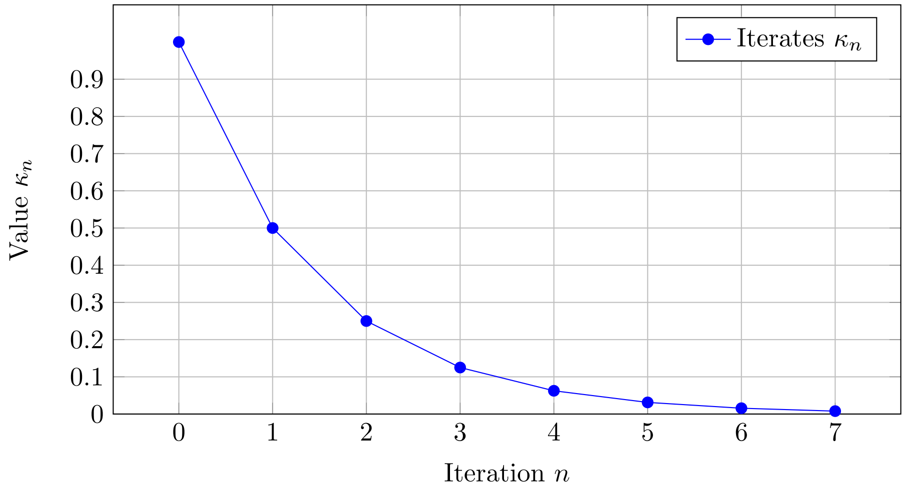
Finally, we demonstrate the fixed point approximation using the iterative process numerically by

$$\kappa_{n+1} = T(\kappa_n) = \frac{\kappa_n}{2}.$$

Starting with $\kappa_0 = 1$, we generate the sequence

n	κ_n
0	1
1	0.5
2	0.25
3	0.125
4	0.0625
5	0.03125
6	0.015625
7	0.0078125

Clearly, $\kappa_n \to 0$ as $n \to \infty$, which is the unique fixed point of T. So, we can illustrate the convergence plot of the iteration as below.



Since all assumptions of Theorem 4.15 are satisfied, the mapping $T(\kappa) = \dfrac{\kappa}{2}$ has a unique fixed point $\kappa^* = 0$, and the iterative sequence converges to it.

5. Applications

Applications of a fixed point theory in fuzzy supermetric spaces can be found in fields such as optimization, control theory, and mathematical modeling, where the presence of uncertainty and imprecision in distance measurements necessitates a more flexible approach to defining and analyzing fixed points. Here we demonstrate the existence theorem for the Fredholm integral equation.

Consider $H = C([0,p],\mathbb{R})$ be the space containing all real-valued continuous functions defined on $[0,p]$ for $0 < p \le 1$ and define a function $\lambda : H \times H \to [1,+\infty)$ by $\lambda(\kappa,\varsigma) = \kappa + \varsigma + 2$. The fuzzy supermetric F_λ is defined on H by

$$F_\lambda(\kappa,\varsigma,s) = \frac{1}{e^{\sup_{t\in[0,p]} \frac{|\kappa(t)-\varsigma(t)|}{s}}}, \kappa,\varsigma \in H, s > 0. \tag{5.1}$$

Then, $(H, F_\lambda, \diamondsuit, \delta)$ is a complete fuzzy supermetric space.

Theorem 5.1. *Consider $(H, F_\lambda, \diamondsuit, \delta)$ a complete fuzzy supermetric space satisfying equation (5.1). Assume that a mapping $T : H \to H$ is an integral operator given as*

$$T\kappa(\grave{a}) = \int_0^p F(\grave{a}, b, \beta(\grave{a}))d\grave{a} + s(b), \tag{5.2}$$

where $p \in H$, and $F(\grave{a}, b, \beta(\grave{a})) : [0,p] \times [0,p] \to \mathbb{R}$ is a function that is continuous. Suppose there exists a function $\beta : [0,p] \times [0,p] \to [0,+\infty)$ such that $\beta \in L^1([0,P],\mathbb{R})$ for any $a, b \in [0,p]$, and it satisfies the following:

(i) $|F(\grave{a}, b, \kappa(b)) - F(\grave{a}, b, \varsigma(b))| \le \beta(\grave{a}, b)|\kappa(b) - \varsigma(b)|$;

(ii) the integral $\int_0^{\grave{a}} \beta(\grave{a}, b)db$ is bounded, that is, there exists some $q \in (0,1)$ such that

$$0 < \sup_{\grave{a}\in[0,p]} \int_0^{\grave{a}} \beta(\grave{a}, b)db \le q < 1;$$

(iii) Assume

$$e^{-\sup_{\grave{a}\in[0,p]}\dfrac{q|\kappa(\grave{a})-\varsigma(\grave{a})|}{s}} \geq 2e^{-\sup_{\grave{a}\in[0,p]}\dfrac{|\kappa(\grave{a})-\varsigma(\grave{a})|}{s}}.$$

Then, the integral equation (5.2) has a solution.

Proof. We know that $(H, F_\lambda, \diamondsuit, \delta)$ is a complete fuzzy supermetric space. Assume that a mapping $T : H \to H$ is an integral operator given as

$$T\kappa(\grave{a}) = \int_0^p F(\grave{a}, b, \beta(\grave{a}))d\grave{a} + s(b).$$

Further, letting that the following condition is satisfied:
$|F(\grave{a}, b, \kappa(b)) - F(\grave{a}, b, \varsigma(b))| \leq \beta(\grave{a}, b)|\kappa(b) - \varsigma(b)|$ for each $\grave{a}, b \in [0, p]$ and $\kappa, \varsigma \in H$. Utilizing that integral equation (5.2), present a solution, and T fulfills every requirement of the theorem (3.12). For any $\kappa, \varsigma \in H$, then by equation (5.1), we obtain

$$\begin{aligned}
F_\lambda(T\kappa, T\varsigma, s) &= \frac{1}{e^{\sup_{\grave{a}\in[0,p]}\dfrac{|T\kappa(\grave{a})-T\varsigma(\grave{a})|}{s}}} = e^{-\sup_{\grave{a}\in[0,p]}\dfrac{|T\kappa(\grave{a})-T\varsigma(\grave{a})|}{s}} \\
&\geq e^{-\sup_{\grave{a}\in[0,p]}\dfrac{\int_0^{\grave{a}}|F(\grave{a},b,\kappa(b))-F(\grave{a},b,\varsigma(b))|db}{s}} \\
&\geq e^{-\sup_{\grave{a}\in[0,p]}\dfrac{\int_0^{\grave{a}}\beta(\grave{a},b)|\kappa(b)-\varsigma(b)|db}{s}} \\
&\geq e^{-\sup_{\grave{a}\in[0,p]}\dfrac{|\kappa(b)-\varsigma(b)|\sup\int_0^{\grave{a}}\beta(\grave{a},b)db}{s}} \\
&\geq e^{-\sup_{\grave{a}\in[0,p]}\dfrac{q|\kappa(b)-\varsigma(b)|}{s}} \\
&\geq 2e^{-\sup_{\grave{a}\in[0,p]}\dfrac{q|\kappa(b)-\varsigma(b)|}{s}} \\
&= 2(F_\lambda(\kappa, \varsigma, s)).
\end{aligned} \tag{5.3}$$

To demonstrate that T is a fuzzy contractive mapping, we must prove that equation (3.2) is true for $\mu \in (0, 1)$, which yields that

$$\begin{aligned}
\left(\frac{1}{F_\lambda(T\kappa_n, T\kappa_{n+1}, s)} - 1\right) &\leq \mu\left(\frac{1}{F_\lambda(\kappa_n, \kappa_{n+1}, s)} - 1\right) \\
&\leq \mu\left(\mu\left(\frac{1}{F_\lambda(\kappa_{n-1}, \kappa_n, s)} - 1\right)\right) \\
&\ \ \vdots \\
&\leq \mu^n\left(\frac{1}{F_\lambda(\kappa_0, \kappa_1, s)} - 1\right).
\end{aligned} \tag{5.4}$$

Equation (5.4) goes to 0 as n goes to ∞, ensures that T is a fuzzy contractive mapping, which confirms that all the requirements of Theorems (3.12) and (5.1) are satisfied. Hence, the operator T has a fixed point $\kappa^* \in C([0,1],\mathbb{R})$, which is a solution of the integral Equation (5.2).

6. Conclusion

In this study, we introduced and developed the concept of fuzzy supermetric spaces as an extension of classical supermetric spaces, thereby establishing a more generalized and flexible mathematical framework. By constructing and analyzing various types of contractive mappings, we established new fixed-point theorems within this framework. These results significantly generalize several existing classical and contemporary theorems, highlighting the strength and applicability of the fuzzy supermetric structure.

The theoretical contributions presented here offer a robust approach for addressing nonlinear and uncertain problems, particularly in areas where traditional metric structures may fall short. Applications to real-world systems such as those in physics, engineering, decision sciences, neural networks, robotics, control systems, and information theory underscore the practical relevance of the proposed models. In particular, we demonstrated the effectiveness of our findings by verifying the existence of a solution to the Fredholm integral equation, emphasizing the utility of fuzzy supermetric spaces in solving complex integral problems.

Future research will explore several promising directions. One potential avenue is the development of multi-valued or hybrid contractive mappings in fuzzy supermetric spaces. Another possibility is to expand the current findings to fuzzy supermetric-type spaces that include probabilistic, intuitionistic, or interval-valued fuzzy elements. Moreover, studying dynamical systems, optimization problems, or differential equations under the proposed structure could lead to novel theoretical insights and impactful applications in science and technology. Investigating computational methods and algorithms compatible with fuzzy supermetric space frameworks is also a hopeful area that could further bridge theory with practical implementation.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Fréchet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo 22 (1906) 1–74.

[2] S.S. Chang, A common fixed point theorem for commuting mappings, Proceedings of the American Mathematical Society 83 (1981) 645–652. Available from: https://doi.org/10.2307/2044139.

[3] L.B. Ćirić, On Presic type generalization of the Banach contraction mapping principle, Acta Mathematica Universitatis Comenianae 76 (2007) 143–147.

[4] J. Danes, Two fixed point theorems in topological and metric spaces, Bulletin of the Australian Mathematical Society 14 (1976) 259–265. Available from: https://doi.org/10.1017/S0004972700025077.

[5] B.C. Dhage, Generalized metric spaces and mapping with fixed points, Bulletin of Calcutta Mathematical Society 84 (1992) 329–336.

[6] W.A. Kirk, N. Shahzad, Generalized metrics and Caristis theorem, Fixed Point Theory and Applications (2013) 1–9.
[7] I. Ahmed, P. Kumam, J. Tariboon, A. Yusuf, Theoretical analysis for a generalized fractional-order boundary value problem. In Fixed Point Theory and Fractional Calculus: Recent Advances and Applications (2022) 175–187.
[8] M.J. Ibrahim, I. Ahmed, A.S. Muhammad, A Caputo Proportional Fractional Differential Equation with Multi-point Boundary Condition, Bangmod International Journal of Mathematical and Computational Science 8 (2022) 56–64.
[9] I. Ahmed, P. Kumam, J. Tariboon, A. Ibrahim, P. Borisut, M.A. Demba, Generalized Nonlocal Boundary Condition for Factional Pantograph Differential Equation via Hilfer Fractional Derivative, Journal of Nonlinear Analysis & Optimization: Theory & Applications 12(2) (2021) 45–60.
[10] E. Karapinar, A. Fulga, Contraction in Rational Forms in the Framework of Super Metric Spaces, Mathematics 10 (2022) 3077. Available from: https://doi.org/10.3390/math10173077.
[11] P. Shahi, V.N. Mishra, Expansive Fixed Point Results in Super Metric spaces, Journal of Ramanujan Society of Mathematics & Mathematical Sciences 10(2) (2023) 143–160. Available from: https://doi.org/10.56827/JRSMMS.2023.1002.11.
[12] N. Hooda, M. Sihag, P. Kumar, Fixed point results via a control function in Super Metric Space, Electronic Journal of Mathematical Analysis and Applications 12(1) (2024) 1–10. Available from: 10.21608/EJMAA.2024.253576.1107.
[13] K. Abodayeh, S.K. Shah, M. Sarwar, C. Promsakon, T. Sitthiwirattham, Ćirić-type generalized F-contractions with integral inclusion in super metric spaces, Results in Control and Optimization 16 (2024) 100443. Available from: 10.1016/j.rico.2024.100443.
[14] S.K. Shah, M. Sarwar, M. Hleili, M.E. Samei, T. Abdeljawad, Generalized (ψ, ϕ)-contraction to investigate Volterra integral inclusions and fractal fractional PDEs in super-metric space with numerical experiments, Nonlinear Engineering 14(1) (2025) 20240032. Available from: https://doi.org/10.1515/nleng-2024-0032.
[15] S. Saxena, U.C. Gairola, Fixed point results for $\phi - \psi$-interpolative mappings in super metric space with an application, Filomat 39(1) (2025) 173–184. Available from: https://doi.org/10.2298/FIL2501173S.
[16] S. Alexander, George-Veeramani Fuzzy Metric Revised, Axioms 7(3) (2018) 60. Available from: https://doi.org/10.3390/axioms7030060.
[17] M.S. Ashref, R. Ali, N. Hussain, New Fuzzy Fixed Point Results in Generalized Fuzzy Metric Spaces with Applications to Integral Equations, IEEE Access 8 (2020) 91653–91660. Available from: 10.1109/ACCESS.2020.2994130.
[18] A. George, P. Veeramani, On Some Results of Fuzzy Metric Spaces,Fuzzy Sets and Systems 64(3) (1994) 395–399. Available from: https://doi.org/10.1016/0165-0114(94)90162-7.
[19] J.R. Kider, Completion of Standard Fuzzy Metric Spaces, Open Journal of Mathematical Sciences 3(2) (2015) 19–25.
[20] A.E. Michalek, Metric Spaces in Fuzzy Set Theory, Academic Press 69 (1979) 205–230. Available from: https://doi.org/10.1016/0022-247X(79)90189-6.
[21] I. Ahmed, A. Yusuf, M. A. Sani, F. Jarad, W. Kumam, P. Thounthong, Analysis of a caputo HIV and Malaria co-infection epidemic model, Thai Journal of Mathematics 19(3) (2021) 897–912.

[22] I.A. Baba, I. Ahmed, Q.M. Al-Mdallal, F. Jarad, S. Yunusa, Numerical and theoretical analysis of an awareness COVID-19 epidemic model via generalized Atangana-Baleanu fractional derivative. Journal of Applied Mathematics and Computational Mechanics 21(1) (2022) 7–18. Available from: https://doi.org/10.17512/jamcm.2022.1.01.

[23] I. Ahmed, A. Yusuf, A. Ibrahim, P. Kumam, M.J. Ibrahim, A mathematical model of the ongoing coronavirus disease (COVID-19) pandemic: a case study in Turkey, Science & Technology Asia 27(4) (2022) 248–258. Available from: https://doi.org/10.14456/scitechasia.2022.84.

[24] A. Hussain, I. Ahmed, A. Yusuf, M.J. Ibrahim, Existence and stability analysis of a fractional-order COVID-19 model, Bangmod International Journal of Mathematical and Computational Science 7 (2021) 102–125.

[25] I. Ahmed, A. Yusuf, J. Tariboon, M. Muhammad, F. Jarad, B.B. Mikailu, A dynamical and sensitivity analysis of the Caputo fractional-order Ebola virus model: Implications for control measures, Science & Technology Asia 28(4) (2023) 26–37. Available from: https://doi.org/10.14456/scitechasia.2023.69.

[26] I. Ahmed, C. Kiataramkul, M. Muhammad, J. Tariboon, Existence and sensitivity analysis of a Caputo fractional-order diphtheria epidemic model, Mathematics 12(13) (2024) 2033. Available from: https://doi.org/10.3390/math12132033.

[27] I. Kramosil, J. Michalek, Fuzzy Metric and Statistical Metric Space, Kybernetika 11(5) (1975) 326–334. Available from: http://dml.cz/dmlcz/125556.

[28] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis 7(2) (2006) 289–297.

[29] I.A. Bakhtin, The contraction mapping principle in almost metric spaces, Functional Analysis 30 (1989) 26–37.

[30] H. Huang, B. Carić, T. Došenović, D. Rakić, M. Brdar, Fixed-Point Theorems in Fuzzy Metric Spaces via Fuzzy F-Contraction, Mathematics 9(6) (2021) 641. Available from: https://doi.org/10.3390/math9060641.

[31] M. Grabiec, Fixed points in Fuzzy Metric Spaces, Fuzzy Sets and Systems 27 (1988) 385–389. Available from: https://doi.org/10.1016/0165-0114(88)90064-4.

[32] V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002) 245–252. Available from: https://doi.org/10.1016/S0165-0114(00)00088-9.

[33] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\beta-\theta-$contractive type mappings, Nonlinear Analysis 75 (2012) 2154–2165. Available from: https://doi.org/10.1016/j.na.2011.10.014.

[34] E. Karapinar, Discussion on (α,ψ) contractions on generalized metric spaces, Abstract and Applied Analysis (2014) 962784. Available from: http://dx.doi.org/10.1155/2014/962784.

[35] H. Aydi, E. Karapinar, B. Samet, Fixed points for generalized (α,ψ)-contractions on generalized metric spaces, Journal of Inequalities and Applications 2014 (2014) 229. Available from: http://dx.doi.org/10.1186/1029-242X-2014-229.