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The Hybrid Steepest Descent Method with Implicit Double Midpoint for Solving Variational Inequality over Triple Hierarchical Problems

Sangkhae Suwansoontorn[1], **Winate Sanhan**[1], **Thanyarat Jitpeera**[2,*], **Chirasak Mongkolkeha**[1,*]
[1] *Department of Computational Science and Digital Technology, Faculty of Liberal Arts and Sciences, Kasetsart University, Kamphaeng-Saen Campus, Nakhonpathom 73140, Thailand*
E-mails: faassky@ku.ac.th; faaswns@ku.ac.th; faascsm@ku.ac.th
[2] *Department of Science, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiangrai Campus, Chiangrai 57120, Thailand*
E-mails: thanyarat@rmutl.ac.th

**Corresponding author.*

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Abstract Because of the importance of the variational inequality problem that related to many other problems in various branches of science and engineering, it becomes one of the most popular topics which many researchers pay deeply attention to study on the way to solve and the way to apply. In this research, we study the monotone variational inequality over triple hierarchical problem. We propose a new implicit algorithm to find the solution with the strong convergence theorem which is proved and applied to guarantee its solution under some weak assumptions. Our results enhance those of Xu et al., Ke and Ma, Dhakal and Wutiphol and many other authors.

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1. Introduction

The theory of variational inequalities develop rapidly and its applications are highly productive. In this research, we study the monotone variational inequality over triple hierarchical problem. Throughout this paper, let C be a closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. The weak convergence and strong convergence are denoted by $\rightharpoonup$ and $\to$, respectively.

Before we mention on problems in this research, we recall some mapping definitions. A mapping $f : C \to C$ is called *ρ-contraction* if there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \ \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

A mapping $T : C \to C$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle .$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty [15].

One of the most interesting problems is the variational inequality which has been extensively studied by many researchers due to its applications in various disciplines such as engineering, economics and many others. Exactly, the well-known problem *Hartmann-Stampacchia variational inequality* [7] was introduced in 1966, its aim is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C, \tag{1.1}$$

where A is a nonlinear mapping. The set of solutions of (1.1) is denoted by $VI(C, A)$. That is, $VI(C, A) = \{x \in C : \langle Ax, y - x \rangle \geq 0, \ \forall y \in C\}$. The methodologies for solving this problem has been widely used and improved as shown in the literature [3, 5, 20–23, 27, 29].

Later, the more complicated problem, that is the variational inequality problem over the fixed point set of a nonexpansive mapping, was introduced and it is well-known in the name of *hierarchical problem*. Since it has been discovered, there are many extended results which have been published continuously (see [2, 6, 10, 25, 26, 30–33]). This problem was state as follow:

For a continuous monotone mapping $A : H \to H$ and a nonexpansive mapping $T : H \to H$, find $x^* \in VI(F(T), A) = \Big\{x^* \in F(T) : \langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in F(T)\Big\}$, where $F(T) \neq \emptyset$. The solution set of the hierarchical problem is denoted by S.

Moreover, there is the variational inequality problem over the solution set of the variational inequality problem over the fixed point set of a nonexpansive mapping (see [8, 9, 11–13, 28]), which is called *triple-hierarchical problem*. Let $A : H \to H$ be an inverse-strongly monotone, $B : H \to H$ a strongly monotone and Lipschitz continuous and $T : H \to H$ a nonexpansive mapping. The triple hierarchical problem is to find $x^* \in VI(S, B) = \Big\{x^* \in S : \langle Bx^*, x - x^* \rangle \geq 0, \ \forall x \in S\Big\}$, where $S := VI(F(T), A) \neq \emptyset$.

A mapping $A : H \to H$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be α-*strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \ \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be β-*inverse-strongly monotone* if there exists a positive real number β such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2, \ \forall x, y \in H.$$

A mapping $A : H \to H$ is said to be L-*Lipschitz continuous* if there exists a positive real number L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \ \forall x, y \in H.$$

A linear bounded operator A is said to be *strongly positive* on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \forall x \in H.$$

The various methods for solving the triple hierarchical problem were widely proposed. In 2000, Moudafi [18] introduced the viscosity approximation method to solve the fixed point problems by both implicit and explicit methods, which are stated as follows:

$$x_n = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T x_n, \ \forall n \in \mathbb{N}, \tag{1.2}$$

and

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T x_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where a self mapping $T : C \to C$ is nonexpansive, $f : C \to C$ is a contraction and $\varepsilon_n \in (0, 1)$ for all $n \in \mathbb{N}$.

Later in 2015, Xu et.al [35] considered the following the viscosity method to the implicit midpoint rule,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \ \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\alpha_n \in (0, 1)$ satisfies certain assumptions and f is a contraction. They verified that the above iterative sequence $\{x_n\}$ converges to a unique fixed point.

Next, Ke and Ma [14] introduced the following generalized viscosity implicit rule

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n f(x_n) - (1 - \alpha_n) T(s_n x_n + (1 - s_n) x_{n+1}), \ \forall n \geq 0, \end{cases} \tag{1.5}$$

where $\alpha_n, s_n \in (0, 1)$ satisfy some certain conditions, $f : C \to C$ is a contraction. They proved the sequence converges to a unique fixed point.

Moreover, in 2011, Dhakal and Sintunavarat [4] extended the previous idea by proposing the viscosity method to the implicit double midpoint rule for a nonexpansive mapping. They constructed the algorithm by generating the sequence $\{x_n\}$ by the following:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n f\left(\frac{x_n + x_{n+1}}{2}\right) - (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \ \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\alpha_n \in (0,1)$, $f : C \to H$ be a contraction satisfying some conditions. Their strong convergence theorem is proved under some control condition to gaurantee the solution of the mentioned problem.

Owing to the motivation from the previous studies, in this paper, we establish an algorithm for solving the variational inequality over the triple hierarchical problem as shown below.

Let $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Find $x^* \in \Omega$ such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \ \forall x \in \Omega, \tag{1.7}$$

where $\Omega := VI(F(T), A - \gamma f) \neq \emptyset$, T is a nonexpansive mapping, $A : C \to H$ is a strongly positive linear bounded operator and $f : C \to H$ is a ρ-contraction. This solution set of (1.7) is denoted by $\Upsilon := VI(\Omega, B)$. The strong convergence result is also proved under some weak assumptions.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the projection P_C from H onto C, mapping each $x \in H$ to the unique point in C, satisfies the following property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

We sometimes call this projection as the nearest point of x in C and denote it by $P_C x$. Next, we state some lemmas which will be used in the rest of this paper.

Lemma 2.1. *The function $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$ for all $\lambda > 0$.*

Lemma 2.2. *For a given $z \in H$, $u \in C$, $u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0, \quad \forall v \in C$. It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0. \tag{2.2}$$

Lemma 2.3. [3] *There holds the following inequality in an inner product space H*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \ \forall x, y \in H.$$

Lemma 2.4. [1] *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.5. [16] *Assume A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.6. [19] *Each Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

hold for all $y \in H$ with $y \neq x$.

Lemma 2.7. [24]*Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.8. [31]*Let* $B : H \to H$ *be* β*-strongly monotone and* L*-Lipschitz continuous and let* $\mu \in (0, \frac{2\beta}{L^2})$. *For* $\lambda \in [0,1]$, *define* $T_\lambda : H \to H$ *by* $T_\lambda(x) := x - \lambda\mu B(x)$ *for all* $x \in H$. *Then, for all* $x, y \in H$,

$$\|T_\lambda(x) - T_\lambda(y)\| \leq (1-\lambda\tau)\|x-y\|$$

hold, where τ *is defined by* $1 - \sqrt{1-\mu(2\beta - \mu L^2)}$ *which is in the interval* $(0,1]$.

Lemma 2.9. [16, 34]*Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \ \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathcal{R}$ *such that*
(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main result

In this section, we introduce our new iterative algorithm which is generated to solve the monotone variational inequality over triple hierarchical problem and exactly proved its convergence theorem that can gaurantee the convergence to the solution of the problem.

Theorem 3.1. *Let* H *be a real Hilbert space,* C *be a closed convex subset of* H. *Let* $A : C \to H$ *be a strongly positive linear bounded operator,* $f : C \to H$ *be a* ρ*-contraction,* γ *be a positive real number such that* $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho} < \frac{1}{\rho}$ *where* $\bar{\gamma} \in (0,\infty)$. *Let* $T : C \to C$ *be a nonexpansive mapping,* $B : C \to C$ *be a* β*-strongly monotone and* L*-Lipschitz continuous. Let* $\phi : C \to C$ *be a* k*-contraction mapping with* $k \in [0,1)$. *Suppose* $\{x_n\}$ *is a sequence generated by the following algorithm, for an arbitrary* $x_0 \in C$,

$$\begin{cases} z_n = TP_C[I - \delta_{n+1}(A - \gamma f)](w_n x_n + (1-w_n)x_{n+1}), \\ x_{n+1} = \alpha_n\phi(r_n x_n + (1-r_n)x_{n+1}) + (1-\alpha_n)(I - \mu\beta_n B)z_n, \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0,1]$. *If* $\mu \in (0, \frac{2\beta}{L^2})$ *is used and if* $\{\beta_n\} \subset (0,1]$ *satisfy the following conditions:*
(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;
(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;
(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty}\alpha_n = 0$;
(C4): $\delta_n \leq \beta_n$ *and* $\beta_n \leq \alpha_n$.

Then $\{x_n\}$ *converges strongly to* $x^* \in \Upsilon$, *which is the unique solution of the variational inequality:*

$$\text{Find } x^* \in \Upsilon \text{ such that } \langle (I-\phi)x^*, x - x^*\rangle \geq 0, \ \forall x \in \Upsilon, \tag{3.2}$$

where Υ *is the set* $VI(\Omega, B)$ *which is* $VI\Big(VI(F(T), A - \gamma f), B\Big)$.

Proof. First, we aim to show the existence of a sequence $\{x_n\}$ defined by (3.1). Consider the mapping $S_n : C \to C$ by

$$S_n x = \alpha_n \phi(r_n v+(1-r_n)x)+(1-\alpha_n)(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n v+(1-w_n)x)$$

for all $x \in C$. We can verify that the mapping S_n is a contraction for all $n \in \mathbb{N}$ and $x, y \in C$ as shown below.

$$\begin{aligned}
& \|S_n x - S_n y\| \\
= & \Big\|\alpha_n \phi(r_n v + (1-r_n)x) \\
& +(1-\alpha_n)(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n v + (1-w_n)x) \\
& -\alpha_n \phi(r_n v + (1-r_n)y) \\
& -(1-\alpha_n)(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n v + (1-w_n)y)\Big\| \\
\le & \ \alpha_n \|\phi(1-r_n)x - \phi(1-r_n)y\| \\
& +(1-\alpha_n)\big\|(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](1-w_n)x \\
& -(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](1-w_n)y\big\| \\
\le & \ \alpha_n(1-r_n)k\|x-y\| + (1-\alpha_n)(1-\beta_n\tau) * \\
& \big\|TP_C[I-\delta_{n+1}(A-\gamma f)](1-w_n)x - TP_C[I-\delta_{n+1}(A-\gamma f)](1-w_n)y\big\| \\
\le & \ \alpha_n(1-r_n)k\|x-y\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-w_n)\big\|\delta_{n+1}(\gamma f x - \gamma f y) + (I-\delta_{n+1}A)(x-y)\big\| \\
\le & \ \alpha_n(1-r_n)k\|x-y\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-w_n)\big(\delta_{n+1}\gamma\rho\|x-y\| + (1-\delta_{n+1}\bar{\gamma})\|x-y\|\big) \\
= & \ \Big(\alpha_n(1-r_n)k + (1-\alpha_n)(1-\beta_n\tau)(1-w_n)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)\Big)\|x-y\|
\end{aligned}$$

where $\acute{\alpha} = \alpha_n(1-r_n)k + (1-\alpha_n)(1-\beta_n\tau)(1-w_n)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big) \in [0,1)$ for all $n \in \mathbb{N}$. This shows that the mapping S_n is a contraction for all $n \in \mathbb{N}$. According to the Banach contraction principle, we can conclude that S_n has a unique fixed point for all $n \in \mathbb{N}$. Thus we can verify the existence of a sequence $\{x_n\}$ defined by (3.1). To make an easy description of the proof, we will divide into the following four steps.

Step 1. We will prove that $\{x_n\}$ is bounded. For any $q \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n \phi(r_n x_n + (1-r_n)x_{n+1}) + (1-\alpha_n)(I-\mu\beta_n B)z_n - q\| \\
&= \|\alpha_n \phi(r_n x_n + (1-r_n)x_{n+1}) \\
&\quad +(1-\alpha_n)(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)] \\
&\quad (w_n x_n + (1-w_n)x_{n+1}) - q\| \\
&\le \alpha_n\|\phi(r_n x_n + (1-r_n)x_{n+1}) - q\| \\
&\quad +(1-\alpha_n)\|(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)] \\
&\quad (w_n x_n + (1-w_n)x_{n+1}) - q\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|\phi(r_nx_n+(1-r_n)x_{n+1})-\phi q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)\Big\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1}) \\
&\quad -(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)]q\Big\| \\
&\leq \alpha_nk\|(r_nx_n+(1-r_n)x_{n+1})-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)* \\
&\quad \Big\|TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-TP_C[I-\delta_{n+1}(A-\gamma f)]q\Big\| \\
&\leq \alpha_nk\Big(r_n\|x_n-q\|+(1-r_n)\|x_{n+1}-q\|\Big)+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)* \\
&\quad \Big\|[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-[I-\delta_{n+1}(A-\gamma f)]q\Big\| \\
&\leq \alpha_nkr_n\|x_n-q\|+\alpha_nk(1-r_n)\|x_{n+1}-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\Big(\delta_{n+1}\|\gamma f(w_nx_n+(1-w_n)x_{n+1})-\gamma fq\| \\
&\quad +(1-\delta_{n+1}\bar{\gamma})\|(w_nx_n+(1-w_n)x_{n+1})-q\|\Big) \\
&\leq \alpha_nkr_n\|x_n-q\|+\alpha_nk(1-r_n)\|x_{n+1}-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\Big(\delta_{n+1}\gamma\rho\|w_nx_n+(1-w_n)x_{n+1}-q\| \\
&\quad +(1-\delta_{n+1}\bar{\gamma})\|w_nx_n+(1-w_n)x_{n+1}-q\|\Big) \\
&= \alpha_nkr_n\|x_n-q\|+\alpha_nk(1-r_n)\|x_{n+1}-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)\|w_nx_n+(1-w_n)x_{n+1}-q\| \\
&= \alpha_nkr_n\|x_n-q\|+\alpha_nk(1-r_n)\|x_{n+1}-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)\big(w_n\|x_n-q\|+(1-w_n)\|x_{n+1}-q\|\big) \\
&= \alpha_nkr_n\|x_n-q\|+\alpha_nk(1-r_n)\|x_{n+1}-q\|+\alpha_n\|\phi q-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)w_n\|x_n-q\| \\
&\quad +(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)\|x_{n+1}-q\| \\
&= \Big(\alpha_nkr_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)w_n\Big)\|x_n-q\|+\alpha_n\|\phi q-q\| \\
&\quad +\Big(\alpha_nk(1-r_n)+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)\Big)\|x_{n+1}-q\| \\
&= \Big(\alpha_nkr_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)w_n\Big)\|x_n-q\|+\alpha_n\|\phi q-q\| \\
&\quad +\Big(\alpha_nk-\alpha_nkr_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)\Big)\|x_{n+1}-q\|.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\|x_{n+1}-q\| \\
&\leq \frac{\alpha_nkr_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)w_n}{1-\alpha_nk+\alpha_nkr_n-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\|x_n-q\|
\end{aligned}
$$

$$+\frac{\alpha_n}{1-\alpha_n k+\alpha_n k r_n-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\|\phi q-q\|.$$

Since

$$\begin{aligned}
&\alpha_n k+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)-\alpha_n k(1-r_n)\\
&-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)\\
&=\alpha_n k+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)-\alpha_n k+\alpha_n k r_n\\
&-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)\\
&=\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-(1-w_n))\\
&=\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)w_n\\
&>0,
\end{aligned}$$

and $1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)-\alpha_n>0$. Then, we have

$$\frac{1-\alpha_n k-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\in(0,1)$$

and

$$\frac{\alpha_n}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\in(0,1).$$

Hence

$$\begin{aligned}
&\|x_{n+1}-q\|\\
=\;&\left(1-\frac{1-\alpha_n k-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\right)\|x_n-q\|\\
&+\frac{\alpha_n}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}\|\phi q-q\|\\
=\;&(1-\lambda_n)\|x_n-q\|+d_n\|\phi q-q\|,
\end{aligned}$$

where λ_n is definded by $\frac{1-\alpha_n k-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}$ and d_n is defined by $\frac{\alpha_n}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)\big(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}\big)(1-w_n)}$. Then, by mathematical induction implies that

$$\|x_n-q\| \quad\leq\quad \|x_0-q\|,\ \forall n\geq 0.$$

Therefore $\{x_n\}$ is bounded. Consequently, we also obtain $\{\phi(r_n x_n+(1-r_n)x_{n+1})\}$ and $\{TP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n+(1-w_n)x_{n+1})\}$ are bounded.

Step 2. We claim that $\lim_{n\to\infty}\Big\|x_{n+1}-x_n\|=0$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. From (3.1), we have, for each $n>1$,

$$
\begin{aligned}
& \|x_{n+1}-x_n\| \\
= & \Big\|\alpha_n\phi(r_nx_n+(1-r_n)x_{n+1}) \\
& +(1-\alpha_n)(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1}) \\
& -\alpha_{n-1}\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n) \\
& -(1-\alpha_{n-1})(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\| \\
\leq & \alpha_n\|\phi(r_nx_n+(1-r_n)x_{n+1})-\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +(1-\alpha_n)\Big\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1}) \\
& -(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\| \\
& +|\alpha_n-\alpha_{n-1}|\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\| \\
\leq & \alpha_nk\|(r_nx_n+(1-r_n)x_{n+1})-(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +|\alpha_n-\alpha_{n-1}|\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)\Big(\Big\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1}) \\
& -(I-\mu\beta_nB)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\| \\
& +\Big\|(I-\mu\beta_nB)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n) \\
& -(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\|\Big) \\
\leq & \alpha_nk(1-r_n)\|x_{n+1}-x_n\|+\alpha_nkr_{n-1}\|x_n-x_{n-1}\| \\
& +|\alpha_n-\alpha_{n-1}|\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)\Big\|[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1}) \\
& -[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\| \\
& +(1-\alpha_n)\mu|\beta_n-\beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\| \\
= & \alpha_nk(1-r_n)\|x_{n+1}-x_n\|+\alpha_nkr_{n-1}\|x_n-x_{n-1}\| \\
& +|\alpha_n-\alpha_{n-1}|\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\| \\
& +|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)* \\
& \Big\|\delta_{n+1}\big(\gamma f(w_nx_n+(1-w_n)x_{n+1})-\gamma f(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\big) \\
& +(I-\delta_{n+1}A)(w_nx_n+(1-w_n)x_{n+1}-w_{n-1}x_{n-1}-(1-w_{n-1})x_n) \\
& +(\delta_{n+1}-\delta_n)\gamma f(w_{n-1}x_{n-1}+(1-w_{n-1})x_n) \\
& +(\delta_{n+1}-\delta_n)A(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\Big\| \\
& +(1-\alpha_n)\mu|\beta_n-\beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \alpha_n k(1-r_n)\|x_{n+1}-x_n\| + \alpha_n k r_{n-1}\|x_n - x_{n-1}\| \\
& +|\alpha_n - \alpha_{n-1}|\|\phi(r_{n-1}x_{n-1} + (1-r_{n-1})x_n)\| \\
& +|\alpha_n - \alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)\Big(\delta_{n+1}\gamma\rho\|w_n x_n + (1-w_n)x_{n+1} - w_{n-1}x_{n-1} - (1-w_{n-1})x_n\| \\
& +(1-\delta_{n+1}\bar{\gamma})\|w_n x_n + (1-w_n)x_{n+1} - w_{n-1}x_{n-1} - (1-w_{n-1})x_n\| \\
& +|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\|\Big) \\
& +(1-\alpha_n)\mu|\beta_n - \beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
\leq\ & \alpha_n k(1-r_n)\|x_{n+1}-x_n\| + \alpha_n k r_{n-1}\|x_n - x_{n-1}\| \\
& +|\alpha_n - \alpha_{n-1}|\|\phi(r_{n-1}x_{n-1} + (1-r_{n-1})x_n)\| \\
& +|\alpha_n - \alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho)) * \\
& \|w_n x_n + (1-w_n)x_{n+1} - w_{n-1}x_{n-1} - (1-w_{n-1})x_n\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)\mu|\beta_n - \beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
=\ & \alpha_n k(1-r_n)\|x_{n+1}-x_n\| + \alpha_n k r_{n-1}\|x_n - x_{n-1}\| \\
& +|\alpha_n - \alpha_{n-1}|\|\phi(r_{n-1}x_{n-1} + (1-r_{n-1})x_n)\| \\
& +|\alpha_n - \alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))\|(1-w_n)(x_{n+1}-x_n) + w_{n-1}(x_n - x_{n-1})\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)\mu|\beta_n - \beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
\leq\ & \alpha_n k(1-r_n)\|x_{n+1}-x_n\| + \alpha_n k r_{n-1}\|x_n - x_{n-1}\| \\
& +|\alpha_n - \alpha_{n-1}|\|\phi(r_{n-1}x_{n-1} + (1-r_{n-1})x_n)\| \\
& +|\alpha_n - \alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)\|x_{n+1}-x_n\| \\
& +(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_{n-1}\|x_n - x_{n-1}\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\| \\
& +(1-\alpha_n)\mu|\beta_n - \beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1} + (1-w_{n-1})x_n)\|.
\end{aligned}
$$

It implies that, for each $n > 1$,

$$
\begin{aligned}
& \|x_{n+1}-x_n\| \\
& \leq \frac{\alpha_n k r_{n-1} + (1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_{n-1}}{1-\alpha_n k(1-r_n) - (1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}\|x_n - x_{n-1}\| \\
& + \frac{|\alpha_n - \alpha_{n-1}|\|\phi(r_{n-1}x_{n-1} + (1-r_{n-1})x_n)\|}{1-\alpha_n k(1-r_n) - (1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}
\end{aligned}
$$

$$+\frac{|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}$$
$$+\frac{(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}$$
$$+\frac{(1-\alpha_n)(1-\beta_n\tau)|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}$$
$$+\frac{(1-\alpha_n)\mu|\beta_n-\beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{1-\alpha_n k(1-r_n)-(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(1-w_n)}.$$

Then, we have

$$\|x_{n+1}-x_n\|$$
$$\leq\left(1-\frac{\alpha_n k(r_n-r_{n-1})+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(w_n-w_{n-1})+\xi_n}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}\right)$$
$$*\|x_n-x_{n-1}\|$$
$$+\frac{|\alpha_n-\alpha_{n-1}|\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\|}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}$$
$$+\frac{|\alpha_n-\alpha_{n-1}|\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}$$
$$+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}$$
$$*\left(\frac{|\delta_{n+1}-\delta_n|\|\gamma f(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right)$$
$$+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}$$
$$*\left(\frac{|\delta_{n+1}-\delta_n|\|A(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right)$$
$$+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}$$
$$*\left(\frac{\mu|\beta_n-\beta_{n-1}|\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|}{(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right),$$

where ξ_n is defined by $1-\alpha_n k-(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))$. This yields that, for $n>1$,

$$\|x_{n+1}-x_n\|$$
$$\leq\left(1-\frac{\alpha_n k(r_n-r_{n-1})+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))(w_n-w_{n-1})+\xi_n}{\alpha_n kr_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}\right)$$
$$*\|x_n-x_{n-1}\|$$

$$
\begin{aligned}
&+\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}M_1\\
&+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}*\left(\frac{|\delta_{n+1}-\delta_n|M_2}{(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right)\\
&+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}*\left(\frac{|\delta_{n+1}-\delta_n|M_3}{(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right)\\
&+\frac{(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}{\alpha_n k r_n+(1-\alpha_n)(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n+\xi_n}\\
&*\left(\frac{\mu|\beta_n-\beta_{n-1}|M_4}{(1-\beta_n\tau)(1-\delta_{n+1}(\bar{\gamma}-\gamma\rho))w_n}\right),
\end{aligned}
$$

where $M_1=\sup_{n\geq 0}\Big\{\|\phi(r_{n-1}x_{n-1}+(1-r_{n-1})x_n)\|+\|(I-\mu\beta_{n-1}B)TP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|\Big\}$, $M_2=\|\gamma f(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|$, $M_3=\|A(w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|$ and $M_4=\|BTP_C[I-\delta_n(A-\gamma f)](w_{n-1}x_{n-1}+(1-w_{n-1})x_n)\|$. From (C1)-(C3) and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{Ax_n\}$, $\{Bz_n\}$, $\{\phi(x_n)\}$ and $\{f(x_n)\}$. By Lemma 2.9, then we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| \quad = \quad 0. \tag{3.3}$$

Later, we need to show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. For each $n\in\mathbb{N}$, we obtain

$$
\begin{aligned}
&\|x_n-Tx_n\|\\
\leq\;&\|x_n-x_{n+1}\|+\big\|x_{n+1}-TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\big\|\\
&+\big\|TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-Tx_n\big\|\\
\leq\;&\|x_n-x_{n+1}\|+\big\|\alpha_n\phi(r_nx_n+(1-r_n)x_{n+1})+(1-\alpha_n)(I-\mu\beta_nB)z_n\\
&-TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\big\|\\
&+\big\|P_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-P_C[I-\delta_{n+1}(A-\gamma f)]x_n\big\|\\
\leq\;&\|x_n-x_{n+1}\|+\Big\|\alpha_n\phi(r_nx_n+(1-r_n)x_{n+1})\\
&+(1-\alpha_n)(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\\
&-TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\Big\|\\
&+\big\|[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-[I-\delta_{n+1}(A-\gamma f)]x_n\big\|\\
\leq\;&\|x_n-x_{n+1}\|+\Big\|\alpha_n\phi(r_nx_n+(1-r_n)x_{n+1})\\
&+(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\\
&-\alpha_n(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\\
&-TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})\Big\|\\
&+\big\|\delta_{n+1}\big(\gamma f(w_nx_n+(1-w_n)x_{n+1})-\gamma fx_n\big)\\
&+(1-\delta_{n+1}\bar{\gamma})(w_nx_n+(1-w_n)x_{n+1}-x_n)\big\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \|x_n - x_{n+1}\| + \Big\|\alpha_n\phi(r_n x_n + (1-r_n)x_{n+1}) \\
&\quad -\mu\beta_n BTP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1}) \\
&\quad -\alpha_n(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\Big\| \\
&\quad +\delta_{n+1}\|\gamma f(w_n x_n + (1-w_n)x_{n+1}) - \gamma f x_n\| \\
&\quad +(1-\delta_{n+1}\bar{\gamma})\|w_n x_n + (1-w_n)x_{n+1} - x_n\| \\
&\leq \|x_n - x_{n+1}\| - \beta_n\|\mu BTP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\| \\
&\quad +\alpha_n\big\|\phi(r_n x_n + (1-r_n)x_{n+1}) \\
&\quad -(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\big\| \\
&\quad +\delta_{n+1}\gamma\rho\|(w_n x_n + (1-w_n)x_{n+1}) - x_n\| \\
&\quad +(1-\delta_{n+1}\bar{\gamma})\|w_n x_n + (1-w_n)x_{n+1} - x_n\| \\
&\leq \|x_n - x_{n+1}\| - \beta_n\|\mu BTP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\| \\
&\quad +\alpha_n\big\|\phi(r_n x_n + (1-r_n)x_{n+1}) \\
&\quad -(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\big\| \\
&\quad +(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})\|w_n x_n + (1-w_n)x_{n+1} - x_n\| \\
&\leq \|x_n - x_{n+1}\| - \beta_n\|\mu BTP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\| \\
&\quad +\alpha_n\big\|\phi(r_n x_n + (1-r_n)x_{n+1}) \\
&\quad -(I-\mu\beta_n B)TP_C[I-\delta_{n+1}(A-\gamma f)](w_n x_n + (1-w_n)x_{n+1})\big\| \\
&\quad +(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(1-w_n)\|x_{n+1} - x_n\|.
\end{aligned}$$

By (C3)-(C4), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and it follows that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.4}$$

Step 3. First, $\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0$ is proved. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - x^*, \gamma f(x^*) - Ax^*\rangle = \lim_{i\to\infty}\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle.$$

The boundedness of $\{x_{n_i}\}$ implies the existences of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\hat{x} \in H$ such that $\{x_{n_{i_j}}\}$ converges weakly to $\hat{x}$. We may assume without loss of generality that $\lim_{i\to\infty}\langle x_{n_i}, w\rangle = \langle \hat{x}, w\rangle,\ w \in H$. Assume $\hat{x} \neq T(\hat{x})$. By $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ with $F(T) \neq \emptyset$ guarantee that

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - \hat{x}\| &< \liminf_{i\to\infty}\|x_{n_i} - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - T(x_{n_i}) + T(x_{n_i}) - T(\hat{x})\| \\
&= \liminf_{i\to\infty}\|T(x_{n_i}) - T(\hat{x})\| \\
&\leq \liminf_{i\to\infty}\|x_{n_i} - \hat{x}\|,
\end{aligned}$$

which is a contradiction. Therefore $\hat{x} \in F(T)$. From $x^* \in VI(F(T), A - \gamma f)$, we find

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - x^*, \gamma f(x^*) - Ax^*\rangle &= \lim_{i\to\infty}\langle x_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle \\ &= \langle \hat{x} - x^*, \gamma f(x^*) - Ax^*\rangle \\ &\leq 0.\end{aligned}$$

Setting $u_n = [I - \delta_n(A - \gamma f)]x_n$ and by (C3)-(C4), we notice that

$$\|u_n - x_n\| \leq \delta_n\|(A - \gamma f)\| \to 0.$$

Hence,we get

$$\limsup_{n\to\infty}\langle u_n - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0. \tag{3.5}$$

Second, the proof of $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle \leq 0$ is shown. From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ guarantees the existences of a subsequence $\{x_{n_k+1}\}$ of $\{x_{n_k}\}$ and a point $\bar{x} \in H$ such that $\limsup_{n\to\infty}\langle x^* - x_{n+1}, Bx^*\rangle = \lim_{k\to\infty}\langle x^* - x_{n_k+1}, Bx^*\rangle$ and $\lim_{k\to\infty}\langle x_{n_k}, w\rangle = \lim_{k\to\infty}\langle x_{n_k+1}, w\rangle = \langle \bar{x}, w\rangle,\ w \in H$. By the same discussion as in the proof of $\hat{x} \in F(T)$, we have $\bar{x} \in F(T)$. Let y be an arbitary fixed point in $F(T)$. Then, it follows from $T : C \to C$ is a nonexpansive mappings with $F(T) \neq \emptyset$, $A : C \to H$ be a strongly positive linear bounded operator and $f : C \to H$ be a contraction that, for all $n \in \mathbb{N}$. From (3.1)

$$\begin{aligned}\|z_n - y\| &= \|TP_C u_n - TP_C y\| \\ &\leq \|u_n - y\|.\end{aligned} \tag{3.6}$$

By (C3)-(C4), it follows that

$$\begin{aligned}\|u_n - y\| &= \|[I - \delta_n(A - \gamma f)]x_n - y\| \\ &\leq \|x_n - y\| + \delta_n\|(A - \gamma f)x_n\| \\ &\leq \|x_n - y\|.\end{aligned} \tag{3.7}$$

Using (3.6) and (3.7)

$$\begin{aligned}\|u_n - y\|^2 &= \|[I - \delta_n(A - \gamma f)]x_n - y\|^2 \\ &= \Big\|\delta_n(\gamma f(x_n) - Ay) + (I - \delta_n A)(x_n - y)\Big\|^2 \\ &\leq (1 - \delta_n\bar{\gamma})^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(x_n) - Ay, u_n - y\rangle \\ &\leq (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 + 2\delta_n\gamma\rho\|x_n - y\|\|u_n - y\| \\ &\quad +2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &\leq (1 - 2\delta_n\bar{\gamma} + \delta_n^2\bar{\gamma}^2)\|x_n - y\|^2 \\ &\quad +2\delta_n\gamma\rho\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &= [1 - 2\delta_n(\bar{\gamma} - \gamma\rho)]\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}0 &\leq \Big(\|x_n - y\|^2 - \|u_n - y\|^2\Big) - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 \\ &\quad +2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &= (\|x_n - y\| + \|u_n - y\|)(\|x_n - y\| - \|u_n - y\|) - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 \\ &\quad +\delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle \\ &\leq M_5\|x_n - u_n\| - 2\delta_n(\bar{\gamma} - \gamma\rho)\|x_n - y\|^2 + \delta_n^2\bar{\gamma}^2\|x_n - y\|^2 + 2\delta_n\langle \gamma f(y) - Ay, u_n - y\rangle,\end{aligned}$$

where $M_5 = \sup\{\|x_n - y\| + \|u_n - y\| : n \in \mathbb{N}\} < \infty$, for every $n \in \mathbb{N}$. By the weak convergence of $\{u_{n_i}\}$ to $\bar{x} \in F(T)$, $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and (C3)-(C4), we get $\langle(\gamma f - A)y, \bar{x} - y\rangle \leq 0$ for all $y \in F(T)$. A mapping A be a strongly positive linear bounded operator and f be a contraction ensures $\langle(\gamma f - A)y, \bar{x} - y\rangle \leq 0$ for all $y \in F(T)$, that is, $\bar{x} \in VI(F(T), A - \gamma f)$. Thus $x^* \in VI(VI(F(T), A - \gamma f), B)$, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle x^* - x_n, Bx^*\rangle &= \limsup_{i\to\infty} \langle x^* - x_{n_i}, Bx^*\rangle \\ &= \langle x^* - \bar{x}, Bx^*\rangle \\ &\leq 0. \end{aligned} \tag{3.8}$$

From (3.8), we notice that

$$\limsup_{n\to\infty} \langle x^* - y_n, Bx^*\rangle \leq 0. \tag{3.9}$$

Thus, $\limsup_{n\to\infty} \langle x_n - x^*, \phi(x^*) - x^*\rangle \leq 0$ is proved. Choose a subsequence $\{x_{n_g}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - x^*, \phi(x^*) - x^*\rangle = \lim_{g\to\infty} \langle x_{n_g} - x^*, \phi(x^*) - x^*\rangle.$$

The boundedness of $\{x_{n_g}\}$ implies the existences of a subsequence $\{x_{n_{g_h}}\}$ of $\{x_{n_g}\}$ and a point $\tilde{x} \in H$ such that $\{x_{n_{g_h}}\}$ converges weakly to $\tilde{x}$. By $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\lim_{h\to\infty} \langle x_{n_{g_h}+1}, w\rangle = \langle \tilde{x}, w\rangle,\ w \in H$. We may assume without loss of generality that $\lim_{i\to\infty} \langle x_{n_g}, w\rangle = \langle \tilde{x}, w\rangle,\ w \in H$. Assume $\tilde{x} \neq T(\tilde{x})$. By $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ with $F(T) \neq \emptyset$, it guarantees that

$$\begin{aligned} \liminf_{g\to\infty} \|x_{n_g} - \tilde{x}\| &< \liminf_{g\to\infty} \|x_{n_g} - T(\tilde{x})\| \\ &= \liminf_{g\to\infty} \|x_{n_g} - T(x_{n_g}) + T(x_{n_g}) - T(\tilde{x})\| \\ &= \liminf_{g\to\infty} \|T(x_{n_g}) - T(\tilde{x})\| \\ &\leq \liminf_{g\to\infty} \|x_{n_g} - \tilde{x}\|. \end{aligned}$$

This is a contradiction, that is, $\tilde{x} \in F(T)$. From $x^* \in VI\Big(VI(VI(F(T), A - \gamma f), B), I - \phi\Big)$, we find

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - x^*, \phi(x^*) - x^*\rangle &= \lim_{g\to\infty} \langle x_{n_g} - x^*, \phi(x^*) - x^*\rangle \\ &= \langle \tilde{x} - x^*, \phi(x^*) - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.10}$$

Step 4. Finally, we prove $\lim_{n\to\infty}\|x_n - x^*\| = 0$. By Lemma 2.8, we compute

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
= \;& \|\alpha_n\phi(r_nx_n + (1-r_n)x_{n+1}) + (1-\alpha_n)(I-\mu\beta_nB)z_n - x^*\|^2 \\
= \;& \|\alpha_n\phi(r_nx_n + (1-r_n)x_{n+1}) \\
& +(1-\alpha_n)(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\|^2 \\
= \;& \Big\|\alpha_n\Big(\phi(r_nx_n + (1-r_n)x_{n+1}) - x^*\Big) \\
& +(1-\alpha_n)(I-\mu\beta_nB)\Big(TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\Big)\Big\|^2 \\
\le \;& \alpha_n^2\|\phi(r_nx_n + (1-r_n)x_{n+1}) - x^*\|^2 \\
& +(1-\alpha_n)^2\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\|^2 \\
& +2\alpha_n(1-\alpha_n)\Big\langle\phi(r_nx_n + (1-r_n)x_{n+1}) - x^*, \\
& (I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\Big\rangle \\
\le \;& \alpha_n^2\|\phi(r_nx_n + (1-r_n)x_{n+1}) - x^*\|^2 \\
& +(1-\alpha_n)^2\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) \\
& -(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)]x^*\|^2 \\
& +2\alpha_n(1-\alpha_n)\Big\langle\phi(r_nx_n + (1-r_n)x_{n+1}) - \phi x^*, \\
& (I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\Big\rangle \\
& +2\alpha_n(1-\alpha_n)\Big\langle\phi x^* - x^*, \\
& (I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\Big\rangle \\
\le \;& (1-\alpha_n)^2\|(w_nx_n + (1-w_n)x_{n+1}) - x^*\|^2 \\
& +2\alpha_n(1-\alpha_n)\|\phi(r_nx_n + (1-r_n)x_{n+1}) - \phi x^*\| \\
& *\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) - x^*\| + \eta_n \\
\le \;& (1-\alpha_n)^2\|w_n(x_n - x^*) + (1-w_n)(x_{n+1} - x^*)\|^2 \\
& +2k\alpha_n(1-\alpha_n)\|r_nx_n + (1-r_n)x_{n+1} - x^*\| \\
& *\Big\|(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n + (1-w_n)x_{n+1}) \\
& -(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)]x^*\Big\| + \eta_n \\
\le \;& (1-\alpha_n)^2\Big\{w_n^2\|x_n - x^*\|^2 + 2w_n(1-w_n)\|x_n - x^*\|\|x_{n+1} - x^*\| \\
& +(1-w_n)^2\|x_{n+1} - x^*\|^2\Big\} + 2k\alpha_n(1-\alpha_n)\{r_n\|x_n - x^*\| + (1-r_n)\|x_{n+1} - x^*\|\} \\
& *(1-\beta_n\tau)(1-(\bar\gamma-\gamma\rho)\delta_{n+1})\{\|w_nx_n + (1-w_n)x_{n+1} - x^*\|\} + \eta_n \\
\le \;& (1-\alpha_n)^2\Big\{w_n^2\|x_n - x^*\|^2 + w_n(1-w_n)\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \\
& +(1-w_n)^2\|x_{n+1} - x^*\|^2\Big\} \\
& +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar\gamma-\gamma\rho)\delta_{n+1})\{r_n\|x_n - x^*\| \\
& +(1-r_n)\|x_{n+1} - x^*\|\}\cdot\{w_n\|x_n + x^*\| + (1-w_n)\|x_{n+1} - x^*\|\} + \eta_n
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2\Big\{w_n^2\|x_n-x^*\|^2+w_n(1-w_n)\|x_n-x^*\|^2+w_n(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +(1-w_n)^2\|x_{n+1}-x^*\|^2\Big\} \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})\Big\{r_nw_n\|x_n-x^*\|^2 \\
&\quad +(1-r_n)w_n\|x_{n+1}-x^*\|\|x_n-x^*\|+r_n(1-w_n)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad +(1-r_n)(1-w_n)\|x_{n+1}-x^*\Big\}+\eta_n \\
&\leq (1-\alpha_n)^2\{(w_n^2+w_n-w_n^2)\|x_n-x^*\|^2+(1-w_n)(w_n+1-w_n)\|x_{n+1}-x^*\|^2\} \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})r_nw_n\|x_n-x^*\|^2 \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(1-r_n)(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}) \\
&\quad *((1-r_n)w_n+r_n(1-w_n))\|x_{n+1}-x^*\|\|x_n-x^*\|+\eta_n \\
&\leq (1-\alpha_n)^2w_n\|x_n-x^*\|^2+(1-\alpha_n)^2(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})r_nw_n\|x_n-x^*\|^2 \\
&\quad +2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(1-r_n)(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})((1-r_n)w_n+r_n(1-w_n)) \\
&\quad *\{\|x_{n+1}-x^*\|^2+\|x_n-x^*\|^2\}+\eta_n \\
&\leq (1-\alpha_n)^2w_n\|x_n-x^*\|^2+(1-\alpha_n)^2(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2r_nw_n+(1-r_n)w_n+r_n(1-w_n)) \\
&\quad *\|x_n-x^*\|^2+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1}) \\
&\quad *(2(1-r_n)(1-w_n)+(1-r_n)w_n+r_n(1-w_n))\|x_{n+1}-x^*\|^2+\eta_n \\
&\leq (1-\alpha_n)^2w_n\|x_n-x^*\|^2+(1-\alpha_n)^2(1-w_n)\|x_{n+1}-x^*\|^2 \\
&\quad +k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)\|x_n-x^*\|^2 \\
&\quad +k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)\|x_{n+1}-x^*\|^2+\eta_n \\
&\leq \{(1-\alpha_n)^2w_n+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)\}\|x_n-x^*\|^2 \\
&\quad +\{(1-\alpha_n)^2(1-w_n)+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)\} \\
&\quad *\|x_{n+1}-x^*\|^2+\eta_n,
\end{aligned}
$$

where $\eta_n=\alpha_n^2\|\phi(r_nx_n+(1-r_n)x_{n+1})-x^*\|^2+2\alpha_n(1-\alpha_n)\big\langle\phi x^*-x^*,(I-\mu\beta_nB)TP_C[I-\delta_{n+1}(A-\gamma f)](w_nx_n+(1-w_n)x_{n+1})-x^*\big\rangle$ which implies that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\leq \frac{(1-\alpha_n)^2w_n+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)}{1-\{(1-\alpha_n)^2(1-w_n)+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)\}} \\
&\quad *\|x_n-x^*\|^2 \\
&\quad +\frac{\eta_n}{1-\{(1-\alpha_n)^2(1-w_n)+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)\}} \\
&\leq \frac{(1-\alpha_n)^2w_n+k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)}{1-(1-\alpha_n)^2(1-w_n)-k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)}
\end{aligned}
$$

$$
\begin{aligned}
& *\|x_n - x^*\|^2 \\
& + \frac{\eta_n}{1-(1-\alpha_n)^2(1-w_n) - k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(2-r_n-w_n)} \\
\leq\ & \frac{(1-\alpha_n)^2 w_n + k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)}{(1-\alpha_n)^2 w_n + k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)+\zeta_n}\|x_n - x^*\|^2 \\
& + \frac{\eta_n}{(1-\alpha_n)^2 w_n + k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)+\zeta_n} \\
\leq\ & \left(1 - \frac{\alpha_n(2-\alpha_n) - 2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})}{(1-\alpha_n)^2 w_n + k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)+\zeta_n}\right) \\
& *\|x_n - x^*\|^2 \\
& + \frac{\eta_n}{(1-\alpha_n)^2 w_n + k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})(w_n+r_n)+\zeta_n},
\end{aligned}
$$

where $\zeta_n = \alpha_n(2-\alpha_n) - 2k\alpha_n(1-\alpha_n)(1-\beta_n\tau)(1-(\bar{\gamma}-\gamma\rho)\delta_{n+1})$. Applying Lemma 2.9, we can conclude that $x_n \to x^*$. This completes the proof. ∎

4. An Example

Next, the following example shows that all conditions of Theorem 3.1 are satisfied.

Example 4.1. For instance, let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2n}$ and $\delta_n = \frac{1}{3n}$. We will show that the condition (C1) is achieves. Then, clearly, the sequences $\{\delta_n\}$

$$\Sigma_{n=1}^{\infty}\delta_n = \Sigma_{n=1}^{\infty}\frac{1}{3n} = \infty$$

and

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| &= \Sigma_{n=1}^{\infty}\left|\tfrac{1}{3(n+1)} - \tfrac{1}{3n}\right| \\
&\leq \left|\tfrac{1}{3\cdot1} - \tfrac{1}{3\cdot2}\right| + \left|\tfrac{1}{3\cdot2} - \tfrac{1}{3\cdot3}\right| + \left|\tfrac{1}{3\cdot3} - \tfrac{1}{3\cdot4}\right| + \dots \\
&= \tfrac{1}{3}.
\end{aligned}
$$

The sequence $\{\delta_n\}$ satisfies the condition (C1).
Later, we will show that the condition (C2) is achieved. We compute

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| &= \Sigma_{n=1}^{\infty}\left|\tfrac{1}{2(n+1)} - \tfrac{1}{2n}\right| \\
&\leq \left|\tfrac{1}{2\cdot1} - \tfrac{1}{2\cdot2}\right| + \left|\tfrac{1}{2\cdot2} - \tfrac{1}{2\cdot3}\right| + \left|\tfrac{1}{2\cdot3} - \tfrac{1}{2\cdot4}\right| + \dots \\
&= \tfrac{1}{2}.
\end{aligned}
$$

The sequence $\{\beta_n\}$ satisfies the condition (C2).
Next, we will show that the condition (C3) is achieved. We compute

$$
\begin{aligned}
\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| &= \Sigma_{n=1}^{\infty}\left|\tfrac{1}{n+1} - \tfrac{1}{n}\right| \\
&\leq \left|\tfrac{1}{1} - \tfrac{1}{2}\right| + \left|\tfrac{1}{2} - \tfrac{1}{3}\right| + \left|\tfrac{1}{3} - \tfrac{1}{4}\right| + \dots \\
&= 1
\end{aligned}
$$

and

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{n} = 0,$$

The sequence $\{\alpha_n\}$ satisfies the condition (C3).
Finally, we will show that the condition (C4) is achieved.

$$\frac{1}{3n} < \frac{1}{2n} \text{ and } \frac{1}{2n} < \frac{1}{n}.$$

Corollary 4.2. *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be an inverse-strongly monotone. Let $T : C \to C$ be a nonexpansive mapping. Let $B : C \to C$ be a β-strongly monotone and L-Lipschitz continuous. Suppose $\{x_n\}$ is a sequence generated by the following algorithm, for any $x_0 \in C$,*

$$\begin{cases} z_n = T(I - \delta_{n+1}A)(w_n x_n + (1 - w_n)x_{n+1}), \\ x_{n+1} = (1 - \alpha_n)(I - \mu\beta_n B)z_n, \ \forall n \geq 0, \end{cases} \tag{4.1}$$

$\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0, 1]$ satisfy the following conditions:
(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;
(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;
(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(C4): $\delta_n \leq \beta_n$ and $\beta_n \leq \alpha_n$.
Then $\{x_n\}$ converges strongly to $x^* \in VI(F(T), A)$, which is the unique solution of the variational inequality:

$$\text{Find } x^* \in VI(F(T), A) \text{ such that } \langle Bx^*, x - x^*\rangle \geq 0, \ \forall x \in VI(F(T), A). \tag{4.2}$$

Proof. Setting P_C as an identity mapping and $f, \phi \equiv 0$ in Theorem 3.1, we can obtain the desired conclusion immediately. ∎

Remark 4.3. Corollary 4.2 generalizes and improves the results of Iiduka [8].

Corollary 4.4. *Let H be a real Hilbert space, C be a closed convex subset of H. Let $A : C \to H$ be a strongly positive linear bounded operator, $f : C \to H$ be a ρ-contraction, γ be a positive real number such that $\frac{\bar{\gamma}-1}{\rho} < \gamma < \frac{\bar{\gamma}}{\rho}$ where $\bar{\gamma}$ is a positive constant number and $\rho \in [0, 1)$. Let $T : C \to C$ be a nonexpansive mapping. Suppose $\{x_n\}$ is a sequence generated by the following algorithm $x_0 \in C$ arbitrarily*

$$\begin{cases} z_n = TP_C[I - \delta_{n+1}(A - \gamma f)](w_n x_n + (1 - w_n)x_{n+1}), \\ x_{n+1} = \alpha_n(r_n x_n + (1 - r_n)x_{n+1}) + (1 - \alpha_n)(I - \mu\beta_n B)z_n, \ \forall n \geq 0, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\delta_n\} \subset [0, 1]$. If $\mu \in (0, \frac{2\beta}{L^2})$ is used and if $\{\beta_n\} \subset (0, 1]$ satisfy the following conditions:
(C1): $\Sigma_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$;
(C2): $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;
(C3): $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(C4): $\delta_n \leq \beta_n$ and $\beta_n \leq \alpha_n$.
Then $\{x_n\}$ converges strongly to $x^* \in \Omega$, which is the unique solution of the variational inequality:

$$\text{Find } x^* \in \Omega \text{ such that } \langle Bx^*, x - x^*\rangle \geq 0, \ \forall x \in \Omega. \tag{4.4}$$

Proof. Putting ϕ as an identity mapping in Theorem 3.1, we can obtain the desired conclusion immediately. ∎

Remark 4.5. Corollary 4.4 generalizes and improves the results of Marino and Xu [17].

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