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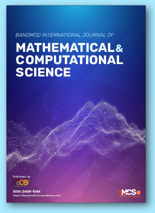

EXISTENCE UNIQUENESS AND ULAM STABILITY OF SOLUTION TO N-TUPLE ORDER Q-FRACTIONAL DIFFERENCE EQUATION

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Abstract In this paper, motivated by Riemann-Liouville quantum fractional operator, Caputo quantum fractional operator, and Hilfer quantum fractional operator, the author will study the n-tuple order definition of q-fractional derivative and the relevant theories. Moreover, the existence, uniqueness, and Ulam stability of the solution to the n-tuple order q-fractional difference equation will be illustrated

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1. Introduction

Fractional calculus was rapidly developed and became one of the focusing points in both mathematics and scientific area (see [1–5, 10–12, 17–19]). Fractional calculus is actually an appendage of mathematical analysis with non-integers order and non-local properties.

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Fractional calculus was developed from integers order derivative. The classical first order derivative is written as follows.

$$D^1 f(t) = \frac{d}{dt} f(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

Thus, the integers order derivative only displays the rate of change of one function around the neighborhood of the inspected point. In other words, the displayed rate of change conforms and is a homogeneous mixture to the time scale. But, in reality, the rate of change, which implies in most natural phenomena, possesses the time-retardation or the time-acceleration in itself. Preferentially, the rate of change in the real world does not fully harmonize with the time scale. And, the non-local operators, especially the fractional operators, are more suitable for dealing with this kind of problem. Consequently, there are various researchers who strive for development in the aspect of operators until now.

From time to time, fractional calculus caught various attentions from several mathematicians. The two original gangster definitions that posseses the most attentiveness until now are Caputo fractional operator and Riemann-Liouville fractional operators, respectively. Subsequently, Hilfer [13], in 2000, introduced the general definition of fractional derivative by interpolated both Caputo fractional derivative and Riemann-Liouville derivative. First and foremost, the first-born child of fractional calculus, the Riemann-Liouville fractional was defined as follows.

Definition 1.1. Let $\alpha > 0$, then the Riemann-Liouville fractional integral is defined as

$$_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s.$$

Subsequenty, determine $n-1 < \alpha < n$ and $\beta \in [0,1]$, the three visualizations of Caputo, Riemann-Liouville and Hilfer derivatives are given as follows.

Definition 1.2. [6] Let $n-1 < \alpha < n$, Riemann-Liouville fractional derivative of function $f(t)$ is defined by

$$_aD_t^\alpha f(t) = D^n {}_aI_t^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s.$$

Definition 1.3. [6] Let $n-1 < \alpha < n$, Caputo fractional derivative of function $f(t)$ is defined by

$${}_a^CD_t^\alpha f(t) = {}_aI_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\mathrm{d}s.$$

Definition 1.4. [13] Let $0 < \alpha < 1$ and $0 \le \beta \le 1$. Then, Hilfer fractional derivative of function $f(t)$ is defined by

$$_aD_t^{\alpha,\beta} f(t) = {}_aI_t^{\beta(1-\alpha)} D^1 {}_aI_t^{(1-\beta)(1-\alpha)} f(t) = {}_aI_t^{\gamma-\alpha} {}_aD_t^\gamma f(t), \gamma = \alpha + \beta - \alpha\beta$$

As it seems, since we set $\beta = 0$, Hilfer fractional derivative will be reduced to Riemann-Liouville fractional derivative. Also, for $\beta = 1$, Hilfer fractional derivative will be reduced to Caputo fractional derivative.

Consequently, these common definitions leads to further tremendous studies in the field of fractional calculus such as ψ-fractional derivatives [7, 19], fractional derivatives with variable order [6, 16], etc. Also, there are several fundamental methodologies used to

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study the well-posedness of the solutions such as Banach fixed point theorem, Schauder fixed point theorem, Ulam stability, Ulam-Hyers stability etc. (see [7, 9, 18])

Back to the basic, the fractional calculus, which was designed to observe real-world phenomena with its dominant non-integers order of operation start, from a historical perspective, is developed from the h-calculus, where h-derivative is defined by

$$D_h f(t) = \frac{f(t+h) - f(t)}{h}.$$

Later, in 1909, Jackson [14] introduced the quantum calculus, where q-derivative is defined as

$$D_q f(t) = \frac{f(qt) - f(t)}{qt - t},$$

and q-integral is

$$I_q f(t) = \int_0^t f(s) \mathrm{d}_q s = (1-q) \sum_{n=0}^{\infty} t q^n f(t q^n).$$

The definition of q-derivative and q-integral are studied and developed through the fractional approaches by various experts. The development of q-fractional derivative and q-fractional integral similarly is the reflection of the development of fractional calculus. The definition of the q-fractional operators was introduced as follows.

Definition 1.5. [20] Let, $q \in (0,1)$ and $\alpha > 0$, then the q-Riemann-Liouville fractional integral is defined as

$$_qI_t^{\alpha} f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s) \mathrm{d}_q s.$$

Where

$$(n-m)^{(k)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{i+k}}, \quad n \neq 0, \quad k \in \mathbb{R},$$

and

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, t \in \mathbb{R} - \{0, -1, -2, ...\}$$

where $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$ with

$$[m]_q = \frac{1 - q^m}{1 - q}, m \in \mathbb{R}.$$

Also, let $\alpha, \beta \geq 0$ and $f(t)$ is a function on $[0, T]$, then there are following properties
(1) ${}_qI_t^{\alpha}\, {}_qI_t^{\beta} f(t) = {}_qI_t^{\alpha+\beta}$ f(t)
(2) ${}_qD_t^{\alpha}\, {}_qI_t^{\alpha} f(t) = f(t)$

Definition 1.6. [8] Let $n - 1 < \alpha < n$, the q-Riemann-Liouville fractional derivative of the function $f(t)$ is defined by ${}_qD_t^{\alpha} f(t) = D_q^n\, {}_qI_t^{n-\alpha} f(t)$

Definition 1.7. [8] Let $n - 1 < \alpha < n$, the q-Caputo fractional derivative of the function $f(t)$ is defined by ${}_q^C D_t^{\alpha} f(t) = {}_qI_t^{n-\alpha} D_q^n f(t)$

Later, in 2021, the operator of the q-Hilfer fractional derivative was introduced as follows.

Definition 1.8. [15] Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $0 < q < 1$ then, the q-Hilfer fractional derivative of the function $f(t)$ is defined by

$$ {}_qD_t^{\alpha,\beta} f(t) = {}_qI_t^{\beta(1-\alpha)} D_{qq}I_t^{(1-\beta)(1-\alpha)} f(t) = {}_qI_t^{\gamma-\alpha} {}_qD_t^{\gamma} f(t), \gamma = \alpha + \beta - \alpha\beta $$

Motivated by the aformentioned definitions, this paper will be constructed as follows. In the next section, the definition of n-tuple order fractional derivative will be introduced. In the third section, the essential conditions, spaces, and vital concepts to display the uniqueness and stability of the solution to the difference equation will be discussed. In the fourth section, the existence and uniqueness of solution will be studied. In the fifth section, Ulam stability of the equation will be displayed. Lastly, in section six, an example will be given to elaborate the result.

2. N-tuple order fractional derivative

According to the fact in the first section, q-Hilfer fractional derivative, the double order q-derivative, nowadays is the most general definition in the sense of order quantity. In this section, the author will introduce the pattern of triple order , quadruple order and n-tuple order q-derivative, which is more general in order quantity aspect.

Definition 2.1. Let $0 < \alpha < 1$ and $\beta, \gamma \in [0,1]$, the triple order q-fractional derivative is defined as

$$ {}_qD_t^{\alpha,\beta,\gamma} f(t) = {}_qI_t^{(\beta+\gamma-\beta\gamma)(1-\alpha)} D_{qq}I_t^{(1-\beta)(1-\alpha)(1-\gamma)} f(t). $$

Denote that $\eta = \alpha + \beta + \gamma - \alpha\beta - \alpha\gamma - \beta\gamma + \alpha\beta\gamma$, the derivative can be written in the reduced form as ${}_aD_t^{\alpha,\beta,\gamma} f(t) = {}_aI_t^{\eta-\alpha} {}_aD_t^{\eta} f(t)$. Obviously, the reduced form is the analogous structure with q-Hilfer fractional derivative.

The triple order q-fractional derivative can be reduced to the other derivatives with following conditions:
(1) Let both $\beta = 0$ and $\gamma = 0$, the derivative become $D_{qq}I_t^{1-\alpha} f(t) = {}_qD_t^{\alpha} f(t)$ as 1.6.
(2) Let $\beta = 1$ or $\gamma = 1$, the derivative become ${}_qI_t^{1-\alpha} D_q f(t) = {}_q^C D_t^{\alpha} f(t)$ as 1.7.
(3) Let $\beta = 0$, the derivative become ${}_qI_t^{\gamma(1-\alpha)} D_{qq}I_t^{(1-\gamma)(1-\alpha)} f(t) = {}_qD_t^{\alpha,\gamma} f(t)$ as 1.8.
(4) Let $\gamma = 0$, the derivative become ${}_qI_t^{\beta(1-\alpha)} D_{qq}I_t^{(1-\beta)(1-\alpha)} f(t) = {}_qD_t^{\alpha,\beta} f(t)$ as 1.8.

At this point, the introduced triple order q-fractional derivative may deemed to be the most general definition in the perspective of order quantity. However, in fact, there infinitely are more general definitions. One of the examples is the quadruple order q-fractional derivative.

Definition 2.2. Let $0 < \alpha < 1$ and $\beta, \gamma, \eta \in [0,1]$, the quadruple order q-fractional derivative is defined as

$$ {}_qD_t^{\alpha,\beta,\gamma,\eta} f(t) = {}_qI_t^{\lambda(1-\alpha)} D_{qq}I_t^{(1-\beta)(1-\alpha)(1-\gamma)(1-\eta)} f(t) $$

where $\lambda = \beta + \gamma + \eta - \beta\gamma - \beta\eta - \gamma\eta + \beta\gamma\eta$. Then, note that $\mu = \alpha + \beta + \gamma + \eta - \alpha\beta - \alpha\gamma - \alpha\eta - \beta\gamma - \beta\eta - \gamma\eta + \alpha\beta\gamma + \alpha\beta\eta + \alpha\gamma\eta + \beta\gamma\eta - \alpha\beta\gamma\eta$, the derivative can be wriiten in the reduced form as ${}_qD_t^{\alpha,\beta,\gamma,\eta} f(t) = {}_qI_t^{\mu-\alpha} {}_qD_t^{\mu} f(t)$. Obviously, the reduced form again is the analogous structure with q-Hilfer fractional derivative.

With restriction of order $\eta \in [0,1]$, the quadruple order q-fractional derivative can be reduced to the other derivatives with following conditions:
(1) Let $\eta = 1$, the derivative become ${}_qI_t^{1-\alpha} D_q f(t) = {}_q^C D_t^{\alpha} f(t)$ as 1.7.

(2) Let $\eta = 0$, the derivative become ${}_qI_t^{(\beta+\gamma-\beta\gamma)(1-\alpha)} D_{q\,q}I_t^{(1-\beta)(1-\alpha)(1-\gamma)} f(t) = {}_qD_t^{\alpha,\beta,\gamma} f(t)$, the q-triple order derivative, following 2.1.
(3) With property of the triple order q-derivative, the quadruple order q-derivative is reducable to both 1.6 and 1.8 as well.

As written that there infinitely are more general definitions, it is possible to infinitely interpolate the fractional order over and over. In this section, the general form of n-tuple order q-fractional derivative will be displayed.

Definition 2.3. Let $0 < \alpha_1 < 1$ and $\alpha_2, \alpha_3, \ldots, \alpha_n \in [0,1]$, the n-tuple order q-fractional derivative is defined as follows.

$$ {}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} f(t) = {}_qI_t^{\left(1-\prod_{i=2}^n(1-\alpha_i)\right)(1-\alpha_1)} D_{qa}I_t^{\prod_{i=1}^n(1-\alpha_i)} f(t). $$

Note that $\Omega = 1 - \prod_{i=1}^n (1-\alpha_i)$, the n-tuple derivative can be written, similar to the triple and the quadruple order derivative, as analogous structure of q-Hilfer derivative as follows.

$$ {}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n} f(t) = {}_qI_t^{\Omega-\alpha_1}\, {}_qD_t^{\Omega} f(t) $$

Obviously, the triple order q-fractional derivative in 2.1 can be generated by set $n = 3$ and let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$. As for the quadruple order q-fractional derivative in 2.2, it can be generated by set $n = 4$ and let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$, $\alpha_4 = \eta$, respectively. This definition implies that if you want to define the quintuple order q-fractional derivative, just easily set $n = 5$ to govern the triple and quadruple ones. Thus, the n-tuple definition is an easy voyage to define novel q-fractional operators.

3. Framework and relevant materials

This section will introduce the necessary space, and concept of q-difference equation.

Definition 3.1. [8] Determine $\mu \in \mathbb{R}$, the subset A of $\mathbb{C}$ is called μ-geometric if $\mu z \in A$ as $z \in A$

Definition 3.2. [8] The function f on A that is q-geometric where $0 \in A$ is called q-regular at 0 if for every $z \in A$ there exist a limit

$$ \lim_{n\to\infty} f(zq^n) = f(0). $$

Definition 3.3. [8] For any $p \geq 1$, the space $L_q^p(a,b)$ is the space of the functions such that

$$ \left(\int_a^b |f(t)|^p \mathrm{d}_q t \right)^{\frac{1}{p}} < \infty $$

For $p = 1$ it can be denoted the space as $L_q(a,b)$.

Definition 3.4. [8] For any $p \in \mathbb{R}^+$, the space $\mathbb{L}_q^p[a,b]$ is the space of the functions on interval $(a,b]$. The space $\mathbb{L}_q^p[a,b]$ is a Banach space with the supremum norm $\|.\|_p$ defined by

$$ \|f\|_p = \sup_{t\in(a,b]} \left(\int_a^b |f(t)|^p \mathrm{d}_q t \right)^{\frac{1}{p}} < \infty $$

For $p = 1$ it can be denoted the space as $\mathbb{L}_q(a,b)$.

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Definition 3.5. [8] The space $C_q^n[a,b]$ is a space of a continuous function on $[a,b]$ such that $D_q^{n-1}f(t) \in C[a,b]$. Also, $C_q^n[a,b]$ is a banach space with supremum norm $\|.\|$ such that

$$\|f\| = \sup_{t\in[a,b]} \sum_{i=0}^{n-1} |D_q^i f(t)| < \infty$$

For $n = 1$ it can be noted the space as $C[a,b]$, and for $q = 1$ as $C^n[a,b]$.

Definition 3.6. [8] The function f on $[0,a]$ is called the absolutely continuous function if f is q-regular at 0 and there exists the constant $K > 0$ such that

$$\sum_{i=0}^{\infty} |f(tq^i) - f(tq^{i+1})| \le K$$

for every $t \in (qa, a]$.

Definition 3.7. [8] Let $AC_q[a,b]$ be a space of the absolutely continuous functions on $[a,b]$, then $f \in AC_q[a,b]$ if and only if there exists an arbitrary constant $\omega \in \mathbb{R}$ and the function $\psi(t) \in \mathbb{L}_q^p[a,b]$ such that

$$f(t) = \omega + \int_a^b \psi(t)\mathrm{d}_q s.$$

For $q = 1$, it can be noted the space as $AC[a,b]$.

Definition 3.8. [8] The space $AC_q^{(n)}[a,b]$ is a space of function on $[a,b]$ such that $D_q^{n-1}f(t) \in AC_q[a,b]$. For $q = 1$, it can be denoted as $AC^{(n)}[a,b]$

Theorem 3.9. [8] Suppose $n-1 < \alpha < n$, $f \in \mathbb{L}_q[0,T]$ with ${}_qI_t^{n-\alpha}f(t) \in AC_q^{(n)}[0,T]$, then

$$ {}_qI_t^{\alpha}\, {}_qD_t^{\alpha} f(t) = f(t) - \sum_{i=0}^{n-1} {}_qI_t^{1+i-\alpha} f(0) \frac{t^{\alpha-i-1}}{\Gamma_q(\alpha-i)}$$

where

$$ {}_qI_t^{1+i-\alpha} f(0) = \lim_{t\to 0^+} {}_qI_t^{1+i-\alpha} f(t).$$

Theorem 3.10. Suppose $0 < \alpha_1 < 1$, $0 \le \alpha_2, \alpha_3, \dots, \alpha_n \le 1$, $f \in \mathbb{L}_q[0,T]$ with ${}_qI_t^{1-\Omega}f(t) \in AC_q^{(1)}[0,T]$, where $\Omega = 1 - \prod_{i=1}^{n}(1-\alpha_i)$ then,

$$ {}_qI_t^{\alpha_1}\, {}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n} f(t) = {}_qI_t^{\Omega}\, {}_qD_t^{\Omega} f(t) = f(t) - {}_qI_t^{1-\Omega} f(0) \frac{t^{\Omega-1}}{\Gamma_q(\Omega)}$$

Proof. The proof is trivial. By property (1) pursuant to the definition 1.5 and the definition 2.3, we obtain ${}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n} f(t) = {}_qI_t^{\Omega-\alpha_1}\, {}_qD_t^{\Omega} f(t)$. Subsequently, applies theorem 3.9 with $n = 1$, we will obtain the illustrated result. ∎

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4. Existence and Uniqueness of solution

In this part, the existence and uniqueness of solution to the following equation will be displayed by Banach contraction principle. The concerning difference equation is as follows.

$$
\begin{aligned}
&{}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n}x(t)=f(t,x(t)),\quad 0<\alpha_1<1,\quad 0\le\alpha_2,\alpha_3,\ldots,\alpha_n\le 1\\
&{}_qI_t^{1-\Omega}x(0)=x_0,\quad \Omega=1-\prod_{i=1}^{n}(1-\alpha_i),\quad t\in(0,T],
\end{aligned}
\tag{4.1}
$$

the mild solution of the equation (4.1) is written as

$$x(t)=\frac{x_0t^{\Omega-1}}{\Gamma_q(\Omega)}+\frac{1}{\Gamma_q(\alpha_1)}\int_0^t(t-qs)^{(\alpha_1-1)}f(s,x(s))\mathrm{d}_qs. \tag{4.2}$$

Theorem 4.1. [8] Suppose $\phi:(0,a]\to\mathbb{R}$ is a function, if $\phi\in\mathbb{L}_q[0,a]$, then ${}_qI_t^{\alpha}\phi\in\mathbb{L}_q[0,a]$, and $\|{}_qI_t^{\alpha}\phi\|_1\le\frac{a^{\alpha}}{\Gamma_q(\alpha+1)}\|\phi\|_1$.

To display the uniqueness of solution, we state the essential assumptions as follows:
(A0) There exists positive constant M_f such that

$$\|f(t,u)-f(t,v)\|\le M_f\|u-v\|$$

for all $u,v\in\mathbb{L}_q[0,T]$.

Theorem 4.2. [15] For any $0<q<1$ and $0<\alpha<1$, the inequality of q-gamma function holds,

$$(1-q)^{\alpha}<\frac{1}{\Gamma_q(\alpha+1)}<\left(\frac{1-q}{1-q^{\frac{\alpha+1}{2}}}\right)^{\alpha}.$$

Theorem 4.3. Suppose the assumptions (A0) is satisfied, then the equation (4.2) is a unique solution of the problem (4.1) in $\mathbb{L}_q[0,T]$ if there exist a contraction constant $M_fT^{\alpha_1+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}}\right)^{\alpha_1}<1$

Proof. We define the contraction mapping $\mathbb{T}:\mathbb{L}_q[0,T]\to\mathbb{L}_q[0,T]$ by $\mathbb{T}x=x$, we get

$$\mathbb{T}x(t)=\frac{x_0t^{\Omega-1}}{\Gamma_q(\Omega)}+\frac{1}{\Gamma_q(\alpha_1)}\int_0^t(t-qs)^{(\alpha_1-1)}f(s,x(s))\mathrm{d}_qs.$$

Then,

$$
\begin{aligned}
\|\mathbb{T}x-\mathbb{T}y\|_1&\le\|f(t,x(t))-f(t,y(t))\|_1\frac{T^{\alpha_1}}{\Gamma_q(\alpha_1+1)}\|{}_qI_t^{\alpha_1}1\|_1\\
&\le\frac{M_fT^{\alpha_1+1}}{\Gamma_q(\alpha_1+1)}\|x-y\|_1\\
&\le M_fT^{\alpha_1+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}}\right)^{\alpha_1}\|x-y\|_1
\end{aligned}
$$

By Banach contraction theorem, since $M_fT^{\alpha_1+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}}\right)^{\alpha_1}<1$ then there exist x as a unique solution on $\mathbb{L}_q[0,T]$. The proof is completed. ∎

5. Ulam-Hyers stability of solution

In this section, the authors will illustrate the Ulam-Hyers stability of the solutions.

Definition 5.1. The equation (4.1) is Ulam-Hyers stable if for any ϵ and for solution $x \in \mathbb{L}_q[0,T]$ of the inequality

$$|{}_qD_t^{\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n}x(t) - f(t,x(t))| \leq \epsilon,$$

there exist a constant $c > 0$ and a solution $u \in \mathbb{L}_q[0,T]$ of the equation (4.1) with

$$|x(t) - u(t)| \leq c\epsilon,$$

Theorem 5.2. Suppose the assumption (A0) is satisfied, the equation (4.1) is Ulan-Hyers stable

Proof. Let $x \in \mathbb{L}_q[0,T]$ satisfies the first inequality in the definition 5.1. It follows that

$$\left| x(t) - \frac{x_0 t^{\Omega-1}}{\Gamma_q(\Omega)} - \frac{1}{\Gamma_q(\alpha_1)} \int_0^t (t-qs)^{(\alpha_1-1)} f(s,x(s)) \mathrm{d}_q s \right| \leq \epsilon \frac{T^{\alpha_1+1}}{\Gamma_q(\alpha_1+1)}$$

Now, suppose $u \in \mathbb{L}_q[0,T]$ be a solution of the equation (4.1). Hence, it satisfies

$$u(t) = \frac{x_0 t^{\Omega-1}}{\Gamma_q(\Omega)} + \frac{1}{\Gamma_q(\alpha_1)} \int_0^t (t-qs)^{(\alpha_1-1)} f(s,u(s)) \mathrm{d}_q s.$$

Thus, we obtain

$$\begin{aligned}
|x(t) - u(t)| &= \left| x(t) - \frac{x_0 t^{\Omega-1}}{\Gamma_q(\Omega)} - \frac{1}{\Gamma_q(\alpha_1)} \int_0^t (t-qs)^{(\alpha_1-1)} f(s,u(s)) \mathrm{d}_q s \right| \\
&\leq \epsilon \frac{T^{\alpha_1+1}}{\Gamma_q(\alpha_1+1)} + \left| \frac{1}{\Gamma_q(\alpha_1)} \int_0^t (t-qs)^{(\alpha_1-1)} [f(s,x(s)) - f(s,u(s))] \mathrm{d}_q s \right| \\
&\leq \epsilon \frac{T^{\alpha_1+1}}{\Gamma_q(\alpha_1+1)} + \frac{M_f T^{\alpha+1}}{\Gamma_q(\alpha+1)} |x(t) - u(t)| \\
&\leq \epsilon T^{\alpha_1+1} \left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}} \right)^{\alpha_1} + M_f T^{\alpha_1+1} \left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}} \right)^{\alpha_1} |x(t) - u(t)|
\end{aligned}$$

Denote that $L = T^{\alpha_1+1} \left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}} \right)^{\alpha_1}$, then

$$|x(t) - u(t)| \leq \epsilon \left(\frac{L}{1 - M_f L} \right) = c\epsilon.$$

Therefore, the equation (4.1) is Ulam-Hyers stable. ∎

6. Example

In this section, we give the examples to illustrate our result. Consider the following equations.

Example 6.1.

$$\begin{aligned}
{}_{\frac{1}{2}}D_t^{\frac{1}{2},\frac{1}{3},\frac{2}{3}} x(t) &= \frac{x(t) + x^2(t)\sin(x(t)) + \sin(x(t))}{20x^2(t) + 20} \\
{}_{\frac{1}{2}}I_t^{\frac{1}{9}} x(0) &= \frac{2}{3}, \quad t \in (0,1].
\end{aligned} \tag{6.1}$$

The mild solution of (6.1) is written as

$$x(t)=\frac{2t^{-\frac{1}{9}}}{3\Gamma_{\frac{1}{2}}\left(\frac{8}{9}\right)}+\frac{1}{\Gamma_{\frac{1}{2}}\left(\frac{1}{2}\right)}\int_0^t\left(t-\frac{1}{2}s\right)^{\left(\frac{-1}{2}\right)}\frac{x(s)+x^2(s)\sin(x(s))+\sin(x(s))}{20x^2(s)+20}\mathrm{d}_{\frac{1}{2}}s.$$

It can be seen that $M_f=\frac{1}{10}$. By mean value theorem, we get

$$\begin{aligned}|f(t,x)-f(t,y)|&=\frac{1}{20}\left(\left|\frac{x}{x^2+1}-\frac{y}{y^2+1}\right|+|\sin(x)-\sin(y)|\right)\\&\leq\frac{1}{10}\|x-y\|.\end{aligned}$$

This mean there exist the contraction constant

$$M_fT^{\alpha_1+1}\left(\frac{1-q}{1-q^{\frac{\alpha_1+1}{2}}}\right)^{\alpha_1}=\frac{1}{10}\left(\frac{1-0.5}{1-0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}}\approx 0.1111<1$$

and the constant

$$c=\frac{\left(\frac{1-0.5}{1-0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}}}{1-\frac{1}{10}\left(\frac{1-0.5}{1-0.5^{\frac{3}{4}}}\right)^{\frac{1}{2}}}\approx 1.2493>0.$$

Thus, the equation (6.1) has a unique solution and is Ulam-Hyers stable.

7. Conclusion

In this work, the author present the novel generalized definition of q-fractional derivative under the aspect of order quantity, which is the n-tuple order q-fractional derivative. The existence and uniqueness of solutions to the n-tuple order q-fractional fractional difference equations are proved by Banach contraction theorem under Lipschitz conditions for nonlinear terms. In addition, the Ulam-Hyers stability of solutions are demonstrated by examples considered.

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