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ON L-FUZZY FIXED POINT RESULTS IN $\mathbb{G}$-METRIC SPACE

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Abstract Among numerous efforts in progressing fuzzy mathematics, a lot of attention has been paid to studying new L-fuzzy analogues of the conventional fixed point results and their various applications. In this paper, some new concepts of L-fuzzy contractions in $\mathbb{G}$-metric space are studied, and sufficient conditions for the existence of L-fuzzy fixed points for such mappings are investigated. Nontrivial examples are provided to support the assumptions of our obtained theorems. It is highlighted that the main ideas established herein refine a few known results in the corresponding literature. Some of these special cases of our results are pointed out and analyzed.

MSC: 47H10; 47H30
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1. Introduction

Fixed point theory is an interdisciplinary topic which can be applied in various disciplines of mathematics and mathematical sciences like mathematical economics, optimization theory, game theory, variational inequalities and approximation theory. Poincare [21] was the first to work in the field of fixed point theory in 1886. After that, Banach [6] proved the existence of unique fixed point for a contractive mapping in a complete metric space. The fixed point theory as well as Banach contraction theorem has been studied and generalized in different spaces and various fixed point theorems were developed (e.g., see [9, 10, 20] and some references therein). Many authors have worked to find the fixed point of several contractive mappings and have also introduced several new contractions. In this regard, several fixed point theorems, common fixed point theorems and coincidence point theorems for both single-valued and fuzzy mappings, satisfying certain contractive conditions have been obtained. In 1968, Nadler [19] generalized the Banach contraction theorem to set valued mappings by using the Hausdorff distance. After that, many authors have studied various fixed point results for multi-valued contraction mappings [11, 16, 17, 29].

The concept of weak contraction was initiated by Alber and Gurre [2] in 1997. In 2004, Berinde [7] introduced the concept of $(\theta, \mathcal{L})$-weak contraction and studied fixed point theorems for the related contraction. Then, Berinde and Berinde [8] extended the concept of $(\theta, \mathcal{L})$-weak contraction from single-valued mappings to multi-valued mappings and presented related fixed point theorems for the Picard iteration associated to a multi-valued weak contraction.

Following the introduction of Banach contraction theorem, mathematicians have made several attempts to generalize the ideas via new concepts of metric spaces. In this regard, different authors have suggested various generalization of metric spaces. Along this line, Mustafa and Sims [11, 18] presented the concept of generalized metric spaces, namely $\mathbb{G}$-metric spaces. After that, various fixed point results have been obtained using various contractive conditions [1, 4, 22, 24].

In an effort to reduce uncertainties in dealing with practical problems for which classical mathematics cannot cope effectively, the evolvement of fuzzy mathematics came up with the introduction of the concepts of fuzzy sets by Zadeh [24] in 1965. Fuzzy set theory is now well-known as one of the mathematical tools for handling information with nonstatistical uncertainty. As a result, the theory of fuzzy sets has gained greater applications in diverse domains such as management sciences, engineering, environmental sciences, medical sciences and in other emerging fields. Meanwhile, the basic notions of fuzzy sets have been modified and improved in different directions; for example, see [5, 12–14, 23, 30]. In 1981, Heilpern [26] employed the concept of fuzzy set to initiate a class of fuzzy mappings and established a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of fixed point theorems due to Nadler [19]. A very interesting extension of fuzzy sets by replacing the interval $[0, 1]$ of range set by a complete distributive lattice was introduced by Goguen [25] and called L-fuzzy sets.

Following the existing literature, we noticed that L-fuzzy fixed point results in G-metric space are not sufficiently examined. Therefore, motivated by the basic ideas in [11, 15, 18, 25], the aim of this paper is to establish new fixed point theorems for L-fuzzy $(\theta, \mathcal{L})$-weak contraction mappings in $\mathbb{G}$-metric space. The presented results herein generalize and subsume some known ideas in the comparable literature.

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2. Preliminaries

This section lists some preliminary notions and results relevant to the main ideas following hereafter.

Definition 2.1. [11] Let $\mathcal{J} \neq \emptyset$ and $\mathbb{G} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathbb{R}^+$ be a function such that the following conditions are satisfied:

($\mathbb{G}$1) $\mathbb{G}(\delta, \rho, \gamma) = 0$ if $\delta = \rho = \gamma$;
($\mathbb{G}$2) $\mathbb{G}(\delta, \delta, \rho) > 0$ for all $\delta, \rho \in \mathcal{J}$ with $\delta \neq \rho$;
($\mathbb{G}$3) $\mathbb{G}(\delta, \delta, \rho) \leq \mathbb{G}(\delta, \rho, \gamma)$ for all $\delta, \rho, \gamma \in \mathcal{J}$ with $\gamma \neq \rho$;
($\mathbb{G}$4) $\mathbb{G}(\delta, \rho, \gamma) = \mathbb{G}(\delta, \gamma, \rho) = \mathbb{G}(\rho, \gamma, \delta) = \cdots$ (symmetric with respect to δ, ρ, γ);
($\mathbb{G}$5) $\mathbb{G}(\delta, \rho, \gamma) \leq \mathbb{G}(\delta, a, a) + \mathbb{G}(a, \rho, \gamma)$ for all $\delta, \rho, \gamma, a \in \mathcal{J}$ (rectangular property).

Then $\mathbb{G}$ is called a generalized function (or $\mathbb{G}$-metric) and $(\mathcal{J}, \mathbb{G})$ is called a generalized metric space (or $\mathbb{G}$-metric space).

Definition 2.2. [11] Let $(\mathcal{J}, \mathbb{G})$ be a $\mathbb{G}$-metric space. A sequence $\{\delta_e\}$ in $\mathcal{J}$ is called $\mathbb{G}$-Cauchy sequence if, for any $\varrho > 0$, there exists $O(\varrho) \in \mathbb{N}$ such that $\mathbb{G}(\delta_\varsigma, \delta_e, \delta_\rho) < \varrho$, for each $\varsigma, e, \rho \geq O(\varrho)$, that is, $\mathbb{G}(\delta_\varsigma, \delta_e, \delta_\rho) \to 0$ as $\varsigma, e, \rho \to \infty$.

Definition 2.3. [11] Let $(\mathcal{J}, \mathbb{G})$ be a $\mathbb{G}$-metric space. A sequence $\{\delta_e\}$ in $\mathcal{J}$ is called $\mathbb{G}$-complete (or complete $\mathbb{G}$-metric space) if every $\mathbb{G}$-Cauchy sequence in $(\mathcal{J}, \mathbb{G})$ is convergent in $\mathcal{J}$.

Lemma 2.4. [11] *Let $(\mathcal{J}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $\{\delta_e\}$ be a sequence in $\mathcal{J}$. Then the following statements are equivalent:*

(i) *$\{\delta_e\}$ is $\mathbb{G}$-convergent to δ;*
(ii) $\mathbb{G}(\delta_e, \delta_e, \delta) \to 0$, *as* $e \to \infty$;
(iii) $\mathbb{G}(\delta_e, \delta, \delta) \to 0$, *as* $e \to \infty$;
(iv) $\mathbb{G}(\delta_e, \delta_\rho, \delta) \to 0$, *as* $e, \rho \to \infty$.

Definition 2.5. [25] A partially ordered set $(L \preceq_L)$ is called:

(i) a lattice : if $\delta \vee \rho \in L$, $\delta \wedge \rho \in L$ for any $\delta, \rho \in L$;
(ii) a complete lattice : if $\vee A \in L$, $\wedge A \in L$ for any A $\subseteq$ L;
(iii) a distributive lattice : if $\delta \vee (\rho \wedge \gamma) = (\delta \vee \rho) \wedge (\delta \vee \gamma)$; $\delta \wedge (\rho \vee \gamma) = (\delta \wedge \rho) \vee (\delta \wedge \gamma)$ for any $\delta, \rho, \gamma \in L$;
(iv) a complete distributive lattice : if $\delta \vee (\wedge \rho_i) = \wedge_i(\delta \wedge \rho_i)$, $\delta \wedge (\vee_i \rho_i) = \vee_i(\delta \wedge \rho_i)$ for any $\delta, \rho_i \in$ L.

Definition 2.6. An L fuzzy set $\mathcal{A}$ in $\mathcal{J}$ is a function whose domain is $\mathcal{J}$ and co-domain is L , where L is a complete distributive lattice with top and bottom elements 1_L and 0_L, respectively. In other words, an L fuzzy set in $\mathcal{J}$ is a function $\mathcal{A} : \mathcal{J} \to L$.

We denote the family of all L-fuzzy sets in $\mathcal{J}$ by $\mathcal{F}_L$.

Definition 2.7. The α_L-level set of an L-fuzzy set $\mathcal{A}$ is denoted by $[\mathcal{A}]_{\alpha L}$ and is defined as follows:

$$[\mathcal{A}]_{\alpha L} = \begin{cases} \overline{\{\beta \in \mathcal{J} : 0_L \preceq_L \mathcal{A}(\beta)\}}, & \text{if } \alpha_L = 0 \\ \{\beta \in \mathcal{J} : \alpha_L \preceq_L \mathcal{A}(\beta)\}, & \text{if } \alpha_L \in L \setminus \{0_L\}. \end{cases}$$

Definition 2.8. Let $\mathcal{J}$ be a non-empty set. The mapping $\Re$: $\mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ is called an L-fuzzy set-valued map. A point $\mathcal{I} \in \mathcal{J}$ is said to be an L-fuzzy fixed point of $\Re$ if $\mathcal{I} \in [\Re\mathcal{I}]_{\alpha_L}$, for some $\alpha_L \in L \setminus \{0_L\}$.

Lemma 2.9. [11] *Let* $(\mathcal{J}, \mathbb{G})$ *be a* $\mathbb{G}$*-metric space, Then* $\mathbb{G}(\delta, \rho, \rho) \leq 2\mathbb{G}(\rho, \delta, \delta)$ *for all* $\delta, \rho \in \mathcal{J}$.

Lemma 2.10. [11] *Let* $(\mathcal{J}, \mathbb{G})$ *be a* $\mathbb{G}$*-metric space and* $\{\delta_e\}$ *be a sequence in* $\mathcal{J}$*. Then the following statements are equivalent:*

(i) $\{\delta_e\}$ *is* $\mathbb{G}$*-Cauchy Sequence*;
(ii) *For every* $\varrho > 0$*, there exists* $O(\varrho) \in \mathbb{N}$ *such that* $\mathbb{G}(\delta_e, \delta_\rho, \delta_\rho) < \varrho$*, for each* $e, \rho \geq O(\varrho)$;
(iii) $\{\delta_e\}$ *is a Cauchy sequence in the* $\mathbb{M}$ *space* $(\mathcal{J}, \xi_{\mathbb{G}})$.

Let $\mathcal{J}$ be a $\mathbb{G}$-metric space and $CB(\mathcal{J})$ be the family of all non empty closed and bounded subsets of $\mathcal{J}$. Then, the Hausdorff $\mathbb{G}$-distance is defined as:

$$\mathcal{H}_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) = \max\left\{ \sup_{\tau\in\mathcal{Z}_1} \mathbb{G}(\tau, \mathcal{Z}_2, \mathcal{Z}_3), \sup_{\tau\in\mathcal{Z}_2} \mathbb{G}(\tau, \mathcal{Z}_1, \mathcal{Z}_3), \sup_{\tau\in\mathcal{Z}_3} \mathbb{G}(\tau, \mathcal{Z}_1, \mathcal{Z}_2) \right\},$$

where

$$\begin{aligned} \mathbb{G}(\tau, \mathcal{Z}_2, \mathcal{Z}_3) =& d_{\mathbb{G}}(\tau, \mathcal{Z}_2) + d_{\mathbb{G}}(\mathcal{Z}_2, \mathcal{Z}_3) + d_{\mathbb{G}}(\tau, \mathcal{Z}_3), \\ d_{\mathbb{G}}(\tau, \mathcal{Z}_2) =& \inf_{\sigma\in\mathcal{Z}_2} d_{\mathbb{G}}(\tau, \sigma), \\ d_{\mathbb{G}}(\mathcal{Z}_1, \mathcal{Z}_2) =& \inf_{\tau\in\mathcal{Z}_1, \sigma\in\mathcal{Z}_2} d_{\mathbb{G}}(\tau, \sigma), \\ \mathbb{G}(\tau, \sigma, \mathcal{Z}_3) =& \inf_{\tau\in\mathcal{Z}_1, \sigma\in\mathcal{Z}_2, \upsilon\in\mathcal{Z}_3} d_{\mathbb{G}}(\tau, \sigma, \upsilon). \end{aligned}$$

Lemma 2.11. [28] *Let* $(\mathcal{J}, \mathbb{G})$ *be a* $\mathbb{G}$*-metric space and* $\mathcal{M}, \mathcal{N} \in CB(\mathcal{J})$*. Then for each* $\delta \in \mathcal{M}$*, we have*

$$\mathbb{G}(\delta, \mathcal{N}, \mathcal{N}) \leq \mathcal{H}_{\mathbb{G}}(\mathcal{M}, \mathcal{N}, \mathcal{N}).$$

Lemma 2.12. [28] *Let* $(\mathcal{J}, \mathbb{G})$ *be a* $\mathbb{G}$*-metric space. If* $\mathcal{M}, \mathcal{N} \in CB(\mathcal{J})$ *and, Then for each* $\epsilon > 0$, $\exists \rho \in \mathcal{N}$ *such that*

$$\mathbb{G}(\delta, \rho, \rho) \leq \mathcal{H}_{\mathbb{G}}(\mathcal{M}, \mathcal{N}, \mathcal{N}) + \epsilon.$$

Definition 2.13. [2] Let $(\mathcal{J}, \xi)$ be a metric space. A map $\phi : \mathcal{J} \to \mathcal{J}$ is called weak contraction if there exist a constant $\theta \in (0, 1)$ and some $\mathcal{L} \geq 0$ such that

$$\xi(\phi\delta, \phi\rho) \leq \theta\xi(\delta, \rho) + \mathcal{L}\xi(\rho, \phi\delta) \tag{2.1}$$

for all $\delta, \rho \in \mathcal{J}$.

Example 2.14. [7] Let $\phi : [0, 1] \to [0, 1]$ be given by $\phi\delta = \frac{2}{3}$, for $\delta \in [0, 1)$ and $\phi 1 = 0$. Then ϕ satisfies 2.1 with $\theta \geq \frac{2}{3}$ and $\mathcal{L} \geq \theta$. Here $\frac{2}{3}$ is a unique fixed point of ϕ.

Definition 2.15. [31] Let T be the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ in such a way that for $\psi \in \mathsf{T}, \sum_{e=1}^{\infty} \psi^e(\vartheta) < \infty$ and $\psi(\vartheta) < \vartheta$ for each $\vartheta > 0$ and ψ^e is the e-th iterate of ψ. Let $(\mathcal{J}, \eta)$ be a metric space. Then, $\phi : \mathcal{J} \to \mathcal{J}$ is an α-ψ-contractive mapping if for a pair functions $\alpha : \mathcal{J} \times \mathcal{J} \to [0, +\infty)$ and $\psi \in \mathsf{T}$, we have

$$\alpha(\delta, \rho)\eta(\phi\delta, \phi\rho) \leq \psi(\mu(\delta, \rho)) \text{ for all } \delta, \rho \in \mathcal{J}.$$

3. Main Results

In this section, we extend the idea of $(\theta, \mathcal{L})$-weak contraction for L-fuzzy mappings in $\mathbb{G}$-metric space and obtain fixed point theorem for L-fuzzy mapping and common fixed point theorem for a pair of L-fuzzy mappings satisfying $(\theta, \mathcal{L})$-weak contraction in $\mathbb{G}$-metric space.

Definition 3.1. [8] Let $(\mathcal{J}, \xi)$ be a metric space and $\phi : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ be an L-fuzzy mapping. Then $\mathcal{S}$ will be called L-fuzzy $(\theta, \mathcal{L})$ weak contraction if and only if there exist two constants $\theta \in (0,1)$ and $\mathcal{L} \geq 0$ such that

$$\mathcal{H}_{\mathbb{G}}([\phi\delta]_{\alpha_L}, [\mathcal{S}\rho]_{\alpha_L}) \leq \theta\xi(\delta, \rho) + \mathcal{L}D(\rho, [\phi\delta]_{\alpha_L}), \text{ for all } \delta, \rho \in \mathcal{J}.$$

Consistent with Definition 3.1, we have the next concept.

Definition 3.2. Let $(\mathcal{J}, \mathbb{G})$ be a $\mathbb{G}$-metric space and $\phi_1, \phi_2 : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ be a pair of L-fuzzy mappings. The pair ϕ_1, ϕ_2 is said to be L-fuzzy $(\theta, \mathcal{L})$-weak contraction if there exist constants $\theta \in (0,1)$ and $\mathcal{L}_1, \mathcal{L}_2 \geq 0$ such that:

(i) $\mathcal{H}_{\mathbb{G}}([\phi_1\delta]_{\alpha_L}, [\phi_2\rho]_{\alpha_L}, [\phi_2\gamma]_{\alpha_L}) \leq \theta\mathbb{G}(\delta, \rho, \gamma) + \mathcal{L}_1\xi_{\mathbb{G}}(\rho, [\phi_1\delta]_{\alpha_L})$ for all $\delta, \rho, \gamma \in \mathcal{F}$.

(ii) $\mathcal{H}_{\mathbb{G}}([\phi_2\delta]_{\alpha_L}, [\phi_1\rho]_{\alpha_L}, [\phi_1\gamma]_{\alpha_L}) \leq \theta\mathbb{G}(\delta, \rho, \gamma) + \mathcal{L}_2\xi_{\mathbb{G}}(\rho, [\phi_2\delta]_{\alpha_L})$ for all $\delta, \rho, \gamma \in \mathcal{F}$.

The following is our first principal result.

Theorem 3.3. *Let $(\mathcal{J}, \mathbb{G})$ be a complete $\mathbb{G}$-metric space and $\phi : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ be an L-fuzzy $(\theta, \mathcal{L})$-weak contraction mapping, that is, there exist two constants $\theta \in (0,1)$ and $\mathcal{L} \geq 0$ such that:*

$$\mathcal{H}_{\mathbb{G}}\Big([\phi\delta]_{\alpha_L}, [\mathcal{S}\rho]_{\alpha_L}, [\phi\gamma]\alpha_L\Big) \leq \theta\mathbb{G}(\delta, \rho, \gamma) + \mathcal{L}\xi_{\mathbb{G}}(\rho, [\phi\delta]_{\alpha_L}). \tag{3.1}$$

Then there exists $\delta^* \in \mathcal{J}$ such that $\delta^* \in [\phi\delta^*]_{\alpha_L}$, that is , δ^* is a fixed point of ϕ.

Proof. Consider $\delta_0 \in \mathcal{J}$. Define $\delta_1 \in [\phi_1\delta_0]_{\alpha_L}$ and $\delta_2 \in [\phi_2\delta_1]_{\alpha_L}$ and so on. Generally,

$$\delta_{e+1} \in [\phi\delta_e]\alpha_L. \ e = 0, 1, 2, \ldots$$

For $k > 0$, let $k\theta = h$. Then by condition 3.1 and Lemma 2.12, we have

$$\begin{aligned}
\mathbb{G}(\delta_1, \delta_2, \delta_2) \leq & k\mathcal{H}_{\mathbb{G}}\Big([\phi\delta_0]_{\alpha_L}, [\phi\delta_1]_{\alpha_L}, [\phi\delta_1]_{\alpha_L}\Big) \\
\leq & k\Big[\theta\mathbb{G}(\delta_0, \delta_1, \delta_1) + \mathcal{L}\xi_{\mathbb{G}}(\delta_1, [\phi\delta_0]_{\alpha_L})\Big] \\
\leq & k\theta(\mathbb{G}\Big(\delta_0, \delta_1, \delta_1)\Big) \\
= & h(\mathbb{G}(\delta_0, \delta_1, \delta_1).
\end{aligned}$$

For $k > 0$, let $k\theta = h$, Then by condition 3.1 and Lemma 2.12, we have

$$\begin{aligned}
\mathbb{G}(\delta_2, \delta_3, \delta_3) \leq & k\mathcal{H}_{\mathbb{G}}([\phi\delta_1]_{\alpha_L}, [\phi\delta_2]_{\alpha_L}, [\phi\delta_2]_{\alpha_L}) \\
\leq & k\Big[\theta\mathbb{G}(\delta_1, \delta_2, \delta_2) + \mathcal{L}\xi_{\mathbb{G}}(\delta_2, [\phi\delta_1]_{\alpha_L})\Big] \\
\leq & k\theta\Big(\mathbb{G}(\delta_1, \delta_2, \delta_2)\Big) \\
= & h(\mathbb{G}(\delta_1, \delta_2, \delta_2).
\end{aligned}$$

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Continuing in this way, we have:

$$\mathbb{G}(\delta_e, \delta_{e+1}, \delta_{e+1}) \leq h^e(\mathbb{G}(\delta_0, \delta_1, \delta_1)).$$

To show that the sequence $\{\delta_e\}$ is Cauchy, consider for $\rho > e$:

$$\begin{aligned}
\mathbb{G}(\delta_e, \delta_\rho, \delta_\rho) \leq & \mathbb{G}(\delta_e, \delta_{e+1}, \delta_{e+1}) + \mathbb{G}(\delta_{e+1}, \delta_{e+2}, \delta_{e+2}) + \cdots + \mathbb{G}(\delta_{\rho-1}, \delta_\rho, \delta_\rho) \\
\leq & h^e \mathbb{G}(\delta_0, \delta_1, \delta_1) + h^{e+1}\mathbb{G}(\delta_0, \delta_1, \delta_1) + \cdots + h^{\rho-1}\mathbb{G}(\delta_0, \delta_1, \delta_1) \\
\leq & (h^e + h^{e+1} + \cdots + h^{\rho-1})\mathbb{G}(\delta_0, \delta_1, \delta_1) \\
\leq & h^e(1 + h^e + \cdots + h^{\rho-e-1})\mathbb{G}(\delta_0, \delta_1, \delta_1) \\
\leq & h^e(\frac{1 - h^{\rho-e-1}}{1-h})\mathbb{G}(\delta_0, \delta_1, \delta_1) \\
\leq & h^e \mathbb{G}(\delta_0, \delta_1, \delta_1) \to 0 \text{ as } e \to \infty.
\end{aligned}$$

This shows that $\{\delta_e\}$ is a Cauchy sequence in $\mathcal{J}$. The completeness of this space yields that there exists $\delta^* \in \mathcal{J}$ such that $\delta_e \to \delta^*$ as $e \to \infty$. Now,

$$\begin{aligned}
\mathbb{G}(\delta_e, [\phi\delta^*]_{\alpha_L}, [\phi\delta^*]_{\alpha_L}) \leq & k\mathcal{H}_{\mathbb{G}}([\phi\delta_{e-1}]_{\alpha_L}, [\phi\delta^*]_{\alpha_L}, [\phi\delta^*]_{\alpha_L}) \\
\leq & k\Big[\theta\mathbb{G}(\delta_{e-1}, \delta^*, \delta^*) + \mathcal{L}\xi_{\mathbb{G}}(\delta^*, [\phi\delta_{e-1}]_{\alpha_L})\Big].
\end{aligned}$$

Applying $\lim e \to \infty$, we get

$$\lim_{e\to\infty} \mathbb{G}(\delta_e, [\phi\delta^*]_{\alpha_L}, [\phi\delta^*]_{\alpha_L}) \leq k\theta \lim_{e\to\infty}\Big[\theta\mathbb{G}(\delta_{e-1}, \delta^*, \delta^*) + \mathcal{L}\xi_{\mathbb{G}}(\delta^*, [\phi\delta_{e-1}]_{\alpha_L})\Big]$$

which gives $\mathbb{G}(\delta_e, [\phi\delta^*]_{\alpha_L}, [\phi\delta^*]_{\alpha_L}) = 0$. This implies $\delta^* \in [\phi\delta^*]_{\alpha_L}$. Hence δ^* is a fixed point of $\mathcal{J}$. ∎

Example 3.4. Let $\mathcal{J} = [0, 1]$ be a $\mathbb{G}$-metric space and $L = \{w_1, w_2\}$ with $w_1 \preceq_L w_2$, Therefore $(L, \preceq_L)$ is a complete distributive lattice. Let $\phi : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ be an L-fuzzy mapping defined as follows:

$$\phi(\delta)(t) = \begin{cases} w_1 \text{ if } 0 \preceq_L t \preceq_L \frac{\delta}{7} \\ w_2 \text{ if } \frac{\delta}{20} \preceq_L t \preceq_L 1. \end{cases}$$

Therefore for $\alpha_L = w_1$, $[\phi(\delta)]_{w_1} = \{t : 0 \preceq_L t \preceq_L \frac{\delta}{7}\}$. Thus, for $\theta = \frac{1}{7}$ and $\mathcal{L} \geq 0$ all conditions of Theorem 3.3 are satisfied. Therefore, there exists $0 \in \mathcal{J}$ such that $0 \in [\phi(0)]_{w_1}$.

Theorem 3.5. *Let $(\mathcal{J}, \mathbb{G})$ be a complete $\mathbb{G}$-metric space and $\phi_1, \phi_2 : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ be a pair of L-fuzzy $(\theta, \mathcal{L})$-weak contraction mappings such that for all $\delta, \rho, \gamma \in \mathcal{J}$ and $\theta \in (0, 1)$, $\mathcal{L}_1, \mathcal{L}_2 \geq 0$,*

$$\mathcal{H}_{\mathbb{G}}([\phi_1\delta]_{\alpha_L}, [\phi_2\rho]_{\alpha_L}, [\phi_2\gamma]_{\alpha_L}) \leq \frac{\theta}{6} \max \mathcal{K}_{i,j} + \mathcal{L}_1\xi_{\mathbb{G}}(\rho, [\mathcal{S}_1\delta]_{\alpha_L}) + \mathcal{L}_2\mathcal{S}_{i,j}, \tag{3.2}$$

where

$$\mathcal{K}_{i,j} = \max \left[\begin{array}{c} 6\mathbb{G}(\delta, \rho, \gamma), \mathbb{G}(\delta, [\phi_i\delta]_{\alpha_L}, [\phi_i\delta]_{\alpha_L}), \\ \mathbb{G}(\rho, [\phi_j\rho]_{\alpha_L}, [\phi_j\rho]_{\alpha_L}), \\ \frac{\mathbb{G}(\rho, [\phi_i\delta]_{\alpha_L}, [\phi_i\delta]_{\alpha_L}) + \mathbb{G}(\delta, [\phi_j\rho]_{\alpha_L}, [\phi_j\rho]_{\alpha_L})}{2} \end{array} \right],$$

$$\mathcal{S}_{i,j} = \min \begin{bmatrix} \mathbb{G}(\rho, [\phi_j\rho]_{\alpha_L}, [\phi_j\rho]_{\alpha_L}), \\ \mathbb{G}(\rho, [\phi_i\delta]_{\alpha_L}, [\phi_i\delta]_{\alpha_L}), \\ \mathbb{G}(\delta, [\phi_j\rho]_{\alpha_L}, [\phi_j\rho]_{\alpha_L}) \end{bmatrix},$$

and $i \neq j$, $i, j = 1, 2$. *Then* $\exists\ \delta^*$ *such that* $\delta^* \in [\phi_1\delta^*]_{\alpha_L}$ *and* $\delta^* \in [\phi_2\delta^*]_{\alpha_L}$.

Proof. Let $\delta_0 \in \mathcal{J}$. Take $\delta_1 \in [\mathcal{S}_1\delta_0]_{\alpha_L}$ and $\delta_2 \in [\phi_2\delta_1]_{\alpha_L}$ and so on. Generally

$$\delta_{2e+1} \in [\phi_1\delta_{2e}]_{\alpha_L}\ \delta_{2e+2} \in [\phi_2\delta_{2e+1}]_{\alpha_L} \qquad e = 0, 1, 2, \ldots$$

For $k > 0$, let $k\theta = h$, Then by condition 3.2 and Lemma 2.12, we have

$$\begin{aligned} \mathbb{G}(\delta_1, \delta_2, \delta_2) \leq & kH_{\mathbb{G}}\Big([\phi_1\delta_0]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L}\Big) \\ \leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta_0, \delta_1, \delta_1), G(\delta_0, [\phi_1\delta_0]_{\alpha_L}, [\phi_1\delta_0]_{\alpha_L}), \\ \mathbb{G}(\delta_1, [\phi_2\delta_1]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L}), \\ \frac{\mathbb{G}(\delta_1, [\phi_1\delta_0]_{\alpha_L}, [\mathcal{S}_1\delta_0]_{\alpha_L}) + \mathbb{G}(\delta_0, [\phi_2\delta_1]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L})}{2} \end{array}\right)\right] \\ & + \mathcal{L}_1\xi_{\mathbb{G}}(\delta_1, [\phi_1\delta_0]_{\alpha_L}) + \mathcal{L}_2 \min \begin{bmatrix} \mathbb{G}(\delta_1, [\phi_2\delta_1]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L}), \\ \mathbb{G}(\delta_0, [\phi_2\delta_1]_{\alpha_L}, [\phi_2\delta_1]_{\alpha_L}), \\ \mathbb{G}(\delta_1, [\phi_1\delta_0]_{\alpha_L}, [\phi_1\delta_0]_{\alpha_L}) \end{bmatrix} \\ \leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta_0, \delta_1, \delta_1), 6\mathbb{G}(\delta_0, \delta_1, \delta_1), 6\mathbb{G}(\delta_1, \delta_2, \delta_2), \\ \frac{6\mathbb{G}(\delta_1, \delta_1, \delta_1) + 6\mathbb{G}(\delta_0, \delta_2, \delta_2)}{2} \end{array}\right)\right] \\ & + \mathcal{L}_1\xi_{\mathbb{G}}(\delta_1, \delta_1) + \mathcal{L}_2 \min\Big[6\mathbb{G}(\delta_1, \delta_2, \delta_2), 6\mathbb{G}(\delta_0, \delta_2, \delta_2), 6\mathbb{G}(\delta_1, \delta_1, \delta_1)\Big]. \\ \leq & k\left[\frac{\theta}{6}\max\left(6\mathbb{G}(\delta_0, \delta_1, \delta_1), 6\mathbb{G}(\delta_1, \delta_2, \delta_2), \frac{6\mathbb{G}(\delta_0, \delta_2, \delta_2)}{2}\right)\right] \\ & + \mathcal{L}_2 \min\Big[6\mathbb{G}(\delta_1, \delta_2, \delta_2), 0, 6\mathbb{G}(\delta_0, \delta_2, \delta_2)\Big]. \\ \leq & k\left[\frac{\theta}{6}\left[\max\left(6\mathbb{G}(\delta_0, \delta_1, \delta_1), 6\mathbb{G}(\delta_1, \delta_2, \delta_2), \frac{6\mathbb{G}(\delta_0, \delta_2, \delta_2)}{2}\right)\right]\right] \end{aligned}$$

Since

$$\frac{6\mathbb{G}(\delta_0, \delta_2, \delta_2)}{2} \leq \frac{6\mathbb{G}(\delta_0, \delta_1, \delta_1) + 6\mathbb{G}(\delta_1, \delta_2, \delta_2)}{2},$$

So we have

$$\frac{\mathbb{G}(\delta_0, \delta_2, \delta_2)}{2} \leq \max\Big[\mathbb{G}(\delta_0, \delta_1, \delta_1), \mathbb{G}(\delta_1, \delta_2, \delta_2)\Big],$$

which gives,

$$\mathbb{G}(\delta_1, \delta_2, \delta_2) \leq k\Big[\theta \max\Big(\mathbb{G}(\delta_0, \delta_1, \delta_1), \mathbb{G}(\delta_1, \delta_2, \delta_2)\Big)\Big].$$

Suppose $\mathbb{G}(\delta_0, \delta_1, \delta_1) < \mathbb{G}(\delta_1, \delta_2, \delta_2)$. By using properties 2.15 of ψ, we get a contradiction. Hence

$$\mathbb{G}(\delta_1, \delta_2, \delta_2) \leq k\theta\Big[\mathbb{G}(\delta_0, \delta_1, \delta_1)\Big].$$

As $h = k\theta$, then

$$\mathbb{G}(\delta_1, \delta_2, \delta_2) \leq h(\mathbb{G}(\delta_0, \delta_1, \delta_1)).$$

Given that $\delta_2 \in [\phi_2\delta_1]_{\alpha_L}$ and $\delta_3 \in [\phi_1\delta_2]_{\alpha_L}$, there exist $k > 0$ and $k\theta = h$. So, again by Lemma 2.12 and condition (3.2), we get

$$\begin{aligned}
\mathbb{G}(\delta_2,\delta_3,\delta_3) \leq & kH_{\mathbb{G}}\Big([\phi_2\delta_1]_{\alpha_L},[\phi_1\delta_1]_{\alpha_L},[\phi_1\delta_2]_{\alpha_L}\Big)\\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta_1,\delta_2,\delta_2), G(\delta_1,[\phi_2\delta_1]_{\alpha_L},[\phi_2\delta_1]_{\alpha_L}),\\ \mathbb{G}(\delta_2,[\phi_1\delta_2]_{\alpha_L},[\phi_1\delta_2]_{\alpha_L}),\\ \frac{\mathbb{G}(\delta_2,[\phi_2\delta_1]_{\alpha_L},[\mathcal{S}_2\delta_1]_{\alpha_L})+\mathbb{G}(\delta_1,[\phi_1\delta_2]_{\alpha_L},[\phi_1\delta_2]_{\alpha_L})}{2}\end{array}\right)\right]\\
& + \mathcal{L}_3\xi_{\mathbb{G}}(\delta_2,[\phi_2\delta_1]_{\alpha_L}) + \mathcal{L}_4\min\left[\begin{array}{c}\mathbb{G}(\delta_2,[\phi_1\delta_2]_{\alpha_L},[\phi_1\delta_2]_{\alpha_L}),\\ \mathbb{G}(\delta_1,[\phi_1\delta_2]_{\alpha_L},[\phi_1\delta_2]_{\alpha_L}),\\ \mathbb{G}(\delta_2,[\phi_2\delta_1]_{\alpha_L},[\phi_2\delta_1]_{\alpha_L})\end{array}\right]\\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\delta_1,\delta_2,\delta_2),\\ 6\mathbb{G}(\delta_1,\delta_2,\delta_2),\\ 6\mathbb{G}(\delta_2,\delta_3,\delta_3),\\ \frac{6\mathbb{G}(\delta_2,\delta_2,\delta_2)+6\mathbb{G}(\delta_1,\delta_3,\delta_3)}{2}\end{array}\right)\right] + \mathcal{L}_3\xi_{\mathbb{G}}(\delta_2,\delta_2)\\
& + \mathcal{L}_4\min\Big[6\mathbb{G}(\delta_2,\delta_3,\delta_3), 6\mathbb{G}(\delta_1,\delta_3,\delta_3), 6\mathbb{G}(\delta_2,\delta_2,\delta_2)\Big]\\
\leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\delta_1,\delta_2,\delta_2),\\ 6\mathbb{G}(\delta_2,\delta_3,\delta_3),\\ \frac{6\mathbb{G}(\delta_1,\delta_3,\delta_3)}{2}\end{array}\right)\right]\\
& + \mathcal{L}_4\min\left[\begin{array}{c}6\mathbb{G}(\delta_2,\delta_3,\delta_3),\\ 0,\\ 6\mathbb{G}(\delta_1,\delta_3,\delta_3)\end{array}\right]\\
\leq & k\left[\frac{\theta}{6}\max\left(6\mathbb{G}(\delta_1,\delta_2,\delta_2), 6\mathbb{G}(\delta_2,\delta_3,\delta_3), \frac{6\mathbb{G}(\delta_1,\delta_3,\delta_3)}{2}\right)\right].
\end{aligned}$$

Since

$$\frac{6\mathbb{G}(\delta_1,\delta_3,\delta_3)}{2} \leq \frac{6\mathbb{G}(\delta_1,\delta_2,\delta_2)+6\mathbb{G}(\delta_2,\delta_3,\delta_3)}{2},$$

So we have

$$\frac{\mathbb{G}(\delta_1,\delta_3,\delta_3)}{2} \leq \max[\mathbb{G}(\delta_1,\delta_2,\delta_2),\mathbb{G}(\delta_2,\delta_3,\delta_3)],$$

which gives,

$$\mathbb{G}(\delta_2,\delta_3,\delta_3) \leq k\Big[\theta\max\Big(\mathbb{G}(\delta_1,\delta_2,\delta_2),\mathbb{G}(\delta_2,\delta_3,\delta_3)\Big)\Big].$$

Suppose $\mathbb{G}(\delta_1,\delta_2,\delta_2) < \mathbb{G}(\delta_2,\delta_3,\delta_3)$. By using properties 2.15 of ψ, we get a contradiction. Hence,

$$\mathbb{G}(\delta_2,\delta_3,\delta_3) \leq k\theta(\mathbb{G}(\delta_1,\delta_2,\delta_2)).$$

As $h = k\theta$, then

$$\mathbb{G}(\delta_2,\delta_3,\delta_3) \leq h(\mathbb{G}(\delta_1,\delta_2,\delta_2)).$$

Now using the above expresion, we can be write

$$\mathbb{G}(\delta_2,\delta_3,\delta_3) \leq h(\mathbb{G}(\delta_1,\delta_2,\delta_2)) \leq h(h(\mathbb{G}(\delta_0,\delta_1,\delta_1))) = h^2(\mathbb{G}(\delta_0,\delta_1,\delta_1)).$$

Continuing in this way, we get a sequence δ_e in $\mathcal{J}$ for $\alpha_L(\delta_e,\delta_{e+1},\delta_{e+1}) \geq 1$ such that

$$\mathbb{G}(\delta_e,\delta_{e+1},\delta_{e+1}) \leq h^e(\mathbb{G}(\delta_0,\delta_1,\delta_1)).$$

To show that the sequence δ_e is Cauchy, Consider for $\rho > e$:

$$\begin{aligned}\mathbb{G}(\delta_e,\delta_\rho,\delta_\rho) \leq & \mathbb{G}(\delta_e,\delta_{e+1},\delta_{e+1}) + \mathbb{G}(\delta_{e+1},\delta_{e+2},\delta_{e+2}) + \cdots + \mathbb{G}(\delta_{\rho-1},\delta_\rho,\delta_\rho)\\ \leq & h^e\mathbb{G}(\delta_0,\delta_1,\delta_1) + h^{e+1}\mathbb{G}(\delta_0,\delta_1,\delta_1) + \cdots + h^{\rho-1}\mathbb{G}(\delta_0,\delta_1,\delta_1)\\ \leq & (h^e + h^{e+1} + \cdots + h^{\rho-1})\mathbb{G}(\delta_0,\delta_1,\delta_1)\\ \leq & h^e(1 + h^e + \cdots + h^{\rho-e-1})\mathbb{G}(\delta_0,\delta_1,\delta_1)\\ \leq & h^e(\frac{1-h^{\rho-e-1}}{1-h})\mathbb{G}(\delta_0,\delta_1,\delta_1)\\ \leq & h^e\mathbb{G}(\delta_0,\delta_1,\delta_1) \to 0 \text{ as } e \to \infty.\end{aligned}$$

This shows that $\{\delta_e\}$ is a Cauchy sequence in $\mathcal{J}$. Since $\mathcal{J}$ is complete so, $\delta^* \in \mathcal{J}$ such that $\delta_e \to \delta^*$ as $e \to \infty$. Using the fact that $\delta_{2e+1} \in [\phi_1\delta_{2e}]_{\alpha_L}$ and $\delta_{2e+2} \in [\phi_2\delta_{2e+1}]_{\alpha_L}$, now we show that $\delta^* \in [\phi_1\delta^*]_{\alpha_L}$ and $\delta^* \in [\phi_2\delta^*]_{\alpha_L}$. Now,

$$\begin{aligned}&\mathbb{G}(\delta_{2e+1},[\phi_2\delta^*]_{\alpha_L},[\mathcal{S}_2\delta^*]_{\alpha_L})\\ &\leq kH_{\mathbb{G}}([\phi_1\delta_{2e}]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})\\ &\leq k\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\delta_{2e},\delta^*,\delta^*),\mathbb{G}(\delta_{2e},[\mathcal{S}_1\delta_{2e}]_{\alpha_L},[\phi_1\delta_{2e}]_{\alpha_L}),\\ \mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ \frac{\mathbb{G}(\delta^*,[\phi_1\delta_{2e}]_{\alpha_L},[\mathcal{S}_1\delta_{2e}]_{\alpha_L})+\mathbb{G}(\delta_{2e},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2}\end{array}\right)\right]\\ &+\mathcal{L}_1\xi_{\mathbb{G}}(\delta^*,[\phi_1\delta_{2e}]_{\alpha_L})+\mathcal{L}_2\min\left[\begin{array}{c}\mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ \mathbb{G}(\delta^*,[\phi_1\delta_{2e}]_{\alpha_L},[\phi_1\delta_{2e}]_{\alpha_L}),\\ \mathbb{G}(\delta_{2e},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})\end{array}\right]\\ \leq & k\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\delta_{2e},\delta^*,\delta^*),6\mathbb{G}(\delta_{2e},\delta_{2e+1},\delta_{2e+1}),\\ \mathbb{G}(\delta^*,[\mathcal{S}_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ \frac{6\mathbb{G}(\delta^*,\delta_{2e+1},\delta_{2e+1})+\mathbb{G}(\delta_{2e},[\mathcal{S}_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2}\end{array}\right)\right]\\ &+\mathcal{L}_1\xi_{\mathbb{G}}(\delta^*,\delta_{2e+1})+\mathcal{L}_2\min\left[\begin{array}{c}\mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\mathcal{S}_2\delta^*]_{\alpha_L}),\\ \mathbb{G}(\delta_{2e},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ 6\mathbb{G}(\delta^*,\delta_{2e+1},\delta_{2e+1})\end{array}\right].\end{aligned}$$

Applying $\lim e \to \infty$, we get

$$\begin{aligned}&\lim_{e\to\infty}\mathbb{G}(\delta_{2e+1},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})\\ &\leq k\lim_{e\to\infty}\left[\frac{\theta}{6}\max\left(\begin{array}{c}6\mathbb{G}(\delta_{2e},\delta^*,\delta^*),6\mathbb{G}(\delta_{2e},\delta_{2e+1},\delta_{2e+1}),\\ \mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ \frac{6\mathbb{G}(\delta^*,\delta_{2e+1},\delta_{2e+1})+\mathbb{G}(\delta_{2e},[\mathcal{S}_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2}\end{array}\right)\right]\\ &+\lim_{e\to\infty}\mathcal{L}_1\xi_{\mathbb{G}}(\delta^*,\delta_{2e+1})+\lim_{e\to\infty}\mathcal{L}_2\min\left[\begin{array}{c}\mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\mathcal{S}_2\delta^*]_{\alpha_L}),\\ \mathbb{G}(\delta_{2e},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L}),\\ 6\mathbb{G}(\delta^*,\delta_{2e+1},\delta_{2e+1}),\end{array}\right].\end{aligned}$$

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This implies:

$$
\begin{aligned}
&\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}) \\
&\leq k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta^*,\delta^*,\delta^*), 6\mathbb{G}(\delta^*,\delta^*,\delta^*), \\ \mathbb{G}(\delta^*, [\mathcal{S}_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), \\ \frac{6\mathbb{G}(\delta^*,\delta^*,\delta^*)+\mathbb{G}(\delta^*,[\mathcal{S}_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2} \end{array}\right)\right] \\
&+\mathcal{L}_1\xi_{\mathbb{G}}(\delta^*,\delta^*)+\mathcal{L}_2\min\left[\begin{array}{c} \mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), \\ \mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), \\ 6\mathbb{G}(\delta^*,\delta^*,\delta^*) \end{array}\right] \\
\leq& k\left[\frac{\theta}{6}\max\left(\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), \frac{\mathbb{G}(\delta^*,[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2}\right)\right] \\
&+\mathcal{L}_2\min[\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), 0]. \\
\leq& k\frac{\theta}{6}\left[\max\left(\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}), \frac{\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L})}{2}\right)\right] \\
\leq& k\frac{\theta}{6}\left[\frac{\mathbb{G}(\delta^*, [\mathcal{S}_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L})}{2}\right].
\end{aligned}
$$

Therefore, $\left[1-\frac{k\theta}{6}\right]\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}) = 0$. Hence $\mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_2\delta^*]_{\alpha_L}) = 0$. This implies that $\delta^* \in [\phi_2\delta^*]_{\alpha_L}$. Now,

$$
\begin{aligned}
&\mathbb{G}(\delta_{2e+2}, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}) \\
&\leq kH_{\mathbb{G}}([\phi_2\delta_{2e+1}]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}) \\
&\leq k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta_{2e+2},\delta^*,\delta^*), \mathbb{G}(\delta_{2e+1}, [\phi_2\delta_{2e+1}]_{\alpha_L}, [\phi_2\delta_{2e+1}]_{\alpha_L}), \\ \mathbb{G}(\delta^*, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}), \\ \frac{\mathbb{G}(\delta^*,[\phi_2\delta_{2e+1}]_{\alpha_L},[\phi_2\delta_{2e+1}]_{\alpha_L})+\mathbb{G}(\delta_{2e+1},[\phi_1\delta^*]_{\alpha_L},[\phi_1\delta^*]_{\alpha_L})}{2} \end{array}\right)\right] \\
&\mathcal{L}_3\xi_{\mathbb{G}}(\delta^*, [\phi_2\delta_{2e+1}]_{\alpha_L})+\mathcal{L}_4\min\left[\begin{array}{c} \mathbb{G}(\delta^*, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}), \\ \mathbb{G}(\delta^*, [\phi_2\delta_{2e+1}]_{\alpha_L}, [\phi_2\delta_{2e+1}]_{\alpha_L}), \\ \mathbb{G}(\delta_{2e+1}, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}) \end{array}\right] \\
\leq& k\left[\frac{\theta}{6}\max\left(\begin{array}{c} 6\mathbb{G}(\delta_{2e+1},\delta^*,\delta^*), 6\mathbb{G}(\delta_{2e+1},\delta_{2e+2},\delta_{2e+2}), \\ \mathbb{G}(\delta^*, [\phi_2\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}), \\ \frac{6\mathbb{G}(\delta^*,\delta_{2e+2},\delta_{2e+2})+\mathbb{G}(\delta_{2e+1},[\phi_2\delta^*]_{\alpha_L},[\phi_2\delta^*]_{\alpha_L})}{2} \end{array}\right)\right] \\
&+\mathcal{L}_3\xi_{\mathbb{G}}(\delta^*,\delta_{2e+2})+\mathcal{L}_4\min\left[\begin{array}{c} \mathbb{G}(\delta^*, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}), \\ \mathbb{G}(\delta_{2e+1}, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}), \\ 6\mathbb{G}(\delta^*,\delta_{2e+2},\delta_{2e+2}) \end{array}\right].
\end{aligned}
$$

Similarly, applying $\lim e \to \infty$, we get $\mathbb{G}(\delta^*, [\phi_1\delta^*]_{\alpha_L}, [\phi_1\delta^*]_{\alpha_L}) = 0$. This implies that $\delta^* \in [\phi_1\delta^*]_{\alpha_L}$. Hence, δ^* is the common fixed point of the mappings ϕ_1 and ϕ_2. ∎

Example 3.6. Let $\mathcal{J} = [0,1]$, $\mathbb{G}(\delta,\rho,\gamma) = |\delta-\rho|+|\rho-\gamma|+|\delta-\gamma|$ for all $\delta,\rho,\gamma \in \mathcal{J}$. Let $L = \{w_1, w_2, w_3, w_4\}$ with $w_1 \preceq_L w_2 \preceq_L w_4$, $w_1 \preceq_L w_3 \preceq_L w_4$ where w_2 and w_3 are not comparable, Therefore $(L, \preceq_L)$ is a complete distributive lattice. Suppose that

$S_1, S_2 : \mathcal{J} \to \mathcal{F}_L(\mathcal{J})$ are L-fuzzy mappings defined as follows:

$$S_1(\delta)(t) = \begin{cases} w_1 \text{ if } 0 \preceq_L t \preceq_L \frac{\delta}{60}, \\ w_2 \text{ if } \frac{\delta}{60} \preceq_L t \preceq_L \frac{\delta}{40} \\ w_3 \text{ if } \frac{\delta}{40} \preceq_L t \preceq_L \frac{\delta}{20} \\ w_4 \text{ if } \frac{\delta}{20} \preceq_L t \preceq_L 1 \end{cases}$$

and

$$S_2(y)(t) = \begin{cases} w_1 \text{ if } 0 \preceq_L t \preceq_L \frac{\delta}{60}, \\ w_2 \text{ if } \frac{\delta}{60} \preceq_L t \preceq_L \frac{\delta}{400} \\ w_3 \text{ if } \frac{\delta}{400} \preceq_L t \preceq_L \frac{\delta}{200} \\ w_4 \text{ if } \frac{\delta}{200} \preceq_L t \preceq_L 1. \end{cases}$$

Therefore, for $\alpha_L = w_1$, $[S_1(\delta)]_{w_1} = \{t : 0 \preceq_L t \preceq_L \frac{\delta}{60}\}$ and $[S_2(y)]_{w_1} = \{t \in [0,1] : 0 \preceq_L t \preceq_L \frac{\delta}{60}\}$, we have $\mathcal{H}_{\mathbb{G}}([S_1(\delta)]_{w_1}, [S_2(y)]_{w_1}, [S_2(z)]_{w_1}) = \mathcal{H}_{\mathbb{G}}([S_2(\delta)]_{w_1}, [S_1(y)]_{w_1}, [S_1(z)]_{w_1})$ $\leq k(|\delta - y| + |y - z| + |\delta - z|)$ for $\frac{1}{60} < k \leq 1$.Thus, all conditions of Theorem 3.5 are satisfied. There exist a $0 \in \mathcal{J}$ such that $0 \in [S_1(0)]_{w_1} \cap [S_2(0)]_{w_1}$.

4. Application

Consider the integral equation

$$s(t) = \int_0^Z \mathcal{J}(t,u) f(u, s(u)) du, \ t \in [0, Z], \tag{4.1}$$

where $Z > 0$, $f : [0, Z] \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} : [0, Z] \times [0, Z] \to \mathbb{R}$ are continuous function. Let $\mathcal{F} = C[0, Z]$ be the set of all continuous functions on $[0, Z]$ with

$$\mathbb{G}(\delta, \rho, \gamma) = \sup_{\delta \in [0,Z]} |\delta(\delta) - \rho(\delta)| + \sup_{\delta \in [0,Z]} |\rho(\delta) - \gamma(\delta)| + \sup_{\delta \in [0,Z]} |\delta(\delta) - \gamma(\delta)|.$$

This section's goal is to provide an existence theorem for a solution to the integral equation mentioned above.

Theorem 4.1. *Consider the mapping $F : C[0, Z] \longrightarrow C[0, Z]$ defined by*

$$Fs(\delta) = \int_0^Z \mathcal{J}(\delta, u) f(u, s(u)) du. \tag{4.2}$$

Clearly s^* is a solution of (4.1) if and only if s^* is fixed point of F.

Suppose that the following hypotheses are hold:

(A) $|f(u,\delta) - f(u,\rho)| + |f(u,\delta) - f(u,\gamma)| + |f(u,\gamma) - f(u,\rho)| \leq |\delta - \rho| + |\delta - \gamma| + |\gamma - \rho|$ *for all $u \in [0, Z]$ and $\delta, \rho, \gamma \in \mathbb{R}$.*

(B) $\sup_{t \in [0,Z]} \int_0^Z \mathcal{J}(t,u) du = r < 1.$

Then, the integral equation (4.1) has a solution.

Proof.

$$
\begin{aligned}
&\mathbb{G}(F\delta, F\rho, F\gamma)\\
&= \sup_{\delta\in[0,Z]} |F\delta(\delta) - F\rho(\delta)| + \sup_{\delta\in[0,Z]} |F\rho(\delta) - F\gamma(\delta)|\\
&+ \sup_{\delta\in[0,Z]} |F\delta(\delta) - F\gamma(\delta)|\\
&= \sup_{\delta\in[0,Z]} |\int_0^Z \mathcal{J}(\delta,u)(f(u,\delta(u)) - f(u,F\delta(u)))du|\\
&+ \sup_{\delta\in[0,Z]} |\int_0^Z \mathcal{J}(\delta,u)(f(u,\rho(u)) - f(u,F\rho(u)))du|\\
&+ \sup_{\delta\in[0,Z]} |\int_0^Z \mathcal{J}(\delta,u)(f(u,\gamma(u)) - f(u,F\gamma(u)))du|\\
&\le \sup_{\delta\in[0,Z]} \int_0^Z |\mathcal{J}(\delta,u)|(|\delta(u) - F\rho(u)| + |\rho(u) - F\gamma(u)| + |\delta(u) - F\gamma(u)|)du\\
&\le \mathbb{G}(\delta,\rho,\gamma) \sup_{\delta\in[0,Z]} \int_0^Z |\mathcal{J}(\delta,u)|du\\
&\le r\mathbb{G}(\delta,\rho,\gamma).
\end{aligned}
$$

Hence, all the conditions of Theorem 3.3 are satisfied, so F has a fixed point in $\mathcal{J}$. ∎

5. Conclusion

In this paper, the notion of L-fuzzy contractions in $\mathbb{G}$-metric space is presented. Sufficient conditions for the existence of L-fuzzy fixed points for such mappings are established. It is noted that the main ideas proposed herein improve and include some known results in the related literature.

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