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The Plethystic Exponentials Created in Certain Domains of the Complex Plane and Related Implications

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Abstract The primary aim of this scientific investigation is to introduce basic information about plethystic exponential forms, which play significant roles in both mathematics and related fields. It then aims to reorganize these concepts within specific regions of the complex plane, focus on new complex ideas, and present various propositions, along with their implications and potential applications, for consideration by relevant researchers.

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1. Introduction and Special Information

First of all, we begin our special investigation by offering specific information about the plethystic exponential(s) and related topics in mathematics. Specifically, in algebra, the special term "plethysm" denotes a special operation on symmetric functions first introduced by D.E. Littlewood. Morever, a lambda ring (or λ-ring) is a commutative ring equipped with operations λ that mimic the exterior powers of vector spaces, as introduced by Grothendieck. For further exploration of those special terms, one may refer to the earlier studies cited in the references in [3, 4, 15].

Specifically, in metamathematics, the plethystic exponential is a special operator defined on formal power series. Similar to the standard exponential function, it converts addition into multiplication. This operator arises naturally in the realm of symmetric functions, providing a concise link between the generating series for elementary, complete, and power sums of homogeneous symmetric polynomials in several variables. Its name also originates from the operation called "plethysm", which is defined within the framework of λ-ring.

Additionally, the plethystic exponential (calculus), particularly within combinatorics, acts as a generating function for many well-studied sequences involving integers, polynomials or power series. It also plays a crucial role in enumerative combinatorics, particularly in the context of unlabeled graphs and various other combinatorial objects. It also plays a pivotal role in enumerative combinatorics concerning unlabeled graphs and various other combinatorial objects indicated as in [8, 19].

At the same time, various different viewpoints in the realms of geometry and topology reveal that the plethystic exponential of a particular geometric-topological invariant of a space determines the corresponding invariant of its symmetric products, as outlined in [17].

It is noted here that our main objective will also be to consider these specific computations in the context of transformations between complex planes and to reorganize some possible definitions, theories, and examples. Now, in line with our goal, let us now detail the special information.

2. Definitions, Properties and Special Examples

In this section, we will first revisit a variety of fundamental definitions, then explore various relationships, and provide specific examples related to the plethystic exponential (function).

Let $\mathcal{R}[s]$ be a ring of formal power series, which consists of the independent variable s defined on the set $\mathbb{R}$ of real numbers, with coefficients in a commutative ring $\mathbb{R}$. We also denote by

$$\mathcal{R}^*[s] \subset \mathcal{R}[s]$$

the ideal being of power series without constant term. Then, for any real valued function given by

$$v := v(s) \in \mathcal{R}^*[s],$$

its plethystic exponential (form) is generally symbolized by

$$\mathbf{PE}[v(s)] \quad or \quad \mathbf{PE}[v](s) \tag{2.1}$$

and also defined by the following-exponential form:

$$\mathbf{PE}[v](s) = exp\left(\sum_{n=1}^{\infty} \frac{v(s^n)}{n}\right) \equiv exp\left(\sum_{n\geq 1} \frac{v(s^n)}{n}\right), \tag{2.2}$$

where the familiar notation $exp(\cdot)$ is the well-known exponential function (with real variable s.)

As some special relations, in the light of the main properties of both the exponential function and the special status of its variable s, it can be readily verified that the following fundamental properties:

$$\mathbf{PE}[0](s) = 1 \tag{2.3}$$

$$\mathbf{PE}[v+u](s) = \Big(\mathbf{PE}[v](s)\Big) \cdot \Big(\mathbf{PE}[u](s)\Big) \tag{2.4}$$

and

$$\mathbf{PE}[-v](s) = \Big(\mathbf{PE}[v](s)\Big)^{-1}. \tag{2.5}$$

As a number of elementary examples related to the plethystic exponential, for all $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, the following can also be determined:

$$v(s) = s^n \;\Rightarrow\; \mathbf{PE}[v](s) = \frac{1}{1-s^n} \tag{2.6}$$

and

$$v(s) = \frac{s}{1-s} \;\Rightarrow\; \mathbf{PE}[v](s) = 1 + \sum_{n\geq 1} \chi(n) s^n, \tag{2.7}$$

where the parameter s is a real number with $|s| < 1$ and also $\chi(n)$ is number of partitions of n $(n \in \mathbb{N})$.

By taking into account the series expansion of the related exponential functions, the plethystic exponentials can also be redefined as power series rings in the forms consisting of many different type variables. In the same time, by considering the product-sum form, the relevant plethystic exponential(s) can also offer numerous product-sum identities. This naturally stems from a product formula for the plethystic exponentials themselves. When the mentioned function $v(s) \in \mathcal{R}^*[s]$ is of a formal power series with coefficients ρ_n, which is

$$v(s) = \sum_{j=1}^{\infty} \rho_j s^j \equiv \sum_{j\geq 1} \rho_j s^j, \tag{2.8}$$

it is not difficult to show that the following result:

$$\mathbf{PE}[v](s) = \prod_{j=1}^{\infty} (1-s^j)^{-\rho_j} \equiv \prod_{j\geq 1} (1-s^j)^{-\rho_j}. \tag{2.9}$$

In addition, of course, the equivalent product expression also applies in scenarios involving multiple variables. One particularly intriguing connection arises in its association with integer partitions and the cycle index of the symmetric group, as discussed in [7]. It is worth noting that this relationship is among several others with symmetric functions.

Some of these connections can be further elucidated as follows.

For those, by making use of the parameters:

$$s_1,\ s_2,\ s_3, \cdots, s_n,$$

denoted by τ_ℓ the complete homogeneous symmetric polynomial, which is the sum of all monomials of the degree ℓ in the related parameters s_ℓ, similarly, denoted by δ_ℓ the elementary symmetric polynomials. Then, by using plethystic exponential(s), the mentioned expressions τ_ℓ and δ_ℓ are coupled with the power sum polynomials given by

$$\kappa_\ell = s_1^\ell + s_2^\ell + \cdots + s_n^\ell \tag{2.10}$$

by the identities of Newton, that can be written briefly as follows:

$$\sum_{\ell \geq 0} \tau_\ell s^\ell = \mathbf{PE}[\kappa_1 s] = \mathbf{PE}[s_1 s + s_2 s + \cdots + s_\ell s] \tag{2.11}$$

and

$$\sum_{\ell \geq 0} (-1)^\ell \delta_\ell s^\ell = \mathbf{PE}[-\kappa_1 s] = \mathbf{PE}[-s_1 s - s_2 s - \cdots - s_\ell s]. \tag{2.12}$$

In the same time, in the light of the special information above, it can be also linked to the formula of Macdonald for the symmetric products. For those, Let S be a finite CW complex, namely the cellular complex (*or* the cell complex), of dimension d, with Poincaré polynomial which is also defined by

$$P_S(s) = \sum_{\ell=0}^{d} b_\ell(S) s^\ell, \tag{2.13}$$

where $b_\ell(S)$ denotes the familiar ℓth Betti number. Then, the Poincaré polynomial of the special form of the nth symmetric product of S, which is denoted by $P_{Sym^\ell(S)}$, is also designated as the form of the series expansion given by

$$\begin{aligned} \mathbf{PE}[P_S(-s)r] &= \prod_{\ell=0}^{d} (1 - rs^n)^{(-1)^{\ell+1} b_\ell(S)} \\ &= \sum_{\ell \geq 0} P_{Sym^\ell(S)}(-s) q^\ell. \end{aligned} \tag{2.14}$$

In particular, as supplementary insight into the plethystic program within the realm of physics, a team of theoretical physicists, comprised of Bo Feng, Amihay Hanany, and Yang-Hui He, introduced a systematic approach to enumerate single and multi-trace gauge-invariant operators in supersymmetric gauge theories. For further details, one may refer to the scientific research given in [6]. Moreover, in the context of quiver gauge theories describing D-branes probing Calabi-Yau singularities, this enumeration is encapsulated by the plethystic exponential of the Hilbert series of the singularity.

3. The Complex Plethystic Exponential (Forms), Main Results and Various Implications

As it has been emphasized previously, the information introduced in the preceding Sections 1 and 2 has been presented with real parameters (or variables) determined by the subjective expressions between (2.1) and (2.14). In this esential section, extra information will be provided to reconstruct the previously presented information in a complex form. By considering that both the theory of complex functions and transformations in certain domains of the familiar complex plane will be pivotal in such configurations, the comprehensive sources mentioned in [1, 2, 5, 10, 18, 22, 23, 25] will offer indispensable-detailed information as some fundamental references. Let us now proceed towards our objective again.

Firstly, by considering the notation and the definition of the plethystic exponential (function) introduced by (2.1) and (2.2), we aim to examine its complex form, namely the plethystic exponential (function) with complex parameter (variable) or the complex plethystic exponential function given by the form:

$$\mathbf{PE}[v](z) = exp\left(\sum_{n=1}^{\infty}\frac{f(z^n)}{n}\right) \equiv exp\left(\sum_{n\geq 1}\frac{f(z^n)}{n}\right), \tag{3.1}$$

where, here and in parallel with this study, each one of the values of the complex powers just above is taken as its principal value.

In particular, we also find it useful to emphasize that, for relevant researchers, the earlier studies presented by the references in [10, 11, 13, 14, 24] will play an important role in extending the more complex structures between (2.10) and (2.14) to different type functions with complex variables and (some of) their comprehensive applications in mathematics and many engineering sciences.

Additionally, in consideration of the fundamental properties of both the exponential functions and the logarithmic functions, which are possible inverse functions of each other, with the dependent-independent complex variables in the familiar classical forms generally used by

$$z = x + iy \quad (x, y \in \mathbb{U})$$

and

$$w = u + iv \quad (u, v \in \mathbb{R};\ w \in \mathbb{C}^* := \mathbb{C} - 0):$$

$$w = w(z) = exp(z) \quad \Leftrightarrow \quad z = ln(w),$$

and using both the following-elementary relationships presented by

$$\begin{aligned}
\mathbf{PE}[f](z) &= exp\left(\sum_{n\geq 1}\frac{f(z^n)}{n}\right) \\
&= exp\left(\frac{f(z^1)}{1} + \frac{f(z^2)}{2} + \cdots + \frac{f(z^n)}{n} + \cdots\right) \qquad (3.2) \\
&= exp\left(\frac{f(z^1)}{1}\right) \cdot exp\left(\frac{f(z^2)}{2}\right) \cdot \cdots \cdot exp\left(\frac{f(z^n)}{n}\right) \cdot \cdots
\end{aligned}$$

$$= \prod_{n\geq 1} \left\{ exp\left(\frac{f(z^n)}{n}\right)\right\}$$

$$\Longleftrightarrow \quad \ln\Big(\mathbf{PE}[f](z)\Big) = \sum_{n\geq 1} \frac{f(z^n)}{n}, \tag{3.3}$$

and the well-known series expansion of the (complex) logarithm given by

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots - \frac{1}{n}(-z)^n + \cdots, \tag{3.4}$$

where each one of the complex assertions above is analytic in any suitable region of the complex plane:

$$\mathbb{U}_\rho = \big\{z \in \mathbb{C} : |z| \leq \rho < 1\big\},$$

the details of the quite simply propositions, which are (just) below, can then be demonstrated. They are omitted in this paper. Specially, as references, in [5, 10, 16, 20], some basic resources related to series concepts are also presented.

Through the instrumentality of the information between (3.1)-(3.4), a large number of propositions, which are related to the various complex forms of the plethistic exponentials indicated in the abstract of this special study, can also be constituted. Now, let us introduce a selected set of those propositions, which are more fundamental information for our special investigation.

Proposition 1. Let z be any member of the set $\mathbb{U}_\rho$. Then, the following assertions (in relation with the complex series expansions and/or the complex plethystic exponentials) are always provided:

- **i)** $\sum_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$.
- **ii)** $\sum_{n=1}^{\infty} z^n = \dfrac{z}{1-z}$.
- **iii)** $\sum_{n=0}^{\infty} \dfrac{z^n}{n!} = exp\,(z)\,.$
- **iv)** $\sum_{n=1}^{\infty} \dfrac{z^n}{n} = \ln\big(\dfrac{1}{1-z}\big)\big(= -\ln(1-z)\big)$.
- **v)** $\mathbf{PE}[f](z) = \prod_{n\geq 1} exp\left(\dfrac{f(z^n)}{n}\right)$.
- **vi)** $$\Big|\mathbf{PE}[f](z)\Big| = \left|\prod_{n\geq 1} exp\left(\frac{f(z^n)}{n}\right)\right| = \prod_{n\geq 1}\left|exp\left(\frac{f(z^n)}{n}\right)\right| = \prod_{n\geq 1} exp\left\{\frac{1}{n}\,\Re e\Big(f(z^n)\Big)\right\} = exp\left\{\Re e\Big(\sum_{n\geq 1}\frac{f(z^n)}{n}\Big)\right\}$$

$$= exp\left\{\sum_{n\geq 1}\frac{1}{n}\,\Re e\Big(f(z^n)\Big)\right\}.$$

vii) $$arg\Big(\mathbf{PE}[f](z)\Big) = arg\left\{\prod_{n\geq 1} exp\left(\frac{z^n}{n}\right)\right\}$$
$$= \sum_{n\geq 1} arg\left\{exp\left(\frac{f(z^n)}{n}\right)\right\}$$
$$= \sum_{n\geq 1} \Im m\Big(\frac{f(z^n)}{n}\Big)$$
$$= \sum_{n\geq 1}\left\{\frac{1}{n}\Im m\Big(f(z^n)\Big)\right\}.$$

viii) $$\ln\Big(\mathbf{PE}[f](z)\Big) = \sum_{n\geq 1}\frac{f(z^n)}{n}.$$

ix) $$\left|\ln\Big(\mathbf{PE}[f](z)\Big)\right| \leq \sum_{n\geq 1}\left|\frac{f(z^n)}{n}\right|$$
$$= \sum_{n\geq 1}\frac{1}{n}\left|f(z^n)\right|.$$

x) $$\Re e\Big\{\ln\Big(\mathbf{PE}[f](z)\Big)\Big\} = \Re e\Big(\sum_{n\geq 1}\frac{f(z^n)}{n}\Big)$$
$$= \sum_{n\geq 1}\left\{\frac{1}{n}\,\Re e\Big(f(z^n)\Big)\right\}.$$

xi) $$\Im m\Big\{\ln\Big(\mathbf{PE}[f](z)\Big)\Big\} = \Im m\Big(\sum_{n\geq 1}\frac{f(z^n)}{n}\Big)$$
$$= \sum_{n\geq 1}\left\{\frac{1}{n}\,\Im m\Big(f(z^n)\Big)\right\}.$$

We would now like to consider (or concentrate on) several significant assertions related to functions with complex variable that are analytic in specific domains of the familiar complex plane, particularly those involving unusual complex-type transformations. These insights will also provide different perspectives on various complex transformations, which will be redefined based on the definition established in light of the essential equation given in (3.1).

Examples 1. Below are various special relationships dealing with some special transformations between the mentioned complex planes z and w. Let us now concentrate on each one of them:

i) Let $w := u + iv$, $\mathbb{U} := \mathbb{U}_1$ and $z \in \mathbb{E}_1 := \left\{z \in \mathbb{U} : \Im m(z) = \frac{1}{2}\right\}$. Then, through the instrument of the complex relationships consisting of the familiar variables z and w, which are given by

$$w := \Phi_1(z) = \frac{1}{1-z} \iff z = 1 - \frac{1}{w}$$
$$\iff \Im m\big(z\big) = \Im m\Big(1 - \frac{1}{w}\Big),$$

the special assertions given by

$$\Im m(z) = \Im m\Big(1 - \frac{1}{w}\Big) \iff \frac{1}{2} = \frac{v}{u^2+v^2}$$

can easy be obtained, which immediately infers that the function $\Phi_1(z)$ is a complex function including an interesting result in relation to the special condition between the z-plane and the w-plane:

$$\Phi_1\big(\mathbb{E}_1\big) = \mathbb{E}_2 := \big\{z \in \mathbb{C} \,:\, |z-i| = 1\big\}.$$

ii) Let $w := u + iv$ and $z \in \mathbb{U}$. Then, by the help of the following implications between the complex variables z and w :

$$\begin{aligned} w := \Phi_2(z) = \frac{z}{1-z} \quad &\iff \quad z = \frac{w}{w+1} \\ &\iff \quad \Big|\frac{w}{w+1}\Big| = |z| \\ &\iff \quad \Big|\frac{w}{w+1}\Big| < 1 \quad \iff \quad \big|w\big| < \big|w+1\big|, \end{aligned}$$

it can readily be arrived at:

$$\begin{aligned} \big|u+iv\big| < \big|u+1+iv\big| \quad &\iff \quad \sqrt{(u+1)^2+v^2} < \sqrt{u^2+v^2} \\ &\iff \quad 2u > -1\,, \end{aligned}$$

which also gives us that the function $\Phi_1(z)$ is a complex function being of the following assertion:

$$\Phi_2\big(\mathbb{U}\big) = \mathbb{E}_3 := \Big\{z \in \mathbb{C} \,:\, \Re e(z) > -\frac{1}{2}\Big\}.$$

iii) Let $w := u + iv$ and $z \in \mathbb{E}_4 := \Big\{z \in \mathbb{U} \,:\, Arg(z) = -\frac{\pi}{8}\Big\}$. Then, when taking into consideration the special relationships between those standard variables:

$$\begin{aligned} w := \Phi_3(z) = \frac{1}{1-z^2} \quad &\iff \quad z^2 = 1 - \frac{1}{w} \\ &\iff \quad Arg\big(z^2\big) = Arg\Big(1 - \frac{1}{w}\Big)\,, \end{aligned}$$

the following connections:

$$\begin{aligned} Arg\big(z^2\big) = arg\Big(1 - \frac{1}{w}\Big) \;&=\; Arg\Big(\frac{u-1+iv}{u+iv}\Big) \\ &\iff \quad 2Arg(z) = \tan^{-1}\Big(\frac{v}{u-u^2-v^2}\Big) \\ &\implies \quad \frac{-v}{u^2+v^2-u} = \tan\Big(-\frac{\pi}{4}\Big) \\ &\iff \quad u^2 - u + v^2 - v = 0 \end{aligned}$$

can then be determined. Therefore, it instantly shows that the mentioned-complex function $\Phi_3(z)$ is of the following condition:

$$\Phi_3\big(\mathbb{E}_4\big) = \mathbb{E}_5 := \Big\{z \in \mathbb{C} \,:\, |2z-1-i| = \sqrt{2}\Big\}.$$

iv) Let $w := u + iv$ and $z \in \mathbb{U}$. Then, by considering the following-special relationship between the complex variables z and w, which is given by

$$w := \Phi_4(z) = \frac{z^2}{z^2 - 1} \quad \Longleftrightarrow \quad z^2 = \frac{w}{w-1},$$

the following assertions:

$$\left|\frac{w}{w-1}\right| = \left|z^2\right| \quad \Longleftrightarrow \quad \left|\frac{w}{w-1}\right| = |z|^2$$

$$\Longleftrightarrow \quad \left|\frac{w}{w-1}\right| < 1 \quad \Longleftrightarrow \quad |w| < |w-1|,$$

or, equivalently,

$$|u + iv| < |u - 1 + iv| \quad \Longrightarrow \quad u < \frac{1}{2}$$

can easily be determined, which immediately gives us

$$\Phi_4\big(\mathbb{U}\big) = \mathbb{E}_6 := \left\{z \in \mathbb{C} \,:\, \Re e(z) < \frac{1}{2}\right\}.$$

As it is known, occasionally, various mathematical functions defined between (different) planes are also taken into account as transformations. In this context, naturally, when considering those relationships as complex planes, such as the familiar z-plane and w-plane, the relationship(s) between those relevant planes can also be expressed as a complex transformation(s). In this case, in view of the elementary results determined in Example 1, as great numbers of possible results, it is highly likely that the complex transformations, which will be created by the help of the information presented in (2.8), (2.9) and (3.1), lead us to comprehensive implications. Some of them are constituted as propositions just below and their details are also omitted there.

Proposition 2. In consideration of the definition of the complex-type exponential transformation given in (3.1), each of the following propositions holds:

i) Let $z \in \mathbb{E}_1$, $f(z) := z$ and $\mathcal{H}_1[f](z) := \mathbf{PE}[f](z)$. Then, the complex-type transformation $\mathcal{H}_1[f](z)$ has the form given by

$$\mathcal{H}_1[f](z) \equiv exp\left(\sum_{n\geq 1}\frac{z^n}{n}\right)\left(= e^{-\ln(1-z)}\right) = \frac{1}{1-z},$$

which turns $\mathbb{E}_1$ onto $\mathbb{E}_2$.

ii) Let $z \in \mathbb{U}$, $f(z) := z$ and $\mathcal{H}_2[f](z) := z\mathbf{PE}[f](z)$. Then, the complex-special transform $\mathcal{H}_2[f](z)$ possesses the complex form given by

$$\mathcal{H}_2[f](z) \equiv z\, exp\left(\sum_{n\geq 1}\frac{z^n}{n}\right)\left(= ze^{-\ln(1-z)}\right) = \frac{z}{1-z},$$

which turns $\mathbb{U}$ onto $\mathbb{E}_3$.

iii) Let $z \in \mathbb{E}_4$, $f(z) := z^2$ and $\mathcal{H}_3[f](z) := \mathbf{PE}[f](z)$. Then, the special transformation $\mathcal{H}_3[f](z)$ owns the following complex form given by

$$\mathcal{H}_3[f](z) \equiv exp\left(\sum_{n\geq 1} \frac{\left(z^2\right)^n}{n}\right)\left(= e^{-\ln(1-z^2)}\right) = \frac{1}{1-z^2}\,,$$

which turns $\mathbb{E}_4$ onto $\mathbb{E}_5$.

iv) Let $z \in \mathbb{U}$, $f(z) := z^2$ and $\mathcal{H}_4[f](z) := -z^2\mathbf{PE}[f](z)$. Then, the special-complex-type transform $\mathcal{H}_4[f](z)$ has possession of the following form:

$$\mathcal{H}_4[f](z) \equiv -z^2\, exp\left(\sum_{n\geq 1} \frac{\left(z^2\right)^n}{n}\right)\left(= -z^2 e^{-\ln(1-z^2)}\right) = \frac{z^2}{z^2-1}\,,$$

which turns $\mathbb{U}$ onto $\mathbb{E}_6$.

4. Conclusion and Recommendations

As conclusion and recommendations, in addition to the complex functions similar to Φ_j $(j = 1, 2, 3, 4)$ in the special examples constituted in the Examples 1, which may be closely related to mathematics and many engineering sciences, a number of new comprehensive examples can be also created. For instance, by considering the complex form of the function together with its plethystic exponential, which is given by the equation in (2.6), it is also possible to derive several analytic and/or geometric properties related to the more general assertions given by

$$\begin{aligned}\mathbf{PE}[f](z) = \frac{1}{1-z^n} \quad &\Longleftrightarrow \quad z^n = 1 - \frac{1}{\mathbf{PE}[f](z)} \\ &\Longleftrightarrow \quad z^n = 1 - \mathbf{PE}[-f](z)\,,\end{aligned}$$

for all $n \in \mathbb{N}$ and for some $z \in \mathbb{U}$.

In the same time, each of those significant examples can be also reexamined on the basis of various transformations like the mentioned complex forms $\mathcal{H}_j[f](z)\,(j = 1, 2, 3, 4)$. Particularly, for extra results together with their possible implications, we believe that it would be beneficial for interested researchers to concentrate on the relevant comprehensive information given by both the main works given in [9, 12, 24] and the different-type special studies presented in [1, 6, 7, 10, 11, 13, 14, 18, 25–27].

Finally, more comprehensive transformations can also be identified by the help of the mentioned-complex transformation in (3.1) and various implications of our essential results. For example, firstly, for the analytic functions $g := g(z)$ and $f := f(z)$, a complex transformation such as

$$\mathbf{T}[g(z); f(z)] \equiv \mathbf{T}[g; f](z) = g(z)\,\mathbf{PE}[f](z)$$

can be reorganized and then it can be center upon its comprehensive possible implications, where $z \in \mathbb{U}_\rho \subseteq \mathbb{U} \subseteq \mathbb{C}$. In such a case, it is clear that a special relationship between them such as

$$\mathbf{T}[1; f](z) = \mathbf{PE}[f](z)$$

also manifests itself.

Ethics declarations

Conflict of interest

It is declared that the authors have no conflict of interest as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Contributions

Both authors have the same contribution.

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